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THE EXPECTED JAGGEDNESS OF ORDER IDEALS

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Abstract

The *jaggedness* of an order ideal I in a poset P is the number of maximal elements in I plus the number of minimal elements of P not in I. A probability distribution on the set of order ideals of P is *toggle-symmetric* if for every $p \in P$, the probability that p is maximal in I equals the probability that p is minimal not in I. In this paper, we prove a formula for the expected jaggedness of an order ideal of P under any toggle-symmetric probability distribution when P is the poset of boxes in a skew Young diagram. Our result extends the main combinatorial theorem of Chan–López–Pflueger–Teixidor [*Trans. Amer. Math. Soc.*, forthcoming. 2015, arXiv:1506.00516], who used an expected jaggedness computation as a key ingredient to prove an algebro-geometric formula; and it has applications to homomesies, in the sense of Propp–Roby, of the antichain cardinality statistic for order ideals in partially ordered sets.

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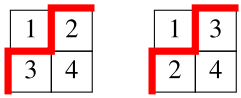
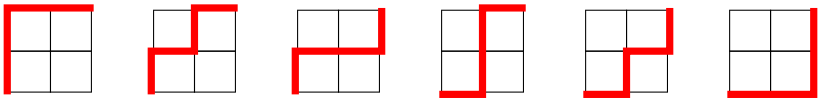
1. Introduction

Fix an $a \times b$ grid. Consider the set $\mathcal{S}$ of $\binom{a+b}{a}$ lattice paths from the lower-left corner of this grid to the upper-right corner. Given a standard Young tableau T of shape $a \times b$ and using English notation, we say a path $s \in \mathcal{S}$ is *compatible*

with T if all the labels of T northwest of s are smaller than all of the labels of T southeast of s. For example, the two standard Young tableaux of shape 2×2 are compatible with the path marked in bold red:



Define a probability distribution on $\mathcal{S}$ as follows: a path occurs with probability proportional to the number of $a \times b$ standard Young tableaux with which it is compatible. We call this distribution μ_{lin}, the *linear* distribution on lattice paths, since it comes from linear orderings of the ab boxes in the grid. For example, the six lattice paths in a 2×2 grid occur in μ_{lin} with the probabilities shown below:



1/5 1/5 1/10 1/10 1/5 1/5

We define the *jaggedness* of such a lattice path to be its number of turns. This is the same as the jaggedness of the order ideal to its northwest (see Definition 2.1). We may ask: what is the expected jaggedness of a lattice path chosen according to μ_{lin}? The answer is surprisingly simple.

THEOREM 1.1. *The expected jaggedness of a lattice path in an $a \times b$ grid, chosen under the distribution μ_{lin}, is exactly $2ab/(a+b)$, the harmonic mean of a and b.*

Theorem 1.1 and a generalization thereof appeared recently in [6] as the key combinatorial result underlying the computation of the genera of *Brill–Noether curves* (Brill–Noether loci of dimension 1). Briefly: it is used to compute the average vertex degree in the dual graph of a nodal degeneration, parametrizing Eisenbud–Harris limit linear series, of a given Brill–Noether curve. (For background on limit linear series, see [8].) Noted in [6] is the unexpected appearance of the harmonic mean, as well as the observation that if the distribution μ_{lin} is replaced by the *uniform* distribution, the answer is still the harmonic mean.

The purpose of this paper is to give a vast generalization of Theorem 1.1, in particular explaining the seeming coincidence above, and putting it in its proper combinatorial context: order ideals in arbitrary posets, and *toggle-symmetric* probability distributions on them. In particular, we would like to put forth these distributions as an interesting class of distributions to study, especially in relation

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to the developing area of dynamical algebraic combinatorics. We define toggle-symmetric distributions on order ideals of posets, and give, with proof, many natural examples, in Section 2. The word 'toggle' refers to the procedure of adding or removing an element from a set if it is permissible to do so. The term was coined by Striker–Williams [17] in describing Cameron and Fon-Der-Flaass' involutions on sets of order ideals of posets [5]. Indeed, our results have direct applications to homomesy results for order ideals under special compositions of toggles, as we discuss.

In Section 3, we prove our main result: a formula for expected jaggedness that applies to all skew Young diagrams, not just rectangles, and any toggle-symmetric distribution. Here is our main theorem:

THEOREM 1.2. *Let σ be a connected skew shape with height a and width b. Let μ be any toggle-symmetric probability distribution on the subshapes of σ. The expected jaggedness of a subshape of σ with respect to μ is*

$$\frac{2ab}{a+b}\left(1+\sum_{c\in C(\sigma)}\delta(c)\,\mathbb{P}_{\mu}(c)\right). \tag{1.1}$$

Here:

- $C(\sigma)$ is the set of *outward corners* of σ, and $\mathbb{P}_{\mu}(c)$ is the probability (according to μ) that the edges of the outward corner c are both included in the lattice path that 'cuts out' the subshape (see Definition 3.2 for details);
- the *displacement* $\delta(c)$ is proportional to the signed distance between the corner c and the antidiagonal of the partition (Definition 3.3).

For now, the main point is that the expected jaggedness can be calculated as the harmonic mean of a and b, plus a sum of correction terms that can be completely understood in terms of μ and σ. (When σ is a rectangle, there are no correction terms and (1.1) gives the harmonic mean exactly, for *any* toggle-symmetric distribution.)

There are several key differences between Theorem 1.2 and the corresponding result [6, Theorem 2.8] of Chan *et al*. First, our theorem applies to *any* toggle-symmetric distribution. Moreover, it is fully symmetric with respect to interchanging rows and columns, which is not the case in [6]. Indeed, our result makes explicit that the only dependence is on the outer corners and their displacements. This will allow us to immediately derive that for any *balanced* shape, the expected jaggedness is always the harmonic mean; see Corollary 3.8.

We also note that our results, combined with theorems of Striker [17], have direct applications to homomesy results under the operations of rowmotion and

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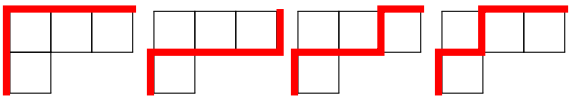
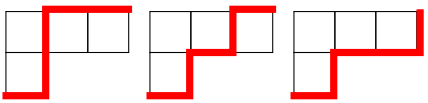
gyration on posets. In particular, they allow us to recover and generalize a theorem of Propp and Roby [12] on homomesies for antichain cardinalities. We explain these applications in Section 3.2. In Section 4, we give four open questions we would like to see explored.

We close this section by giving an example that illustrates Theorem 1.2.

EXAMPLE 1.3. Consider the Young diagram shape $\sigma = (3, 1)$. The seven subshapes of σ, equivalently the lattice paths in σ, are depicted below. The numbers below each path indicate that subshape's jaggedness, along with the probability of that subshape's occurrence according to the linear distribution. Then we can calculate directly that $\mathbb{E}_{\mu_{\text{lin}}}(\text{jag}) = 34/15$.



(1; 1/5) (2; 1/15) (3; 2/15) (3; 1/5)



(2; 1/15) (3; 2/15) (2; 1/5)

Now, let us use Theorem 1.2 instead to compute $\mathbb{E}_{\mu_{\text{lin}}}(\text{jag})$, using the fact that μ_{lin} is toggle-symmetric by Proposition 2.9 and Remark 2.11. The corner c occurring at $(1, 1)$ is the only outward corner of σ (see Section 3 for an explanation of our coordinate system). Its displacement $\delta(c)$ is $-1/6$, as in Definition 3.3. Finally, formula (3.7) obtained in Section 3.1 implies that $\mathbb{P}_{\mu_{\text{lin}}}(c) = 1/3$. Plugging these values into (1.1) yields

$$\mathbb{E}(\text{jag}) = (12/5)(1 - 1/6 \cdot 1/3) = 34/15,$$

as expected.

2. Toggle-symmetric distributions

For background on posets see [16, Section 3]. Fix a finite poset $(P, \leqslant)$. An *order ideal* of P is a subset $I \subseteq P$ such that for every $p \in I$ and every $q \in P$ with $q \leqslant p$, we have $q \in I$. We denote the set of order ideals of P by $\mathcal{J}(P)$. If $P = P_1 \sqcup P_2$ then the set of order ideals decomposes as $\mathcal{J}(P) = \mathcal{J}(P_1) \times \mathcal{J}(P_2)$ so we assume from now on that P is connected. We do not consider the empty

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poset connected. Let $I \in \mathcal{J}(P)$ and let $p \in P$ be any element. We say p can be *toggled in* to I if p is a minimal element not in I, and that p can be *toggled out* of I if p is a maximal element in I. Equivalently, p can be toggled in to I if $p \notin I$ and $I \cup \{p\} \in \mathcal{J}(P)$, and p can be toggled out of I if $p \in I$ and $I \setminus \{p\} \in \mathcal{J}(P)$.

