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ALGEBRAIC DEGENERACY THEOREM FOR HOLOMORPHIC MAPPINGS INTO SMOOTH PROJECTIVE ALGEBRAIC VARIETIES

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§ 1. Introduction

The famous Picard theorem states that a holomorphic mapping $f: C \to P^1(C)$ omitting distinct three points must be constant. Borel [1] showed that a non-degenerate holomorphic curve can miss at most $n+1$ hyperplanes in $P^n(C)$ in general position, thus extending Picard's theorem ($n = 1$). Recently, Fujimoto [3], Green [4] and [5] obtained many Picard type theorems using Borel's methods for holomorphic mappings. In [3] and [4], they proved that a holomorphic mapping $f: C^m \to P^n(C)$ omitting any $n+2$ hyperplanes in general position must have the image lying in a hyperplane, especially Green showed that the same result holds under the condition that hyperplanes are distinct. Furthermore, in [5] he proved that a holomorphic mapping f of C^m into a projective algebraic variety V of dimension n omitting $n+2$ non-redundant hypersurface sections must be algebraically degenerate. On the other hand, in the equidimensional case, Carlson and Griffiths [2] obtained a generalization of Nevanlinna's defect relation for holomorphic mappings of C^n into an n-dimensional smooth projective algebraic variety V. By their results, a holomorphic mapping $f: C^n \to P^n(C)$ having the Nevanlinna's deficiency $\delta(D) = 1$ for a hypersurface $D \subset P^n(C)$ of degree $\geqq n+2$ with simple normal crossings, must be degenerate in the sence that $J_f \equiv 0$ on C^n. While, Noguchi [6] obtained an inequality of the second main theorem type for holomorphic curves in algebraic varieties, thus a holomorphic curve f in an algebraic variety V which has the Nevanlinna's deficiency $\delta(\Sigma) = 1$ for hypersurfaces Σ with some conditions in V must be algebraically degenerate. In this paper, we shall show that for $n+2$ ample divisors $\{D_j\}_{j=1}^{n+2}$ with normal crossings, any holomorphic mapping of C^m into an n-dimensional smooth projective algebraic variety

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which has $\delta(D_j) = 1$ $(j = 1, \cdots, n+2)$ must be algebraically degenerate. Hence a holomorphic mapping of C^n into $P^m(C)$ with $\delta(H_j) = 1$ $(j = 1, \cdots, n+2)$ for hyperplanes $\{H_j\}_{j=1}^{n+2}$ in $P^n(C)$ in general position must be linearly degenerate. Our method is different from that of Fujimoto and Green.

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§ 2. Notation and terminology

Let $z = (z_1, \cdots, z_m)$ be the natural coordinate system in C^m. We set $\|z\|^2 = \sum_{j=1}^m z_j \bar{z}_j$, $B(r) = \{z \in C^m \,|\, \|z\| < r\}$, $\partial B(r) = \{z \in C^m \,|\, \|z\| = r\}$, $d^c = (\sqrt{-1}/4\pi)(\bar{\partial} - \partial)$, $\eta = dd^c \log \|z\|^2$, $\eta_k = \eta \wedge \cdots \wedge \eta$ (k-times) and $\sigma = d^c \log \|z\|^2 \wedge \eta_{m-1}$.

For a divisor $D(\not\ni 0)$ in C^m, we write

$$n(D, t) \equiv \int_{D \cap B(t)} \eta_{m-1} \text{ and } N(D, r) \equiv \int_0^r n(D, t)(dt/t) .$$

Let V be an n-dimensional smooth projective algebraic variety and L a line bundle over V. Let $\{U_\alpha\}$ be an open covering of V such that the restriction $L|_{U_\alpha}$ is trivial. Then L is determined by the 1-cocycle $\{f_{\alpha\beta}\}$ which are nowhere vanishing holomorphic functions in $U_\alpha \cap U_\beta$ satisfying $f_{\alpha\beta} = f_{\alpha\gamma} \cdot f_{\gamma\beta}$ in $U_\alpha \cap U_\beta \cap U_\gamma$. A metric h in L is given by positive C^∞ functions h_α in U_α, where $h_\alpha = |f_{\alpha\beta}|^2 h_\beta$ in $U_\alpha \cap U_\beta$. The curvature form ω of h is given by $\omega = \omega_L = dd^c \log h_\alpha$ which represents the first Chern class $c_1(L)$ of L. A holomorphic line bundle L on V is said to be positive, if L has a metric h whose curvature form is everywhere positive definite.

