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CENTRAL AUTOMORPHISMS OF FINITE GROUPS

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This paper considers an aspect of the general problem of how the structure of a group influences the structure of its automorphism group. A recent result of Beisiegel shows that if P is a p-group then the central automorphism group of P has no normal subgroups of order prime to p. So, roughly speaking, most of the central automorphisms are of p-power order. This generalizes an old result of Hopkins that if $\operatorname{Aut} P$ is abelian (so every automorphism is central), then $\operatorname{Aut} P$ is a p-group.

This paper uses a different approach to consider the case when P is a π-group. It is shown that the central automorphism group of P has a normal π'-subgroup only if P has an abelian direct factor whose automorphism group has such a subgroup.

An automorphism α of a group G is said to be central when it commutes with every inner automorphism of G, or equivalently when $g^{-1}\alpha(g)$ lies in the centre $Z(G)$ of G for each g in G. The central automorphisms of G form a normal subgroup of the full automorphisi

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group Aut(G) .

Non-abelian p-groups having abelian automorphism groups have been studied recently ([3], [6], [9]) and not so recently ([5], [7]): in this case of course all automorphisms are central, and the classical result of Hopkins [5] states that the automorphism group is again a p-group. In this paper we obtain results on central automorphisms which extend the work of Hopkins and others in various directions.

Throughout this paper we will consistently use the following notation:

> π will always denote a set of primes, with π' its complement in the set of all primes;
> P will be a finite π-group;
> A will be a π'-subgroup of $\mathrm{Aut}(P)$, the group of automorphisms of P ;
> $O_\pi(H)$ will denote the largest normal π-subgroup of the group H ;
> $\mathrm{Aut}_c(P)$ will denote the group of central automorphisms of P ;
> Q will denote $[P,A]$, that is $\langle x^{-1}x^a : x \in P,\ a \in A\rangle$, where x^a means $a(x)$;
> C will denote $\{x \in P : x^a = x \text{ for all } a \text{ in } A\}$, the centralizer of A in P .

All groups considered are finite. The remaining notation follows that of Gorenstein [4].

We begin with three lemmas. The first is a straightforward generalization of a standard result: see, for example, 5.2.3 and 5.3.5 of Gorenstein [4]. The other two are little more than observations, but are stated separately to avoid repetition and deviation later on.

LEMMA 1. *We have in general:* $P = CQ$ *and* $[Q,A] = Q$. *Moreover if* P *is abelian then* $P = C \times Q$.

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Proof. First note that $Q = [P,A]$ is A-invariant and normal in P. Thus by 6.2.2.(iv) of Gorenstein the centralizer in P/Q of A is just the image in P/Q of C, that is, CQ/Q. On the other hand A in fact centralizes P/Q by the definition of Q. We deduce that $P = CQ$. Now using a standard commutator formula

$$Q = [P,A] = [CQ,A] = [C,A]\,[Q,A]$$

which, since A centralizes C, reduces to $Q = [Q,A]$.

In the special case where P is abelian we can use an "averaging" argument, following Gorenstein 5.2.3 almost word for word, to conclude that the product CQ is direct.

REMARK. The quaternion group of order 8 shows that in the last part of Lemma 1 the restriction on P is essential.

LEMMA 2. *If* $P = U \times V$ *where* V *is abelian and invariant under central automorphisms of* P, *then elements in* V *and* $Z(U)$ *have coprime orders.*

Proof. Suppose if possible that some prime q divides both $|Z(U)|$ and $|V|$. Choose an element z of order q in $|Z(U)|$ and write V as a direct product $W \times X$, where W is a cyclic q-group. Let the map α be defined by

$$\alpha(u) = u \quad \text{for all } u \text{ in } U\ ; \quad \alpha(w) = zw,$$

where w is a generator of W; $\alpha(x) = x$ for all x in X. Then it is easy to verify that α extends to a central automorphism of P. But V is not invariant under α, a contradiction.

LEMMA 3. $O_{\pi'}(\mathrm{Aut}_c(P))$ *and* $O_{\pi'}(\mathrm{Aut}(P))$ *coincide.*

Proof. $O_{\pi'}(\mathrm{Aut}(P))$ is a normal π'-subgroup of $\mathrm{Aut}(P)$, and as such it commutes elementwise with any normal π-subgroup of $\mathrm{Aut}(P)$, in particular, the group $P/Z(P)$ of inner automorphisms of P. Thus $O_{\pi'}(\mathrm{Aut}(P))$ is a subgroup of $\mathrm{Aut}_c(P)$, and the result follows.