DEFINITION 2.1. The *jaggedness* of an order ideal $I \in \mathcal{J}(P)$, denoted $\mathrm{jag}(I)$, is the total number of elements $p \in P$ which can be toggled into I or out of I.

In this paper we consider $\mathcal{J}(P)$ as a discrete probability space and so refer to functions on $\mathcal{J}(P)$ as random variables. Define, for each $p \in P$, two indicator random variables $\mathcal{T}_p^+, \mathcal{T}_p^- \colon \mathcal{J}(P) \to \mathbb{R}$ that record whether p is toggleable-in (respectively toggleable-out) of an order ideal. Explicitly, for every $I \in \mathcal{J}(P)$, we define

$$\mathcal{T}_p^+(I) := \begin{cases} 1 & \text{if } p \text{ can be toggled in to } I, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{T}_p^-(I) := \begin{cases} 1 & \text{if } p \text{ can be toggled out of } I, \\ 0 & \text{otherwise.} \end{cases}$$

These random variables are highly related to Striker's toggleability statistic [17, Definition 6.1]. Indeed, her toggleability statistic $\mathcal{T}_p$ simply decomposes as $\mathcal{T}_p = \mathcal{T}_p^+ - \mathcal{T}_p^-$. Note furthermore that $\mathrm{jag} = \sum_{p \in P} (\mathcal{T}_p^+ + \mathcal{T}_p^-)$. In this paper, we show how certain conditions on $\mathcal{T}_p$ imply conditions on jag, as in the following main definition of the section.

DEFINITION 2.2. Let μ be a probability distribution on $\mathcal{J}(P)$. Given an element $p \in P$, we say that μ is *toggle-symmetric at* p if

$$\mathbb{P}_\mu(\, p \text{ can be toggled in to } I) = \mathbb{P}_\mu(p \text{ can be toggled out of } I).$$

Equivalently, μ is toggle-symmetric at p if

$$\mathbb{E}_\mu(\mathcal{T}_p) = \mathbb{E}_\mu(T_p^+) - \mathbb{E}_\mu(T_p^-) = 0.$$

We say that μ is *toggle-symmetric* if it is toggle-symmetric at every $p \in P$.

We would like to introduce toggle-symmetric probability distributions as an interesting class of distributions on order ideals of posets. We now give plenty of good examples of toggle-symmetric distributions. Throughout, we fix a poset P with $n := \#P$.

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2.1. Toggle-symmetric distributions arising from P-partitions. In this subsection we define several families of toggle-symmetric distributions that arise from P-partitions and related objects like linear extensions. For background on P-partitions see [16, Section 3.15] or the recent historical survey [9].

DEFINITION 2.3. A *linear extension* of P is a bijection

$$\ell\colon P \to \{1, 2, \ldots, n\},$$

such that $p \leqslant q$ for $p, q \in P$ implies $\ell(p) \leqslant \ell(q)$. The *linear distribution* μ_{lin} on $\mathcal{J}(P)$ is defined as follows: for $I \in \mathcal{J}(P)$ we define $\mu_{\text{lin}}(I)$ to be the probability that, choosing a linear extension ℓ of P and $k \in \{0, 1, 2, \ldots, n\}$ uniformly at random, the order ideal $\ell^{-1}(\{1, \ldots, k\})$ is equal to I.

DEFINITION 2.4. A *weak reverse P-partition of height m* is a map

$$\ell\colon P \to \{0, 1, \ldots, m\},$$

such that $p \leqslant q$ for $p, q \in P$ implies $\ell(p) \leqslant \ell(q)$. Fix $m \geqslant 1$. The *weak distribution* $\mu_{m,\leqslant}$ on $\mathcal{J}(P)$ is defined as follows: for $I \in \mathcal{J}(P)$ we define $\mu_{m,\leqslant}(I)$ to be the probability that, choosing a weak reverse P-partition ℓ of height m and $k \in \{1, 2, \ldots, m\}$ uniformly at random, the order ideal $\ell^{-1}(\{0, \ldots, k-1\})$ is equal to I.

REMARK 2.5. There is a bijection between $\mathcal{J}(P)$ and the set of weak reverse P-partitions of height 1 given by sending an order ideal I to $\mathbf{1} - \mathbf{1}_I$ where $\mathbf{1}_X$ is the indicator function of a subset $X \subseteq P$. Thus $\mu_{1,\leqslant}$ is simply the *uniform* distribution on $\mathcal{J}(P)$, which we denote μ_{unif}.

DEFINITION 2.6. A *strict reverse P-partition of height m* is a map

$$\ell\colon P \to \{0, 1, \ldots, m\},$$

such that $p < q$ for $p \neq q \in P$ implies $\ell(p) < \ell(q)$. The *rank* of P, denoted $\text{rk}(P)$, is the maximum length of a chain of P. Given $m \geqslant \text{rk}(P)$, the *strict distribution* $\mu_{m,<}$ on $\mathcal{J}(P)$ is defined as follows: for $I \in \mathcal{J}(P)$ we define $\mu_{m,<}(I)$ to be the probability that, choosing a strict reverse P-partition ℓ of height m and $k \in \{0, 1, \ldots, m+1\}$ uniformly at random, the order ideal $\ell^{-1}(\{0, \ldots, k-1\})$ is equal to I.

REMARK 2.7. We say that P is *ranked* if there exists a *rank function* $\text{rk}\colon P \to \mathbb{Z}_{\geqslant 0}$ such that $\text{rk}(q) = \text{rk}(p) + 1$ whenever q covers p (denoted $p \lessdot q$)

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in P. We always assume that 0 is in the image of the rank function in which case rk is uniquely determined if it exists. We say that P is *graded* if it is ranked and moreover $\mathrm{rk}(p) = 0$ for all minimal elements p of P and $\mathrm{rk}(q) = \mathrm{rk}(P)$ for all maximal elements q of P. Equivalently, P is graded if all maximal chains of P have the same length. When P is graded, the distribution $\mu_{\mathrm{rk}(P),<}$ is easy to describe: it is uniform on the set of all order ideals $\mathrm{rk}^{-1}(\{0, \dots, k-1\})$ for $k \in \{0, \dots, \mathrm{rk}(P)+1\}$. In this case we call $\mu_{\mathrm{rk}(P),<}$ the *rank distribution* and denote it by μ_{rk}.

Now we show that all of the above distributions μ_{lin}, $\mu_{m,\leqslant}$, and $\mu_{m,<}$ are toggle-symmetric. We start by proving toggle symmetry for $\mu_{m,\leqslant}$ and $\mu_{m,<}$.

LEMMA 2.8. *For any poset P and any $m \geqslant 1$, the distribution $\mu_{m,\leqslant}$ on $\mathcal{J}(P)$ is toggle-symmetric. Similarly, for any $m \geqslant \mathrm{rk}(P)$, the distribution $\mu_{m,<}$ on $\mathcal{J}(P)$ is toggle-symmetric. In particular, the uniform distribution μ_{unif} is toggle-symmetric and, if P is graded, the rank distribution μ_{rk} is toggle-symmetric.*

Proof. Let us start by proving the lemma with the weak distribution $\mu_{m,\leqslant}$. For a given $p \in P$, we define an involution τ_p on the set of weak reverse P-partitions of height m, and this involution will verify that μ is toggle-symmetric at p. Let $\widehat{P}$ denote the poset obtained from P by adjoining a minimal element $\widehat{0}$ and a maximal element $\widehat{1}$. Let ℓ be a weak reverse P-partition of height m; we extend ℓ to $\widehat{P}$ by setting $\ell(\widehat{0}) := 0$ and $\ell(\widehat{1}) := m$. Then for $p, q \in P$ we define

$$\tau_p(\ell)(q) := \begin{cases} \ell(q), & q \neq p, \\ \max\{\ell(r) \colon r \lessdot p, r \in \widehat{P}\} + \min\{\ell(r) \colon p \lessdot r, r \in \widehat{P}\} - \ell(p), & q = p. \end{cases} \tag{2.1}$$

Evidently τ_p is an involution and preserves the relevant weak inequalities. For ℓ a weak reverse P-partition of height m and $k \in \{1, 2, \dots, m\}$ we have

$$\mathcal{T}_p^+(\ell^{-1}(\{0, \dots, k-1\})) = \begin{cases} 1 & \text{if } \max\{\ell(r) \colon r \lessdot p, r \in \widehat{P}\} < k \leqslant \ell(p), \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\mathbb{E}_{\mu_{m,\leqslant}}(\mathcal{T}_p^+) = \mathbb{E}(\ell(p) - \max\{\ell(r) \colon r \lessdot p, r \in \widehat{P}\})$$

for ℓ a uniformly random weak reverse P-partition of height m. Similarly,

$$\mathbb{E}_{\mu_{m,\leqslant}}(\mathcal{T}_p^-) = \mathbb{E}(\min\{\ell(r) \colon p \lessdot r, r \in \widehat{P}\} - \ell(p)).$$

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But then observe that

$$\begin{aligned}\mathbb{E}_{\mu_{m,\leqslant}}(\mathcal{T}_p^+) &= \mathbb{E}(\ell(p) - \max\{\ell(r)\colon r \lessdot p, r \in \widehat{P}\}) \\ &= \mathbb{E}(\min\{\ell(r)\colon p \lessdot r, r \in \widehat{P}\} - \tau_p(\ell)(p)) \\ &= \mathbb{E}_{\mu_{m,\leqslant}}(\mathcal{T}_p^-)\end{aligned}$$

and thus indeed $\mu_{m,\leqslant}$ is toggle-symmetric.