Let f be a holomorphic mapping of C^m into V. Let L be a positive line bundle over V and h a metric in L. We define

$$T_f(L, r) \equiv \int_0^r (dt/t) \int_{B(t)} f^*\omega \wedge \eta_{m-1}$$

and call it the characteristic function of f with respect to L, where $f^*\omega$ denotes the pull-back of the form $\omega = dd^c \log h$ under f.

(*) We note that $T_f(L, r)$ is independent of the choice of a metric h in L up to $O(1)$-term. (See Carlson and Griffiths [2], p. 537).

A holomorphic section $\phi = \{\phi_\alpha\}$ of $L \to V$ is given by holomorphic functions ϕ_α in U_α where $\phi_\alpha = f_{\alpha\beta}\phi_\beta$ in $U_\alpha \cap U_\beta$. For a section ϕ, its norm $|\phi|$ is given by $|\phi|^2 = |\phi_\alpha|^2/h_\alpha$ in U_α which is well defined on V. A holomorphic line bundle whose sections defines a projective embedding is called very ample.

Let $\Gamma(V, \mathcal{O}(L))$ denote the space of holomorphic sections of the line bundle L on V and $|L|$ denote the complete linear system of effective divisors on V given by the zeros of a holomorphic section of $L \to V$, i.e.

$$|L| = \{(\phi) | \phi \in \Gamma(V, \mathcal{O}(L))\} ,$$

where (ϕ) denotes the divisor given by the zeros of ϕ.

Let $D \in |L|$ be an effective divisor given by the zeros of a holomorphic section $\phi \in \Gamma(V, \mathcal{O}(L))$ with $|\phi| \leqq 1$ on V. Assume that $\phi(f(z)) \not\equiv 0$. We define the proximity function of D by

$$m(D, r) \equiv \int_{\partial B(r)} \log (1/|\phi|^2(f(z)))\sigma(z) \qquad (\geqq 0) .$$

Carlson and Griffiths [2] proved the following:

Theorem A (Carlson-Griffiths). *Let $D \in |L|$ and $f\colon C^m \to V$ be a holomorphic mapping such that all components of f^*D are divisors. Then*

$$N(f^*D, r) + m(D, r) = T_f(L, r) + O(1) ,$$

where $O(1)$ depends on D but not on r.

In the case where f^*D passes through the origin, the definition of $N(f^*D, r)$ must be modified by means of Lelong numbers.

In the case that V is an n-dimensional complex projective space $P^n(C)$, Stoll [7] and Vitter [8] proved the Nevanlinna's second main theorem for meromorphic mappings of C^m into $P^n(C)$ in the following form.

Theorem B (Stoll, Vitter). *Let $f\colon C^m \to P^n(C)$ be a meromorphic mapping such that $f(C^m)$ is not contained in any hyperplane in $P^n(C)$. Let H be the hyperplane bundle over $P^n(C)$ and $H_1, \cdots, H_q \in |H|$ distinct hyperplanes in general position in $P^n(C)$. Then*

$$(q - n - 1)T_f(H, r) \leqq \sum_{j=1}^{q} N(f^*H_j, r) + S(r) ,$$

where $S(r) \leqq O(\log (r \cdot T_f(H, r)))$ for $r \to \infty$ outside a set of finite Lebesgue measure.