THEOREM A. (a) *Suppose that* A *is a subgroup of* $\mathrm{Aut}_c(P)$. *Then:*

(a1) Q *lies in the centre* $Z(P)$ *of* P *(so that* Q *is abelian);*

(a2) $Q = [Q,A]$;

(a3) $P = C \times Q$;

(a4) A *is isomorphic to a subgroup of* $\mathrm{Aut}(Q)$;

(a5) A *acts fixed-point-freely on* Q *and trivially on* C ;

(a6) $\mathrm{Aut}(Q)$ *is isomorphic to a subgroup of* $\mathrm{Aut}_c(P)$;

(a7) Q *is trivial if and only if* A *is trivial;*

(a8) C *is trivial only if* P *is abelian.*

(b) *Now suppose that* A *is normal in* $\mathrm{Aut}_c(P)$. *Then in addition to the facts in (a) we have:*

(b1) C *and* Q *are* $\mathrm{Aut}_c(P)$*-invariant;*

(b2) $Z(C)$ *and* Q *have coprime orders.*

(c) *Finally, let* A *be normal in* $\mathrm{Aut}(P)$. *Then:*

(c1) A *is in fact a subgroup of* $\mathrm{Aut}_c(P)$ *(so that all the conclusions in (a) and (b) hold);*

(c2) C *and* Q *are characteristic in* P ;

(c3) $A \le O_{\pi'}(\mathrm{Aut}(Q)) \le O_{\pi'}(\mathrm{Aut}_c(P))$;

(c4) $O_{\pi'}(\mathrm{Aut}_c(P)) = O_{\pi'}(M) \times O_{\pi'}(\mathrm{Aut}(Q))$, *where* M *is the group of central automorphisms of* C;

(c5) *in particular, when* $A = O_{\pi'}(\mathrm{Aut}_c(P))$, *we have equalities in (c3) and so* $O_{\pi'}(M) = 1$ *in (c4).*

Proof. (a) Since A consists of central automorphisms of P , (a1) is immediate. (a2) is obvious from Lemma 1.

Lemma 1 also gives $P = CQ$. By (a1) both C and Q are normal in P , so to complete the proof of (a3) we need only check that C and Q intersect trivially. Note that Q is A-invariant abelian and the last part of Lemma 1 gives $Q = R \times [Q,A]$ where R denotes the centralizer in Q of A , that is, the intersection of C and Q . Then by (a2) and the finiteness of Q , $R = 1$, and (a3) follows.

Now the definition of C and the decomposition (a3) of P quickly yield (a4) and (a5).

The proof of (a6) is easy, as any automorphism of Q extends to the product $C \times Q$ in a natural way (acting trivially on C) and gives a central automorphism of P.

(a7) follows from (a2) and (a4).

Finally (a8) is an obvious consequence of (a1) and (a3).

(b) Since A and P are both invariant under $\mathrm{Aut}_c(P)$, (b1) follows easily.

Now (b2) is a simple consequence of Lemma 2.

(c) A is contained in $O_{\pi'}(\mathrm{Aut}(P))$ which by Lemma 3 coincides with $O_{\pi'}(\mathrm{Aut}_c(P))$, so (c1) is immediate.

We are assuming that A is normal in $\mathrm{Aut}(P)$. Hence both $[P,A]$ $(=Q)$ and the centralizer of A in P $(= C)$ are invariant under $\mathrm{Aut}(P)$, that is, characteristic in P. Thus (c2) is proved.

To establish (c3), first note that in (a4) we have identified A with a subgroup, here clearly a normal π'-subgroup, of $\mathrm{Aut}(Q)$, so it lies in $O_{\pi'}(\mathrm{Aut}(Q))$. On the other hand, (a6) tells us that $\mathrm{Aut}(Q)$ is a subgroup of $\mathrm{Aut}_c(P)$, indeed in this case a normal subgroup because Q and C are characteristic in P and $\mathrm{Aut}(P)$ is isomorphic to $\mathrm{Aut}(C) \times \mathrm{Aut}(Q)$. Thus $O_{\pi'}(\mathrm{Aut}(Q)$ is contained in $O_{\pi'}(\mathrm{Aut}_c(P))$. So (c3) is proved.

Since C and Q are characteristic in P, we have $\mathrm{Aut}_c(P) = M \times \mathrm{Aut}(Q)$ and so $O_{\pi'}(\mathrm{Aut}_c(P)) = O_{\pi'}(M) \times O_{\pi'}(\mathrm{Aut}(Q))$. This is (c4), and (c5) follows easily.

REMARKS. An argument of Beisiegel ([2], 4.1) can be adapted to show that provided A centralizes P' (and any group of central automorphisms will) the product $P = CQ$ has the properties $[C,Q] = 1$, $C \cap Q = Q'$ and Q is nilpotent of class at most 2.