The proof of the lemma for the strong distribution is exactly analogous to the weak distribution. Let ℓ be a strict reverse P-partitions of height m; we extend ℓ to $\widehat{P}$ by setting $\ell(\widehat{0}) := -1$ and $\ell(\widehat{1}) := m+1$. For $p \in P$ we define an involution τ_p on the set of strict reverse P-partitions by the exact same formula (2.1) as above. This involution again establishes that $\mathbb{E}_{\mu_{m,<}}(\mathcal{T}_p) = 0$. The last sentence follows from Remarks 2.5 and 2.7. □

PROPOSITION 2.9. *We have*

$$\lim_{m\to\infty} \mu_{m,\leqslant} = \lim_{m\to\infty} \mu_{m,<} = \mu_{\text{lin}}.$$

Proof. To prove this proposition we define an intermediary distribution $\mu_{m,\hookrightarrow}$ on $\mathcal{J}(P)$ based on injective order-preserving maps. It will turn out that $\mu_{m,\hookrightarrow} = \mu_{\text{lin}}$. For $m \geqslant n-1$ and $I \in \mathcal{J}(P)$ we define $\mu_{m,\hookrightarrow}(I)$ to be the probability that, choosing an *injective* order-preserving map $\ell\colon P \to \{0,1,\ldots,m\}$ and $k \in \{0,1,\ldots,m+1\}$ uniformly at random, the order ideal $\ell^{-1}(\{0,\ldots,k-1\})$ is equal to I. First we claim

$$\lim_{m\to\infty} \mu_{m,\leqslant} = \lim_{m\to\infty} \mu_{m,<} = \lim_{m\to\infty} \mu_{m,\hookrightarrow}.$$

This is clear because as $m \to \infty$ the fraction of (weak or strict) order-preserving maps $P \to \{0,1,\ldots,m\}$ that are injective approaches 1. Furthermore, in the case of $\mu_{m,\leqslant}$, as $m \to \infty$ the probability that a uniformly chosen $k \in \{0,\ldots,m+1\}$ actually lands in $\{1,\ldots,m\}$ approaches 1.

Next we claim that $\mu_{m,\hookrightarrow} = \mu_{\text{lin}}$ for all $m \geqslant n-1$. Given an order ideal I, let $L(I)$ denote the set of linear extensions $\ell\colon P \xrightarrow{\cong} \{1,\ldots,n\}$ that are compatible with I; that is, if $p \in I$ and $q \notin I$ then $\ell(p) < \ell(q)$. Let $\Phi_{n,m}$ denote the set of order-preserving injective maps $\phi\colon \{0,\ldots,n+1\} \to \{-1,\ldots,m+1\}$ such that $\phi(0) = -1$ and $\phi(n+1) = m+1$. Then

$$\mathbb{P}_{\mu_{m,\hookrightarrow}}(I) \sim \sum_{\ell\in L(I)} \sum_{\phi\in\Phi_{n,m}} (\phi(\#I+1) - \phi(\#I)),$$

where the sign $\sim$ denotes proportionality up to a constant. The reason is that the order-preserving injective map $\phi|_{\{1,\ldots,n\}} \circ \ell$ gives rise to I if and only if I is

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compatible with ℓ, and furthermore, if I is compatible with the linear extension ℓ, then I arises from the map $\phi|_{\{1,\dots,n\}} \circ \ell$ with probability proportional to $\phi(\#I+1) - \phi(\#I)$.

But now we claim the inner sum $\sum_{\phi \in \Phi_{n,m}} (\phi(\#I+1) - \phi(\#I))$ is a constant, not depending on $\#I$. Indeed, given $\phi \in \Phi_{n,m}$, call the *signature* of ϕ the multiset of consecutive differences

$$\{\phi(1) - \phi(0), \phi(2) - \phi(1), \dots, \phi(n+1) - \phi(n)\}.$$

Then the sum $\sum (\phi(\#I+1) - \phi(\#I))$ restricted to any given signature-equivalence class in $\Phi_{n,m}$ is a constant, not depending on $\#I$. Therefore, the same is true for the sum over all $\phi \in \Phi_{n,m}$. This shows that $\mu_{m,\hookrightarrow}(I)$ is proportional to $\#L(I)$ and so $\mu_{m,\hookrightarrow} = \mu_{\mathrm{lin}}$. □

From Lemma 2.8 and Proposition 2.9 we conclude:

COROLLARY 2.10. *The linear distribution μ_{lin} on $\mathcal{J}(P)$ is toggle-symmetric.*

So for any poset P we have the following 'spectrum' of toggle-symmetric distributions:

$$\mu_{\mathrm{unif}} \xrightarrow{\mu_{m,\leqslant}} \mu_{\mathrm{lin}} \xleftarrow{\mu_{m,<}} \mu_{\mathrm{rk}}$$

where the rightmost distribution μ_{rk} applies only to graded P.

REMARK 2.11. Actually, we can give a much more direct and satisfying proof of Corollary 2.10 which goes by way of defining some interesting involutions σ_p for $p \in P$ on the set of linear extensions of P. Note that Corollary 2.10 was proved for skew shapes in [**6**, Lemma 2.9], using involutions on pairs (linear extension, order ideal). Instead, the involutions σ_p that we use here can be regarded as 'shuffle' operations on the set of linear extensions only, in the spirit of [**2**]. They are defined as follows. Let ℓ be a linear extension of P. Extend ℓ to $\widehat{P}$ by setting $\ell(\widehat{0}) := 0$ and $\ell(\widehat{1}) := n+1$. Let $p \in P$ and $x := \ell(p)$ and $x' := \max\{\ell(r) \colon r \lessdot p, r \in \widehat{P}\} + \min\{\ell(r) \colon p \lessdot r, r \in \widehat{P}\} - x$. Then for $q \in P$, define

$$\sigma_p(\ell)(q) := \begin{cases} x' & \text{if } \ell(q) = x, \\ \ell(q) - 1 & \text{if } x < \ell(q) \leqslant x', \\ \ell(q) + 1 & \text{if } x' \leqslant \ell(q) < x, \\ \ell(q) & \text{otherwise.} \end{cases}$$

It is straightforward to verify that σ_p is indeed an involution on the set of linear extensions of P, and that, exactly analogously to the proof of Lemma 2.8, this involution verifies that $\mathbb{E}_{\mu_{\mathrm{lin}}}(\mathcal{T}_p) = 0$.

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2.2. Toggle-symmetric distributions arising from the toggle group. The toggle group was introduced by Cameron and Fon-der-Flaass [5] in order to study a certain combinatorial map on order ideals that is now called *rowmotion*. For background on the toggle group, rowmotion, and gyration, see [18]. For an element $p \in P$ we define the *toggle at* p, denoted $\tau_p \colon \mathcal{J}(P) \to \mathcal{J}(P)$, by

$$\tau_p(I) := \begin{cases} I \Delta \{p\} & \text{if } I \Delta \{p\} \in \mathcal{J}(P), \\ I & \text{otherwise} \end{cases}$$

where Δ denotes the symmetric difference. These are the same τ_p as defined in the proof of Lemma 2.8 for the weak distribution when $m = 1$ via the bijection mentioned in Remark 2.5. The *toggle group* is the subgroup of the permutation group $\mathfrak{S}_{\mathcal{J}(P)}$ generated by all toggles τ_p for $p \in P$. Recently Striker [17] proved that certain distributions on $\mathcal{J}(P)$ arising from toggle group elements are toggle-symmetric.

DEFINITION 2.12. *Rowmotion* is the element of the toggle group

$$\tau_{\ell^{-1}(1)} \circ \tau_{\ell^{-1}(2)} \cdots \circ \tau_{\ell^{-1}(n)}$$

where ℓ is any linear extension of P. Note that because τ_p and τ_q commute unless $p \lessdot q$ or $q \lessdot p$ this composition indeed gives a well-defined map.

DEFINITION 2.13. Assume P is ranked. *Gyration* is the element of the toggle group

$$\tau_{o_1} \circ \tau_{o_2} \circ \cdots \circ \tau_{o_{n_1}} \circ \tau_{e_1} \circ \tau_{e_2} \circ \cdots \circ \tau_{e_{n_0}}$$

with

$$\{e_1, \ldots, e_{n_0}\} = \{p \in P : \mathrm{rk}(p) \text{ even}\} \text{ and } \{o_1, \ldots, o_{n_1}\} = \{p \in P : \mathrm{rk}(p) \text{ odd}\}.$$

Again, because most toggles commute, this composition gives a well-defined map.