For a divisor $D \in |L|$ on V, we define the deficiency of D by

$$\delta(D, r) \equiv 1 - \limsup_{r\to\infty} (N(f^*D, r)/T_f(L, r)) .$$

Let f be a holomorphic mapping of C^m into a smooth projective algebraic variety V such that $f(C^m)$ is not contained in any divisor belonging to $|L|$. Let $D_1, \cdots, D_\ell$ $(D_j \in |L|)$ be divisors on V given by the zeros of holomorphic sections $\phi_1, \cdots, \phi_\ell$, $\phi_j = \{\phi_{j\alpha}\} \in \Gamma(V, \mathcal{O}(L))$ with $|\phi_j| \leqq 1$ $(j = 1, \cdots, \ell)$ and the system $(\phi_1, \cdots, \phi_\ell)$ has no common zeros on V. Then the function $h = \{h_\alpha\}$, $h_\alpha \equiv \sum_{j=1}^{\ell} |\phi_{j\alpha}|^2$ is a positive C^∞ function on V and satisfies $h_\alpha = |f_{\alpha\beta}|^2 h_\beta$ in $U_\alpha \cap U_\beta$. Hence we may take h as a metric in L.

Note that, if ψ_1 and ψ_2 are two holomorphic sections of $L \to V$, then its ratio ψ_1/ψ_2 is a global meromorphic function on V.

By Theorem A, we have

$$
\begin{aligned}
T_f(L, r) &= N(f^*D_i, r) + m(D_i, r) + O(1) \\
&= N(f^*D_i, r) + \int_{\partial B(r)} \log (h_\alpha(f(z))/|\phi_{i\alpha}|^2(f(z)))\sigma(z) + O(1) \qquad (1)\\
&= N(f^*D_i, r) + \int_{\partial B(r)} \log \left(\sum_{j=1}^{\ell} |\phi_{j\alpha}(f(z))/\phi_{i\alpha}(f(z))|^2\right)\sigma(z) + O(1) .
\end{aligned}
$$

§ 3. Statement of results

Let V be a smooth projective algebraic variety of dimension n and $L \to V$ a fixed positive line bundle over V. We shall prove the following theorem which yields an algebraic degeneracy of holomorphic mappings into V under some conditions on the Nevanlinna's deficiencies.

THEOREM. *Let f: $C^m \to V$ be a holomorphic mapping of C^m into V. Let $D_1, \cdots, D_{n+2}$, $D_j \in |L^{\ell_j}|$, $(l_j \in Z^+)$, be divisors on V such that $\delta(D_j) = 1$ $(j = 1, \cdots, n+2)$ and*

$$(2) \qquad \bigcap_{k=1}^{n+1} \operatorname{supp} D_{j_k} = \emptyset \text{ for every } \{j_1, \cdots, j_{n+1}\} \subset \{1, \cdots, n+2\} .$$

Then f must be algebraically degenerate.

Here $\delta(D_j) = 1 - \limsup_{r\to\infty} (N(f^*D_j, r)/T_f(L^{\ell_j}, r))$ for $D_j \in |L^{\ell_j}|$ and Z^+ denotes the set of all positive integers.

We note that the condition (2) is satisfied for divisors $\{D_j\}_{j=1}^{n+2}$ with normal crossings.

COROLLARY. *Let $S_1, \cdots, S_{n+2}$ be hypersurfaces with $\bigcap_{k=1}^{n+1} S_{j_k} = \emptyset$ in $P^n(C)$ for every $\{j_1, \cdots, j_{n+1}\} \subset \{1, \cdots, n+2\}$. Then any holomorphic mapping f: $C^m \to P^n(C)$ which has $\delta(S_j) = 1$ $(j = 1, \cdots, n+2)$ is algebraically degenerate.*

Remark. In this theorem, the condition (2) can not be replaced by a condition that $D_1, \cdots, D_{n+2}$ are non-redundant, i.e.

$$\operatorname{supp} D_j \not\subset \bigcup_{i \neq j} \operatorname{supp} D_i \qquad \text{for any } j .$$