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We point out that when $\mathrm{Aut}_c(P)$ is abelian then for any π'-subgroup A of $\mathrm{Aut}_c(P)$, the corresponding subgroup Q is cyclic and $\mathrm{Aut}_c(P)$-invariant. From (a6) of Theorem A we know that $\mathrm{Aut}(Q)$ is isomorphic to a subgroup of $\mathrm{Aut}(P)$, so that $\mathrm{Aut}(Q)$ is abelian. But it is well known that the only abelian groups with abelian automorphism groups are cyclic. The fact that Q is $\mathrm{Aut}_c(P)$-invariant follows from Theorem A(b1) since A will always be normal in $\mathrm{Aut}_c(P)$ in this case.

COROLLARY 1. *Suppose* $\pi = \{p\}$. *Assume that* A *is a normal subgroup of* $\mathrm{Aut}_c(P)$. *If* A *is non-trivial then* $P = Q$, *that is,* P *is abelian and* A *acts fixed-point-freely on* P.

Proof. Suppose that A is non-trivial. Then by Theorem A(a7) Q is non-trivial. But $C \times Q = P$ is a p-group and yet $Z(C)$ and Q have coprime orders, by Theorem A(b2). This forces $C = 1$. Now in view of Theorem A(a5) we have A acts fixed-point-freely on P.

COROLLARY 2. *Suppose* $\pi = \{p\}$ *and* P *is non-abelian. Then*

(i) $O_{p'}(\mathrm{Aut}_c(P)) = 1$;

(ii) *if* $\mathrm{Aut}_c(P)$ *is nilpotent then* $\mathrm{Aut}_c(P)$ *is a p-group;*

(iii) *if* $\mathrm{Aut}(P)$ *is nilpotent then* $\mathrm{Aut}(P)$ *is a p-group;*

(iv) *if* $\mathrm{Aut}_c(P)$ *is abelian then* $\mathrm{Aut}_c(P)$ *is a p-group;*

(v) *if* $\mathrm{Aut}(P)$ *is abelian then* $\mathrm{Aut}(P)$ *is a p-group;*

Proof. (i) is just a re-statement of Corollary 1, and (ii) is an easy consequence.

(iii) can be deduced from (ii), with the help of Lemma 3: if $\mathrm{Aut}(P)$ is nilpotent then so is $\mathrm{Aut}_c(P)$, and by (ii) $O_{p'}(\mathrm{Aut}_c(P))$ is 1. Then by Lemma 3, $O_{p'}(\mathrm{Aut}(P))$ is 1 and $\mathrm{Aut}(P)$ is a p-group.

(iv) is a special case of (ii), and finally (v) now follows either as a special case of (iii) or as a consequence of (iv), since in this case $\mathrm{Aut}_c(P) = \mathrm{Aut}(P)$.

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REMARKS. Corollary 2(v) is the old result of Hopkins [5], and (iii) is a generalization due to Ying [10].

All of the statements in Corollary 2 also follow from an elegant theorem of Beisiegel [2], stating that in this situation $O_p(\mathrm{Aut}(P))$ contains its own centralizer.

COROLLARY 3. (i) *If* P *is purely non-abelian, that is, has no abelian direct factors, then* $\mathrm{Aut}_c(P)$ *is a* π*-group and* $O_{\pi'}(\mathrm{Aut}(P))$ *is trivial.*

(ii) *Suppose* π *does not contain the prime 2. Then the following statements are equivalent:*

(a) P *is purely non-abelian;*

(b) $\mathrm{Aut}_c(P)$ *is a* π*-group;*

(c) $\mathrm{Aut}_c(P)$ *is a* $2'$*-group.*

Proof: (i) follows from Theorem A(a7) and Lemma 3.

(ii) Note that by (i), (a) implies (b). Also (b) clearly implies (c).

Finally if P has a non-trivial abelian factor then an inverting automorphism on this factor gives a central automorphism of P, so (c) implies (a).

REMARKS. Corollary 3(i) was first pointed out by Adney and Yen [1], then improved by Sanders [8].

In Corollary 3(i) it is not sufficient just to assume P non-abelian, as it is when $\pi = \{p\}$; for example, consider the direct product of a group of order 11 and a non-abelian group of order 6.

The fact that $O_{\pi'}(\mathrm{Aut}(P))$ is trivial does not imply any of the statements of Corollary 3(ii). Consider the direct product of a cyclic group of order 3 and any non-abelian group of order 27. In this case $\mathrm{Aut}_c(P)$ has order 486 yet the Sylow 2-subgroups in $\mathrm{Aut}_c(P)$ are not normal. Thus $O_{3'}(\mathrm{Aut}_c(P))$ is trivial, and so is $O_{3'}(\mathrm{Aut}(P))$.

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