THEOREM 2.14 (Striker [17]). *Let P be a poset and let $\varphi \colon \mathcal{J}(P) \to \mathcal{J}(P)$ be rowmotion or, in the case where P is ranked, gyration. Then the distribution μ that is supported uniformly on a fixed φ-orbit $\mathcal{O}$ is toggle-symmetric.*

Actually, Striker phrased her result in the language of homomesy. Homomesy is a certain phenomenon in dynamical algebraic combinatorics, recently introduced by Propp and Roby [12], concerning statistical averages along orbits of combinatorial maps.

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DEFINITION 2.15. Let $\mathcal{S}$ be a set of combinatorial objects and $\varphi\colon \mathcal{S} \to \mathcal{S}$ an invertible map. We say that the statistic $f : \mathcal{S} \to \mathbb{R}$ is *homomesic*, or more specifically, *c-mesic*, with respect to the action of φ on $\mathcal{S}$ if there is $c \in \mathbb{R}$ such that $(1/\#\mathcal{O}) \sum_{s\in\mathcal{O}} f(s) = c$ for each φ-orbit $\mathcal{O}$. In other words, we say f is *homomesic* with respect to φ if the average of f is the same for each φ-orbit.

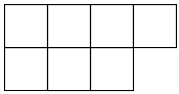
For more about homomesy, see Section 3.2 below, especially Remark 3.12.

What Striker proved was that, for any poset P and any $p \in P$, the signed toggleability statistic $\mathcal{T}_p$ is 0-mesic with respect to rowmotion [**17**, Lemma 6.2] and is 0-mesic with respect to gyration when P is ranked [**17**, Theorem 6.7]. Clearly these results are equivalent to Theorem 2.14 as stated above.

3. The expected jaggedness in skew shapes

In this section, we prove a general result giving a formula for the expected jaggedness of an order ideal in a poset P for any toggle-symmetric distribution whenever P is the poset corresponding to a skew Young diagram.

A partition $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_k)$ is a sequence of weakly decreasing positive integers. Recall that associated to λ is a *Young diagram* consisting of λ_i boxes in the ith row, left-justified. Given two partitions λ and ν of the numbers ℓ and n respectively, we say that $\nu \subseteq \lambda$ if $\nu_i \leqslant \lambda_i$ for all i. We use the usual convention that $\lambda_i = 0$ if i is greater than the number of parts of λ. We use English notation when drawing partitions, so for instance the Young diagram corresponding to the partition $\lambda = (4, 3)$ is



DEFINITION 3.1. Let $\nu \subseteq \lambda$ be two partitions. The diagram obtained by subtracting the Young diagram of ν from the Young diagram of λ is called a *skew Young diagram* or *skew shape*. We write $\sigma = \lambda/\nu$ for this shape.

Let $\sigma = \lambda/\nu$ be a skew shape. Throughout, we let a denote the *height* of σ, that is, the number of rows in σ, and let b denote the *width* of σ, that is, the number of columns. In order to refer to the boxes of σ and their corners, we fix coordinates as follows. Place σ in an $a \times b$ rectangle. Our convention will be that the northwest corner of the rectangle is $(0, 0)$ and the southeast corner is (a, b). The corners of the boxes of σ are then various lattice points in this rectangle. Furthermore, we extend this coordinate system to the boxes of σ by writing $[i, j]$ for the box

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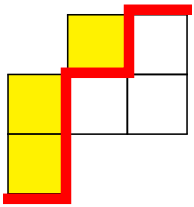

Figure 1. With $\sigma = (3, 3, 1)/(1)$, in this figure we depict $(2, 1, 1)/(1) \in \mathcal{J}(\sigma)$ shaded in yellow and its associated lattice path in bold red.

whose *southeast* corner is (i, j). For example, the upper-leftmost box of a Young diagram is the box $[1, 1]$.

Associated to any skew shape σ is a poset P_σ whose elements are the boxes of σ and with $[i, j] \leqslant [k, l]$ if and only if $i \leqslant k$ and $j \leqslant l$. Note that P_σ is always ranked (where the rank function $\mathrm{rk}([i, j]) = i + j - \kappa$, for an appropriate constant κ, records the diagonal) but is not always graded. All of the general poset-theoretic constructions from Section 2.1 have more common names when specialized to skew shapes, which we record in the following dictionary:

Poset P_σ	Skew shape σ
Order ideals	Subshapes
Linear extensions	Standard Young tableaux
Weak reverse P_σ-partitions	Reverse plane partitions
Strict reverse P_σ-partitions	Increasing tableaux

We will not go through all of these terms in detail, but let us comment for a moment on subshapes. If $\sigma = \lambda/\nu$ is a skew shape, then a *subshape of* σ is a skew shape ρ/ν where ρ is a partition satisfying $\nu \subseteq \rho \subseteq \lambda$. These are clearly the same as order ideals of P_σ and so we use the notation $\mathcal{J}(\sigma)$ for the set of subshapes of σ. We also often identify a subshape $\rho/\nu \in \mathcal{J}(\sigma)$ with its *lattice path*, which is the sequence of steps of the form $(-1, 0)$ and $(0, 1)$ connecting the point $(a, 0)$ to $(0, b)$ (in the coordinate system defined above) given by the southeast border of ρ. In this way $\mathcal{J}(\sigma)$ is in bijection with the set of lattice paths connecting $(a, 0)$ to $(0, b)$ that stay within the diagram of σ. For an example of this bijection see Figure 1.

DEFINITION 3.2. Let σ be a skew shape. We say σ is *connected* if the poset P_σ is connected. Suppose σ is connected. Then an *outward corner* of σ is two consecutive steps along the boundary of σ that do not belong to the same line and

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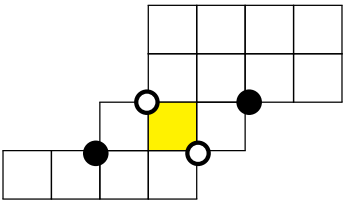

Figure 2. A diagram explaining our notation for corners. Box [3, 4] of σ is shaded yellow and the points where corners $c \in C(\sigma)$ occur are marked with a circle (either filled or unfilled); the points where corners $c \in C_{34}(\sigma)$ occur are the unfilled circles.

do not border a common box of σ. We say that a corner *occurs* at the lattice point (i, j) where its two steps meet. We write $C(\sigma)$ for the set of outward corners of σ.

Note that because σ is a skew shape, the outward corners of σ are either *northwest corners* or *southeast corners*, that is, they comprise part of the northwest border of σ or the southeast border, respectively.

The following notation will be convenient for us: given a box $[i, j] \in \sigma$, we define

$$C_{ij}(\sigma) := \{c \in C(\sigma) \text{ occurring strictly northwest or strictly southeast of } [i, j]\}.$$

When we say that a corner c occurs 'strictly northwest' or 'strictly southeast' of a box $[i, j]$, we mean that it occurs strictly northwest (respectively strictly southeast) of the *center* of that box. For example, a corner at the point (i, j) occurs strictly southeast of the box $[i, j]$. Figure 2 illustrates our notation for corners.

For $c \in C(\sigma)$ and μ a probability distribution on $\mathcal{J}(\sigma)$ we use the notation $\mathbb{P}_\mu(c)$ to mean the probability with respect to μ that a subshape of σ, thought of as a lattice path, includes the two steps of the corner c. It is important to note that if the corner $c \in C(\sigma)$ occurs at (i, j), then saying that the lattice path $\rho \in \mathcal{J}(c)$ includes c is a stronger statement than merely saying that ρ passes through (i, j).

DEFINITION 3.3. Let σ be a connected skew shape with height a and width b. The *main antidiagonal of* σ is the line joining $(a, 0)$ to $(0, b)$. For $(i, j) \in \mathbb{R}^2$ let $\vec{d}(i, j)$ denote the vector from (i, j) to the main antidiagonal of σ and orthogonal to it. For an outward corner $c \in C(\sigma)$ that occurs at (i, j) we define

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the *displacement* of c to be

$$\delta(c) := \begin{cases} \text{the unique } x \in \mathbb{R} \text{ with } \vec{d}(0,0) = x \cdot \vec{d}(i,j) & \text{if } c \text{ is a northwest corner,} \\ \text{the unique } x \in \mathbb{R} \text{ with } \vec{d}(a,b) = x \cdot \vec{d}(i,j) & \text{if } c \text{ is a southeast corner.} \end{cases}$$

Note that $\delta(c)$ is a signed quantity. Explicitly,

$$\delta(c) = \begin{cases} 1 - \dfrac{i}{a} - \dfrac{j}{b} & \text{if } c \text{ is a northwest corner,} \\ -1 + \dfrac{i}{a} + \dfrac{j}{b} & \text{if } c \text{ is a southeast corner.} \end{cases}$$

Now we can state the main theorem of our paper, which computes the expected jaggedness of a subshape of σ for any toggle-symmetric distribution as the harmonic mean of its height and width, up to a sum of signed correction terms.