EXAMPLE. We consider a holomorphic curve $f\colon \boldsymbol{C} \to \boldsymbol{P}^2(\boldsymbol{C})$ given by $f = (1, e^z, ze^z)$ and four hyperplanes $H_j = \{w = (w_1, w_2, w_3) \in \boldsymbol{P}^2(\boldsymbol{C}) | w_j = 0\}$ $(j = 1, 2, 3)$ and $H_4 = \{w \in \boldsymbol{P}^2(\boldsymbol{C}) | w_3 - w_2 = 0\}$. Then we see that $N(f^*H_j, r) = 0$ for $j = 1, 2$ and $N(f^*H_j, r) = o(T_f(H, r))$ for $j = 3, 4$ and hence $\delta(H_j) = 1$ for $j = 1$ to 4. But f is not algebraically degenerate.

Remark. We can construct an example of a non-constant holomorphic curve in $\boldsymbol{P}^2(\boldsymbol{C})$ which satisfies the conditions of the theorem for not all hyperplanes in $\boldsymbol{P}^2(\boldsymbol{C})$.

§ 4. Two lemmas

In order to prove the theorem, we shall use the following two lemmas:

LEMMA 1. *Let $L \to V$ be a very ample line bundle over V and $\psi_1, \cdots, \psi_{n+1}$, $\psi_j = \{\psi_{j\alpha}\} \in \Gamma(V, \mathcal{O}(L))$ holomorphic sections satisfying*

$$\bigcap_{j=1}^{n+1} \operatorname{supp} D_j = \emptyset ,$$

where $D_j = (\psi_j)$ $(j = 1, \cdots, n + 1)$. Then $\psi_1, \cdots, \psi_{n+1}$ are algebraically independent over $\boldsymbol{C}$.

LEMMA 2. *Let $\psi_1, \cdots, \psi_{n+2}$, $\psi_j \in \Gamma(V, \mathcal{O}(L))$ be holomorphic sections of a very ample line bundle $L \to V$ such that*

$$(3) \qquad \bigcap_{k=1}^{n+1} \operatorname{supp} D_{j_k} = \emptyset \text{ for every } \{j_1, \cdots, j_{n+1}\} \subset \{1, \cdots, n + 2\} ,$$

where $D_{j_k} = (\psi_{j_k})$ $(k = 1, \cdots, n + 1)$. Let $R(\psi_1, \cdots, \psi_{n+2}) \equiv \sum_{j=1}^{s} R_j \equiv 0$ be an algebraic relation of an irreducible homogeneous polynomial of degree k in ψ's among $\psi_1, \cdots, \psi_{n+2}$. Then

$$\{p \in V | R_{j_1}(p) = \cdots = R_{j_{s-1}}(p) = 0\} = \emptyset$$

for every $\{j_1, \cdots, j_{s-1}\} \subset \{1, \cdots, s\}$.

Proof of Lemma 1. Let $\zeta_0, \cdots, \zeta_N$ be a basis of global holomorphic sections of L. Since L is very ample, the mapping $\Phi_L = (\zeta_0, \cdots, \zeta_N)$ gives a projective embedding of V into $\boldsymbol{P}^N(\boldsymbol{C})$. We identify V with $\Phi_L(V)$. By means of this embedding, we can identify L with the restriction of the hyperplane bundle H over $P^N(C)$ to V. Hence for each $\psi_j \in \Gamma(V, \mathcal{O}(L))$ there exist global holomorphic sections $\tilde{\psi}_j \in \Gamma(P^N(C), \mathcal{O}(H))$ such that $\tilde{\psi}_j|_V = \psi_j$.