THEOREM 3.4. *Let σ be a connected skew shape with height a and width b. Let μ be any toggle-symmetric probability distribution on $\mathcal{J}(\sigma)$. The expected jaggedness of a subshape of σ with respect to the distribution μ is*

$$\mathbb{E}_\mu(\text{jag}) = \frac{2ab}{a+b}\left(1 + \sum_{c \in C(\sigma)} \delta(c)\, \mathbb{P}_\mu(c)\right). \tag{3.1}$$

In the rest of this section we prove Theorem 3.4. In order to do that, we define a set of random variables R_{ij} that we refer to as *rooks*. The proof of the main theorem involves strategically placing rooks on our skew shape σ.

Let $[i, j]$ be a box in σ. We write $\mathcal{T}^+_{ij}$ and $\mathcal{T}^-_{ij}$ for the toggle-indicator random variables $T^+_{[i,j]}$ and $T^-_{[i,j]}$ on $\mathcal{J}(P_\sigma) = \mathcal{J}(\sigma)$ defined in Section 2. We define the rook random variable $R_{ij} : \mathcal{J}(\sigma) \to \mathbb{R}$ as follows:

$$R_{ij} := \sum_{\substack{i' \leqslant i,\, j' \leqslant j \\ [i',j'] \in \sigma}} \mathcal{T}^+_{i'j'} + \sum_{\substack{i' \geqslant i,\, j' \geqslant j \\ [i',j'] \in \sigma}} \mathcal{T}^-_{i'j'} - \sum_{\substack{i' < i,\, j' < j \\ [i',j'] \in \sigma}} \mathcal{T}^-_{i'j'} - \sum_{\substack{i' > i,\, j' > j \\ [i',j'] \in \sigma}} \mathcal{T}^+_{i'j'}. \tag{3.2}$$

The equation defining R_{ij} is complicated and it is best understood by a drawing as in Figure 3. In this figure, we record the coefficients of the terms $T^+_{i'j'}$ and $T^-_{i'j'}$ in R_{ij} in the northwest and southeast corners, respectively, of the box $[i', j']$. The reason we call R_{ij} a rook is explained by the next lemma, which says that for a toggle-symmetric distribution μ only the toggleability statistics corresponding to boxes in the same row or column as $[i, j]$ contribute to the expectation $\mathbb{E}_\mu(R_{ij})$.

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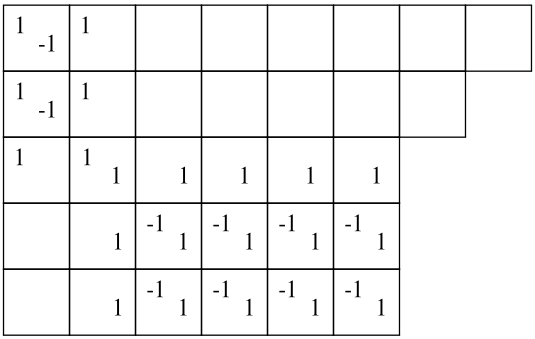

Figure 3. An example of a 'rook' at the box [3, 2].

LEMMA 3.5. *Let σ be a skew shape and μ a toggle-symmetric probability distribution on $\mathcal{J}(\sigma)$. Then for any $[i, j] \in \sigma$ we have*

$$\mathbb{E}_\mu(R_{ij}) = \sum_{[i',j]\in\sigma} \mathbb{E}_\mu(\mathcal{T}^+_{i',j}) + \sum_{[i,j']\in\sigma} \mathbb{E}_\mu(\mathcal{T}^+_{i,j'}).$$

Proof. Expanding formula (3.2),

$$R_{ij} = \sum_{\substack{i'<i,\, j'<j \\ [i',j']\in\sigma}} \mathcal{T}^+_{i'j'} - \sum_{\substack{i'<i,\, j'<j \\ [i',j']\in\sigma}} \mathcal{T}^-_{i'j'} + \sum_{\substack{i'>i,\, j'>j \\ [i',j']\in\sigma}} \mathcal{T}^+_{i'j'} - \sum_{\substack{i'>i,\, j'>j \\ [i',j']\in\sigma}} \mathcal{T}^-_{i'j'} + \sum_{[i,j']\in\sigma} \mathcal{T}^+_{i',j} + \sum_{[i,j']\in\sigma} \mathcal{T}^+_{i,j'}.$$

Since μ is a toggle-symmetric distribution by linearity of expectation we get

$$\mathbb{E}_\mu\left(\sum_{\substack{i'<i,\, j'<j \\ [i',j']\in\sigma}} \mathcal{T}^+_{i'j'} - \sum_{\substack{i'<i,\, j'<j \\ [i',j']\in\sigma}} \mathcal{T}^-_{i'j'}\right) = 0; \quad \mathbb{E}_\mu\left(\sum_{\substack{i'>i,\, j'>j \\ [i',j']\in\sigma}} \mathcal{T}^+_{i'j'} - \sum_{\substack{i'>i,\, j'>j \\ [i',j']\in\sigma}} \mathcal{T}^-_{i'j'}\right) = 0.$$

Hence the claimed expression for $\mathbb{E}_\mu(R_{ij})$ indeed holds. □

LEMMA 3.6. *Let σ be a connected skew shape and μ a probability distribution on $\mathcal{J}(\sigma)$. Then for any $[i, j] \in \sigma$ we have*

$$\mathbb{E}_\mu(R_{ij}) = 1 + \sum_{c\in C_{ij}(\sigma)} \mathbb{P}_\mu(c).$$

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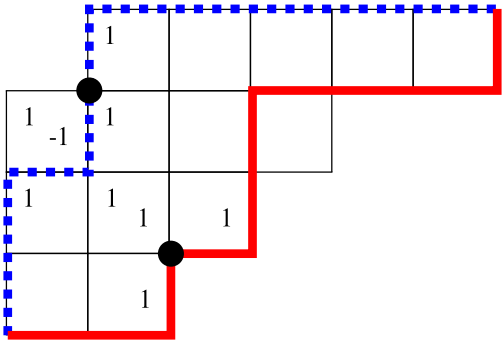

Figure 4. This figure illustrates how each subshape may contribute to $\mathbb{E}(R_{ij})$. Here $[i, j] = [3, 2]$ and the points where corners $c \in C_{ij}$ occur are marked with a circle. Two lattice paths $\rho_1, \rho_2 \in \mathcal{J}(\sigma)$ are drawn in bold blue (dashed) and bold red (solid); one verifies $R_{ij}(\rho_k) = 1 + \#C_{ij}(\rho_k)$ for $k = 1, 2$.

Proof. Let $\rho \in \mathcal{J}(\sigma)$. Let $C_{ij}(\rho)$ be the set of all corners $c \in C_{ij}(\sigma)$ included in the lattice path ρ. We observe that $R_{ij}(\rho) = 1+\#C_{ij}(\rho)$. This observation is again best understood by a picture, as in Figures 3 and 4. In Figure 3, the set $C_{ij}(\sigma)$ is empty, and the claim that $R_{ij}(\rho) = 1$ for any lattice path ρ drawn through the skew shape corresponds to the observation that the turns in ρ always have total weight 1 (with the weights as drawn). As usual, we identify lattice paths and subshapes.

The more general formula $R_{ij}(\rho) = 1 + \#C_{ij}(\rho)$ then corresponds to the fact that any outward corner $c \in C_{ij}(\sigma)$ used by ρ is no longer labeled -1, simply because there is no box at c to be toggled in or toggled out. This is illustrated in Figure 4.

But

$$\mathbb{E}_\mu(\#C_{ij}(\rho)) = \sum_{c \in C_{ij}(\sigma)} \mathbb{P}_\mu(c)$$

and hence the claimed expression for $\mathbb{E}_\mu(R_{ij})$ indeed holds. □

LEMMA 3.7. *For any connected skew shape σ with height a and width b there exist integral coefficients $r_{ij} \in \mathbb{Z}$ for $[i, j] \in \sigma$ such that*

- *for all $1 \leqslant i \leqslant a$, $\sum_{[i,j'] \in \sigma} r_{i,j'} = b$;*
- *for all $1 \leqslant j \leqslant b$, $\sum_{[i',j] \in \sigma} r_{i',j} = a$.*

Proof. If we interpret the coefficient r_{ij} as the number (possibly negative) of rooks placed at box $[i, j] \in \sigma$, the equalities say that each row should be attacked by a total of b rooks and each column by a total of a rooks. There are many possible

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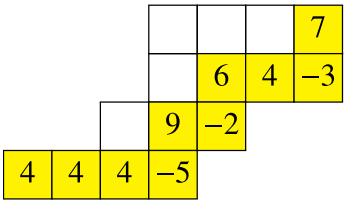

Figure 5. An example of a rook placement that satisfies Lemma 3.7. Here $a = 4$ and $b = 7$; the southeast border strip is shaded in yellow.

such placements. Here is one. Let $\mathcal{B} := \{[i, j] \in \sigma : [i+1, j+1] \notin \sigma\}$ denote the set of boxes in the *southeast border strip of* σ. We claim there is a unique choice of the r_{ij} satisfying the desired equalities with $r_{ij} = 0$ if $[i, j] \notin \mathcal{B}$. Let $b_1, b_2, \ldots, b_m$ be the elements of $\mathcal{B}$ in the unique order so that b_1 is southwesternmost, b_m is northeasternmost, and b_k is adjacent to b_{k+1} for all $1 \leqslant k < m$. Then for each $1 \leqslant k \leqslant m$, exactly one of the following holds:

(I) b_l is not in the same row as b_k for all $l > k$;

(II) b_l is not in the same column as b_k for all $l > k$.

Thus for $k = 1, \ldots, m$ with $b_k = [i_k, j_k]$, we can choose the corresponding coefficients r_{i_k,j_k} in order: if we are in case (I) we choose r_{i_k,j_k} so that $\sum_{[i_k,j]\in\sigma} r_{i_k,j} = b$; if we are in case (II) we choose r_{i_k,j_k} so that $\sum_{[i,j_k]\in\sigma} r_{i,j_k} = a$. For each row or column, there is at least one b_k in that row or column, so in the end all the equations will be satisfied. The result is an assignment of coefficients that looks like Figure 5. □

Proof of Theorem 3.4. Let r_{ij} be the coefficients from Lemma 3.7. Note that the sum of all coefficients is $\sum_{[i,j]\in\sigma} r_{ij} = ab$. Also, for any $[i', j'] \in \sigma$ the sum of coefficients in its row and its column is $\sum_{\substack{[i,j]\in\sigma \\ i=i'}} r_{ij} + \sum_{\substack{[i,j]\in\sigma \\ j=j'}} r_{ij} = a + b$. Using Lemma 3.5, we get

$$\begin{aligned}
\mathbb{E}\left(\sum_{[i,j]\in\sigma} r_{ij} R_{i,j}\right) &= \sum_{[i,j]\in\sigma} r_{ij} \left(\sum_{[i',j]\in\sigma} \mathbb{E}(\mathcal{T}^+_{i',j}) + \sum_{[i,j']\in\sigma} \mathbb{E}(\mathcal{T}^+_{i,j'})\right) \\
&= \sum_{[i,j]\in\sigma} \left(\sum_{[i',j]\in\sigma} r_{i',j} + \sum_{[i,j']\in\sigma} r_{i,j'}\right) \mathbb{E}(\mathcal{T}^+_{i,j}) \\
&= (a+b) \sum_{[i,j]\in\sigma} \mathbb{E}(\mathcal{T}^+_{i,j}). \qquad (3.3)
\end{aligned}$$

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On the other hand, by Lemma 3.6,

$$\begin{aligned}\mathbb{E}\left(\sum_{[i,j]\in\sigma} r_{ij}R_{i,j}\right) &= \sum_{[i,j]\in\sigma} r_{ij}\left(1+\sum_{c\in C_{ij}(\sigma)}\mathbb{P}_\mu(c)\right)\\ &= \sum_{[i,j]\in\sigma} r_{ij} + \sum_{[i,j]\in\sigma} r_{ij}\sum_{c\in C_{ij}(\sigma)}\mathbb{P}_\mu(c)\\ &= ab + \sum_{c\in C(\sigma)}\left(\sum_{\substack{[i,j]\in\sigma \text{ with}\\ C_{ij}(\sigma)\ni c}} r_{ij}\right)\mathbb{P}_\mu(c). \end{aligned} \tag{3.4}$$

As depicted in Figure 6, for any corner $c \in C(\sigma)$ occurring at (x, y) and for any $[i, j] \in \sigma$, we have $c \in C_{ij}(\sigma)$ if and only if ($x \geqslant i$ and $y \geqslant j$) or ($x < i$ and $y < j$). Let $c \in C(\sigma)$ be a southeast corner occurring at (x, y). We have

$$\begin{aligned}\sum_{\substack{[i,j]\in\sigma \text{ with}\\ C_{ij}(\sigma)\ni c}} r_{ij} &= \sum_{[i,j]\in\sigma} r_{ij} - \sum_{\substack{[i,j]\in\sigma\\ i\leqslant x; j>y}} r_{ij} - \sum_{\substack{[i,j]\in\sigma\\ i>x; j\leqslant y}} r_{ij}\\ &= ab - (a-x)b - (b-y)a\\ &= ab\left(\frac{x}{a}+\frac{y}{b}-1\right).\end{aligned}$$

With similar calculations we can see for any $c \in C(\sigma)$ a northeast corner occurring at (x, y) we also have $\sum_{\substack{[i,j]\in\sigma\\ C_{ij}(\sigma)\ni c}} r_{ij} = ab(1 - x/a - y/b)$. In other words, for $c \in C(\sigma)$,

$$\sum_{\substack{[i,j]\in\sigma\\ c\in C_{ij}(\sigma)}} r_{ij} = ab\cdot\delta(c). \tag{3.5}$$

Putting equations (3.3), (3.4) and (3.5) together yields

$$(a+b)\sum_{[i,j]\in\sigma}\mathbb{E}(\mathcal{T}^+_{i,j}(\sigma)) = ab\left(1+\sum_{c\in C(\sigma)}\delta(c)\mathbb{P}(c)\right).$$

But since μ is a toggle-symmetric measure, $\mathbb{E}_\mu(\text{jag}) = 2\sum_{[i,j]\in\sigma}\mathbb{E}(\mathcal{T}^+_{i,j}(\sigma))$. Hence the claimed formula for $\mathbb{E}_\mu(\text{jag})$ holds. □

Let us say a skew shape σ is *balanced* if it is connected and $\delta(c) = 0$ for all $c \in C(\sigma)$. In other words, a connected skew shape is balanced if all outward corners occur at the main antidiagonal. An immediate corollary of our main theorem is the following:

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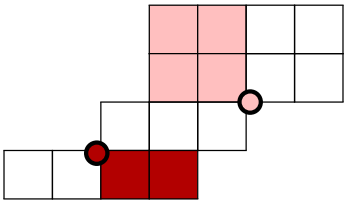

Figure 6. In the above diagram, let X_1 be the set of pink boxes and X_2 the set of dark red boxes. Let c_1 be the corner occurring at $(2, 5)$ (in pink) and c_2 the corner at $(3, 2)$ (in dark red). Then $[i, j] \in X_1$ if and only if $c_1 \in C_{i,j}(\sigma)$ and $[i, j] \in X_2$ if and only if $c_2 \in C_{i,j}(\sigma)$.

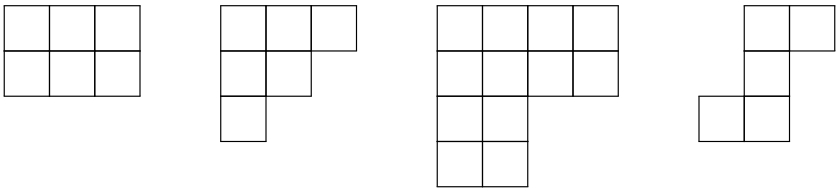

Figure 7. Examples of balanced skew shapes.

COROLLARY 3.8. *Let σ be a balanced skew shape with height a and width b. Let μ be any toggle-symmetric probability distribution on $\mathcal{J}(\sigma)$. The expected jaggedness of a subshape in $\mathcal{J}(\sigma)$ with respect to the distribution μ is $2ab/(a+b)$.*

Some examples of balanced skew shapes are depicted in Figure 7. They include rectangles like (A), staircases like (B), 'stretched' staircases (that is, staircases where we have replaced each box by a $k \times l$ rectangle) like (C), as well as other more general shapes like (D). For any $a, b \geqslant 1$, there are a total of $3^{\gcd(a,b)-1}$ balanced skew shapes with height a and width b (because there are $\gcd(a, b) - 1$ lattice points occurring strictly inside the rectangle $a \times b$ and on the main antidiagonal, and at each of these we can have a northwest outward corner, a southeast outward corner, or neither).