We set $(\tilde{\psi}_j) = \tilde{D}_j$ $(j = 1, \cdots, n + 1)$. Hence the dimension of the algebraic subvarieties

$$V_{jk} \equiv \operatorname{supp} \tilde{D}_j \cap \operatorname{supp} \tilde{D}_k \cap V$$

in V is not less than $(n - 1) + (N - 1) - N = n - 2$, that is, $\dim V_{jk} \geqq n - 2$. Similarly, we see that the dimension of

$$V_{jk\ell} \equiv V_{jk} \cap \operatorname{supp} \tilde{D}_\ell \cap V$$

is not less than $n - 3$. Repeating the same argument as above, we have

$$\dim(\operatorname{supp} D_{j_1} \cap \cdots \cap \operatorname{supp} D_{j_n}) \geqq 0 ,$$

that is,

$$\operatorname{supp} D_{j_1} \cap \cdots \cap \operatorname{supp} D_{j_n} \neq \emptyset .$$

Suppose that $\psi_1, \cdots, \psi_{n+1}$ have an algebraic relation R of homogeneous polynomial of degree k in $\psi_1, \cdots, \psi_{n+1}$ represented by

$$R(\psi_1, \cdots, \psi_{n+1}) \equiv \sum_{i_1 + \cdots + i_{n+1} = k} c_{i_1 \cdots i_{n+1}} \psi_1^{i_1} \cdots \psi_{n+1}^{i_{n+1}} \equiv 0 .$$

Then we see that $c_{0 \cdots 0k} = 0$, since $\psi_{n+1}(p) \neq 0$ for a point $p \in V$ with $\psi_1(p) = \cdots = \psi_n(p) = 0$. Thus the term ψ_{n+1}^k is not contained in the relation R. Similarly, we find that none of the terms $\psi_1^k, \cdots, \psi_{n+1}^k$ belongs to R.

We next consider the curve $\mathcal{L} = \{p \in V | \psi_1(p) = \cdots = \psi_{n-1}(p) = 0\}$. For any point $p \in \mathcal{L}$, we see

$$\sum_{i_n + i_{n+1} = k} c_{0 \cdots 0 i_n i_{n+1}} \psi_n^{i_n} \cdot \psi_{n+1}^{i_{n+1}} \equiv 0 \qquad \text{on } \mathcal{L} . \tag{4}$$

We may assume that all $c_{0 \cdots i_n i_{n+1}}$ are not zero. Then we can rewrite (4) in the form

$$\psi_n^{r_n} \cdot \psi_{n+1}^{r_{n+1}} \{\psi_{n+1}^{k_{n,n+1}} + c_{0 \cdots 0 **} \psi_{n+1}^{k_{n,n+1}-1} \cdot \psi_n + \cdots + c'_{0 \cdots 0 **} \psi_n^{k_{n,n+1}}\} \equiv 0$$

on $\mathcal{L}$, where $r_k = \min i_k$ $(k = n, n + 1)$ and $k_{n,n+1} = k - (r_n + r_{n+1})$, $(\neq 0)$. Since $\psi_n \cdot \psi_{n+1} \not\equiv 0$ on $\mathcal{L}$, we obtain

$$\psi_{n+1}^{k_{n,n+1}} + \cdots + c'_{0 \cdots 0 **} \psi_n^{k_{n,n+1}} \equiv 0 \qquad \text{on } \mathcal{L} - \{(\psi_n = 0) \cup (\psi_{n+1} = 0)\} .$$

By Riemann's extension theorem,

$$(5) \qquad \psi_{n+1}^{k_{n,n+1}} + \cdots + c'_{0\cdots 0**}\psi_n^{k_{n,n+1}} \equiv 0 \quad \text{on } \mathscr{L} .$$