Along the same lines, let us say that a connected skew shape σ is *abundant* if all of its northwest corners occur on or above its antidiagonal and all of its southeast corners occur on or below its antidiagonal. And let us say that σ is *deficient* if all

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of its northwest corners occur on or below the antidiagonal and all of its southeast corners occur on or above the antidiagonal. Then we immediately get:

COROLLARY 3.9. *Let σ be a skew shape of height a and width b and μ be any toggle-symmetric probability distribution on $\mathcal{J}(\sigma)$.*

- *If σ is abundant, then the expected jaggedness of a subshape in $\mathcal{J}(\sigma)$ with respect to the distribution μ is at least $2ab/(a+b)$.*
- *If σ is deficient, then the expected jaggedness of a subshape with respect to μ is at most $2ab/(a+b)$.*

We remark that in the case where σ is a rectangle and $\mu = \mu_{\text{lin}}$ or μ_{unif}, Corollary 3.8 recovers a result of Chan *et al.* [6]. Indeed, [6, Remark 2.16] points out the 'remarkable' fact that for rectangles, the uniform and linear distributions have the same expected jaggedness; Corollary 3.8 is a vast generalization, and perhaps even explanation, of this phenomenon. Theorem 3.4 also gives a reformulation of [6, Theorem 2.8] (which deals with μ_{lin} only) which exhibits more explicitly the way in which expected jaggedness depends on the shape of σ.

3.1. Computing the correction terms for various toggle-symmetric distributions. Although Theorem 3.4 gives an especially nice formula for $\mathbb{E}_\mu(\text{jag})$ when σ is a balanced, we now explain how even when σ is not balanced the correction term $\sum_{c\in C(\sigma)} \delta(c)\mathbb{P}_\mu(c)$ in this formula is easy to compute for all of the 'natural' toggle-symmetric distributions defined in Section 2.1. Of course the displacement $\delta(c)$ for $c \in C(\sigma)$ is easily computed; the issue is computing $\mathbb{P}_\mu(c)$. By Remarks 2.5 and 2.7, the distributions μ_{unif} and μ_{rk} are special cases of $\mu_{m,\leqslant}$ and $\mu_{m,<}$, respectively, so from now on we discuss computing $\mathbb{P}_{\mu_{m,\leqslant}}(c)$, $\mathbb{P}_{\mu_{\text{lin}}}(c)$, and $\mathbb{P}_{\mu_{m,<}}(c)$.

First let us consider $\mu = \mu_{m,\leqslant}$; the other distributions will be similar. Let $\text{RPP}(\sigma; m)$ denote the set of reverse plane partitions of shape σ and height m (recalling the dictionary of terms above). Choose some outward corner $c \in C(\sigma)$ that occurs (i, j). Suppose first that c is a southeast corner. Then

$$\mathbb{P}_{\mu_{m,\leqslant}}(c) = \mathbb{E}\left(\frac{m - \max\{\ell([i+1, j]), \ell([i, j+1])\}}{m}\right)$$

where $\ell \in \text{RPP}(\sigma; m)$ is chosen uniformly at random. But

$$\mathbb{E}(\max\{\ell([i+1, j]), \ell([i, j+1])\}) = m + 1 - \frac{\#\text{RPP}(\sigma \cup \{c\}; m)}{\#\text{RPP}(\sigma; m)}$$

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where $\sigma \cup \{c\}$ denotes the skew shape obtained by adding a box at corner c (that is, for this southeast corner, $\sigma \cup \{c\} := \sigma \cup \{[i+1, j+1]\}$). Indeed, this follows from the same observation as [6, Lemma 2.10]: consider the map $\mathrm{RPP}(\sigma \cup \{c\}; m) \to \mathrm{RPP}(\sigma; m)$ given by forgetting the value at $[i+1, j+1]$; for any $\ell \in \mathrm{RPP}(\sigma; m)$ the size of the fiber of this map at ℓ is $m + 1 - \max\{\ell([i+1, j]), \ell([i, j+1])\}$. Now suppose that c is a northwest corner. Then

$$\mathbb{P}_{\mu_{m,\leqslant}}(c) = \mathbb{E}\left(\frac{\min\{\ell([i+1, j]), \ell([i, j+1])\}}{m}\right)$$

where $\ell \in \mathrm{RPP}(\sigma; m)$ is chosen uniformly at random. By the same reasoning as before,

$$\mathbb{E}(\min\{\ell([i+1, j]), \ell([i, j+1])\}) = \frac{\#\mathrm{RPP}(\sigma \cup \{c\}; m)}{\#\mathrm{RPP}(\sigma; m)} - 1.$$

Whether c is a southeast or northwest corner, we see that

$$\mathbb{P}_{\mu_{m,\leqslant}}(c) = \frac{\#\mathrm{RPP}(\sigma \cup \{c\}; m) - \#\mathrm{RPP}(\sigma; m)}{m \cdot \#\mathrm{RPP}(\sigma; m)}. \tag{3.6}$$

Similar analysis for the other distributions shows

$$\mathbb{P}_{\mu_{\mathrm{lin}}}(c) = \frac{\#\mathrm{SYT}(\sigma \cup \{c\})}{(|\sigma| + 1) \cdot \#\mathrm{SYT}(\sigma)} \tag{3.7}$$

where $\mathrm{SYT}(\sigma)$ denotes the set of standard Young tableaux of shape σ and $|\sigma|$ is the number of boxes in σ, and

$$\mathbb{P}_{\mu_{m,<}}(c) = \frac{\#\mathrm{Inc}(\sigma \cup \{c\}; m) + \#\mathrm{Inc}(\sigma; m)}{(m+2) \cdot \#\mathrm{Inc}(\sigma; m)}, \tag{3.8}$$

where $\mathrm{Inc}(\sigma; m)$ is the set of increasing tableaux of shape σ and height m.

Thus we have reduced the problem of computing $\mathbb{P}_{\mu}(c)$ for μ in the spectrum of toggle-symmetric distributions on $\mathcal{J}(\sigma)$ defined in Section 2.1 to computing the quantities $\#\mathrm{RPP}(\sigma; m)$, $\#\mathrm{SYT}(\sigma)$, and $\#\mathrm{Inc}(\sigma; m)$. Fortunately there are well-known determinantal formulas for these. Let $\sigma = \lambda/\nu$ be a skew shape with $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\nu = (\nu_1, \ldots, \nu_k)$. Then a result of Kreweras [10] says that

$$\#\mathrm{RPP}(\sigma; m) = \det_{i,j=1}^{k}\left[\binom{\lambda_i - \nu_j + m}{i - j + m}\right].$$

Here we interpret $\binom{x}{y} = 0$ for $x < 0$. In the special case $\mu = \emptyset$ the above formula was known to MacMahon [11, page 243]; for more details see [16, Exercise 3.149]. The following formula is due to Aitken [1] (although in fact it is

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a simple consequence of the Jacobi–Trudi identity; see [15, Corollary 7.16.3]):

$$\#\mathrm{SYT}(\sigma) = |\sigma|! \det_{i,j=1}^{k} \left[\frac{1}{(\lambda_i - i - \nu_j + j)!} \right].$$

Here we interpret $1/x! = 0$ if $x < 0$. In the special case $\mu = \emptyset$ we also have the famous hook-length formula, which gives an even better answer for the number of standard Young tableaux; namely,

$$\#\mathrm{SYT}(\lambda) = |\lambda|! \prod_{[i,j]\in\lambda} \frac{1}{h_\lambda(i,j)}$$

where $h_\lambda(i,j)$ is the hook length of box $[i,j]$; see [15, Corollary 7.21.6] for details. As for increasing tableaux of bounded height, it follows from the Reciprocity Theorem for order polynomials (see [16, Corollary 3.15.12]) that if Ω is the unique polynomial satisfying $\Omega(m) = \#\mathrm{RPP}(\sigma; m)$ for all $m \in \mathbb{N}$ then $\#\mathrm{Inc}(\sigma; m) = (-1)^{|\sigma|}\Omega(-m)$. Thus the aforementioned result of Kreweras also allows us to easily compute $\#\mathrm{Inc}(\sigma; m)$.

3.2. Connections to antichain cardinality homomesy. In this subsection we give an application of our main result to the study of homomesies in combinatorial maps. Recall the definitions of rowmotion, gyration, and homomesy from Section 2.2.

Let P be a poset. To any $I \in \mathcal{J}(P)$ we associate the antichain $A(I)$ of P consisting of the maximal elements of I. The *antichain cardinality statistic* is the map $\mathcal{J}(P) \to \mathbb{R}$ given by $I \mapsto \#A(I)$.

COROLLARY 3.10. *If P is the poset associated to the skew shape σ and μ is any toggle-symmetric distribution, then*

$$\mathbb{E}_\mu(\#A(I)) = \frac{ab}{a+b}\left(1 + \sum_{c\in C(\sigma)} \delta(c)\,\mathbb{P}_\mu(c)\right).$$

Proof. This result was already obtained in the proof of Theorem 3.4. Explicitly, the antichain cardinality statistic is just $\sum_{p\in P_\sigma} \mathcal{T}_p^-$, so the average of this statistic is

$$\begin{aligned}\mathbb{E}_\mu\left(\sum_{p\in P_\sigma} \mathcal{T}_p^-\right) &= \frac{1}{2}\left(\mathbb{E}_\mu\left(\sum_{p\in P_\sigma} \mathcal{T}_p^-\right) + \mathbb{E}_\mu\left(\sum_{p\in P_\sigma} \mathcal{T}_p^+\right)\right) \\ &= \frac{1}{2}\mathbb{E}_\mu(\mathrm{jag})\end{aligned}$$

where $\mathbb{E}_\mu(\sum_{p\in P_\sigma} \mathcal{T}_p^-) = \mathbb{E}_\mu(\sum_{p\in P_\sigma} \mathcal{T}_p^+)$ thanks to the toggle symmetry of μ. Now apply Theorem 3.4. □

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COROLLARY 3.11. *For P_σ the poset corresponding to a balanced skew shape σ of height a and width b and $\varphi \in$ {rowmotion, gyration}, the antichain cardinality statistic is $ab/(a+b)$-mesic with respect to the action of φ on $\mathcal{J}(P_\sigma)$.*

Proof. Let $\mathcal{O} \subseteq \mathcal{J}(P_\sigma)$ be a φ-orbit and let μ be the distribution on $\mathcal{J}(P_\sigma)$ that is uniform on $\mathcal{O}$. By Theorem 2.14 we know that μ is toggle-symmetric. Thus by Corollaries 3.8 and 3.10 we conclude that $\mathbb{E}_\mu(\#A(I)) = ab/(a+b)$. □

In the case where $\varphi =$ rowmotion and σ is an $a \times b$ rectangle, Corollary 3.11 recovers a result of Propp and Roby [**12**, Theorem 27]. Actually, Propp and Roby prove a more refined result: they show the cardinality of the intersection of the antichain with any fixed 'fiber' of P_σ is homomesic with respect to rowmotion in this rectangular case. In other words, they show that the statistics $\sum_{[i,j']\in\sigma} T^-_{i,j'}$ for $1 \leqslant i \leqslant a$ and $\sum_{[i',j]\in\sigma} T^-_{i',j}$ for $1 \leqslant j \leqslant b$ are homomesic with respect to the action of rowmotion on $\mathcal{J}(P_\sigma)$. But when σ is an $a \times b$ rectangle and $1 \leqslant i < a$ we have

$$\sum_{[i,j']\in\sigma} T^-_{i,j'} = \sum_{[(i+1),j']\in\sigma} T^+_{(i+1),j'}$$

and similarly for columns. Thus by the toggle symmetry of μ, where μ is as in the proof of Corollary 3.11, we conclude that in this case

$$\mathbb{E}_\mu\left(\sum_{[i_1,j']\in\sigma} T^-_{i_1,j'}\right) = \mathbb{E}_\mu\left(\sum_{[i_2,j']\in\sigma} T^-_{i_2,j'}\right)$$

for any $1 \leqslant i_1, i_2 \leqslant a$, and similarly for columns. In this way we can recover Propp and Roby's refined fiber cardinality result as well. This argument also shows that fiber cardinality is homomesic for gyration acting on rectangular shapes. (But note that the fiber cardinality homomesy does not hold for arbitrary balanced shapes.) At any rate, for nonrectangular, balanced σ when $\varphi =$ rowmotion, and for all balanced σ when $\varphi =$ gyration, the antichain cardinality homomesy result of Corollary 3.11 appears to be new.

REMARK 3.12. The homomesy phenomenon, although introduced only recently, has already been observed in many circumstances [**3**, **7**, **12**, **14**, **17**]. This apparent ubiquity, like that of the more famous cyclic sieving phenomenon [**13**] to which it is related, is already at least a partial motivation for its study. But let us take a moment to explain a more specific motivation for the kind of homomesy result we obtained in this subsection. So let $\varphi : \mathcal{S} \to \mathcal{S}$ be an invertible map from a set $\mathcal{S}$ of combinatorial objects to itself. A natural thing to study is the orbit structure of φ. For example, when φ is rowmotion, a lot of research has focused on proving that

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the order of φ is small for 'nice' classes of posets like products of two chains [4]. The order of φ is of course just the least common multiple of the sizes of the orbits of φ. Here is where homomesy can be useful: if $f\colon \mathcal{S} \to \mathbb{Z}$ is some statistic that takes on *integer* values, and f is c-mesic with respect to the action of φ on $\mathcal{S}$, then, writing $c = p/q$ in reduced terms, we must have that the order of every orbit is divisible by q (this is the only way that q can always appear in the denominator). In other words, f being (p/q)-mesic with respect to the action of φ says that the *greatest common divisor* of the orbit sizes of φ is divisible by q; this addresses a kind of dual problem to that of finding the order of φ. What is remarkable is that often there is no easier way to prove this fact about the gcd of φ orbit sizes than to exhibit homomesy for the appropriate statistic. For instance, Corollary 3.11 above implies that if σ is a balanced skew shape of height a and width b then the gcd of the orbit sizes of rowmotion acting on $\mathcal{J}(P_\sigma)$ is divisible by $(a+b)/\gcd(ab, a+b)$.

4. Open questions

We conclude with some open questions and threads of future research.

(1) For any poset P, the space of toggle-symmetric distributions on $\mathcal{J}(P)$ is some convex polytope. Denote this polytope by $\mathcal{P}(P)$. What is the combinatorial structure of $\mathcal{P}(P)$? Note that $\mathcal{P}(P)$ has dimension $\#\mathcal{J}(P) - 1 - \#P$: specifically, it is the intersection of the standard $\#\mathcal{J}(P)$-simplex in $\mathbb{R}^{\#\mathcal{J}(P)}$ with some linear subspace of codimension $\#P$, and the uniform distribution on $\mathcal{J}(P)$ is an interior point of the simplex that is always toggle-symmetric. It seems that $\mathcal{P}(P)$ can be rather complicated; for example, computation with Sage mathematical software shows that when P_λ is the poset corresponding to the partition $\lambda = (3,3,3)$ the polytope $\mathcal{P}(P_\lambda)$ is 10-dimensional and has 159 vertices. For a specific question about $\mathcal{P}(P)$: are the distributions corresponding to φ-orbits for $\varphi \in \{\text{rowmotion, gyration}\}$ always vertices of $\mathcal{P}(P)$?

(2) Rowmotion and gyration are both elements of the toggle group; moreover, they are both compositions of all of the toggles in some order. Not all such compositions of toggles are 0-mesic with respect to $\mathcal{T}_p$ for all $p \in P$; for instance, Striker [17, Section 6] observes an instance where this fails for *promotion*, another such element of the toggle group. Nevertheless, we could hope that there were other toggle group elements $\varphi\colon \mathcal{J}(P) \to \mathcal{J}(P)$ that are 0-mesic with respect to $\mathcal{T}_p$ for all $p \in P$. It would be interesting to find such φ because then Corollary 3.8 would immediately imply that the antichain cardinality statistic is homomesic with respect to φ.

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(3) For a connected skew shape σ with height a and width b and any $\rho \in \mathcal{J}(\sigma)$ we claim that

$$1 \leqslant \mathrm{jag}(\rho) \leqslant \min\{2a, 2b, a+b-1\} < \frac{4ab}{a+b}.$$

To see this, first note that either ρ is nonempty or $\sigma \setminus \rho$ is nonempty and so there is at least one box of σ that can be toggled in or out, proving $1 \leqslant \mathrm{jag}(\rho)$. Next note that in each column, at most one box can be toggled in and at most one out, and similarly for rows. This proves that $\mathrm{jag}(\rho) \leqslant \min\{2a, 2b\}$. The only case where $a+b-1 < \min\{2a, 2b\}$ is when $a = b$; in this case, note that if a box can be toggled out of every column, then there is no box in the first column that can be toggled in. So indeed the claimed inequality on $\mathrm{jag}(\rho)$ holds. The upshot of this inequality is that for any distribution μ on $\mathcal{J}(\sigma)$,

$$0 < \mathbb{E}_\mu(\mathrm{jag}) < \frac{4ab}{a+b}.$$

If μ is toggle-symmetric then by Theorem 3.4 we conclude

$$-1 < \sum_{c \in C(\sigma)} \delta(c)\mathbb{P}_\mu(c) < 1.$$

It is not obvious a priori that this bound on $\sum_{c \in C(\sigma)} \delta(c)\mathbb{P}_\mu(c)$ should hold for all toggle-symmetric distributions μ. It would be interesting to give a simple explanation for why it does hold, or to offer another expression for $\mathbb{E}_\mu(\mathrm{jag})$ that is evidently strictly between 0 and $4ab/(a+b)$.

A related question, pointed out by Pflueger, is to give a direct explanation for why, for any *balanced* skew shape σ and any toggle-symmetric distribution, the expected jaggedness of a subshape necessarily lies between a and b. (This is true, of course, since the harmonic mean always lies between a and b.)

(4) Our main theorem, Theorem 3.4, which gives a formula for $\mathbb{E}_\mu(\mathrm{jag})$ for toggle-symmetric distributions μ, applies only to posets associated to skew shapes. Can we generalize this result to a broader class of posets? In particular, is there a more general notion of a 'balanced' poset for which all toggle-symmetric distributions have the same expected jaggedness?

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