We now take a point $p_n \in \mathscr{L}$ with $\psi_n(p_n) = 0$. Then we see $\psi_{n+1}(p_n) = 0$ by (5). This is a contradiction. Thus any $c_{0\cdots 0 i_n i_{n+1}}$ equals to zero, that is, no terms $\psi_n^{i_n} \cdot \psi_{n+1}^{i_{n+1}}$ are contained in R. Similarly, we see that no terms $\psi_k^{i_k} \cdot \psi_\ell^{i_\ell}$ are involved in R for any i_k, i_ℓ. We next consider the subvarieties

$$S(j, k, \ell) = \{p \in V \mid \psi_1(p) = \cdots = \hat{\psi}_j(p) = \cdots = \hat{\psi}_k(p) = \cdots = \hat{\psi}_\ell(p) = \cdots = \psi_{n+1}(p) = 0\}$$

and

$$L(j, k) = \{p \in V \mid \psi_1(p) = \cdots = \hat{\psi}_j(p) = \cdots = \hat{\psi}_k(p) = \cdots = \psi_{n+1}(p) = 0\} ,$$

where the $\wedge$ over the ψ_j means that this terms is to be omitted. Then the similar argument to the above implies that no terms of products of three ψ's are involved in R. Repeating the above argument, we have the fact that all coefficients $c_{i_1\cdots i_{n+1}}$ in R are equal to zero, that is, $\psi_1, \cdots, \psi_{n+1}$ are algebraically independent. This completes the proof of Lemma 1.

Proof of Lemma 2. From the condition (3), the mapping $\Psi\colon V \to \boldsymbol{P}^{n+1}(\boldsymbol{C})$ given by $V \ni p \mapsto (\psi_1(p), \cdots, \psi_{n+2}(p)) \in \boldsymbol{P}^{n+1}(\boldsymbol{C})$ is well defined and holomorphic. By Remmert's proper mapping theorem, $\Psi(V)$ is an analytic subset of $\boldsymbol{P}^{n+1}(\boldsymbol{C})$, hence it is algebraic in $\boldsymbol{P}^{n+1}(\boldsymbol{C})$. We note that any $n+1$ ψ's in $\psi_1, \cdots, \psi_{n+2}$ are algebraically independent by Lemma 1. Then using elimination theory, we see that $\Psi(V)$ is an irreducible hypersurface R in $\boldsymbol{P}^{n+1}(\boldsymbol{C})$. We write the R in $\boldsymbol{P}^{n+1}(\boldsymbol{C})$ as

$$(6) \qquad R(x_1, \cdots, x_{n+2}) \equiv \sum_{i_1+\cdots+i_{n+2}=k} a_{i_1\cdots i_{n+2}} x_1^{i_1} \cdots x_{n+2}^{i_{n+2}} \equiv 0$$

for a homogeneous coordinate system $(x_1, \cdots, x_{n+2})$ in $\boldsymbol{P}^{n+1}(\boldsymbol{C})$.

We now consider the point $(1, 0, \cdots, 0) \in \boldsymbol{P}^{n+1}(\boldsymbol{C})$. Then we see $(1, 0, \cdots, 0) \notin R$ from the hypothesis (3) in $\psi_1, \cdots, \psi_{n+2}$.

Thus we see $a_{k0\cdots 0} \neq 0$. Similarly, we have

$$a_{0k\cdots 0} \neq 0, \cdots, a_{0\cdots 0k} \neq 0 .$$

Thus we can rewrite (6) in the form

$$R(x_1, \cdots, x_{n+2}) = a_{k0\cdots 0}x_1^k + \cdots + a_{0\cdots 0k}x_{n+2}^k + \alpha(x_1, \cdots, x_{n+2}) ,$$

where $\alpha(x_1, \cdots, x_{n+2})$ are the remainder terms of R. Hence we obtain

$$R(\psi_1, \cdots, \psi_{n+2}) = a_{k0\cdots 0}\psi_1^k + \cdots + a_{0\cdots 0k}\psi_{n+2}^k + \alpha(\psi_1, \cdots, \psi_{n+2})$$
$$\equiv R_1 + \cdots + R_{n+2} + R_{n+3} + \cdots + R_s\,, \qquad \text{(say)}\,,$$

where $R_j = a_{0\cdots 0\overset{(j)}{k}0\cdots 0}\psi_j^k$ and $a_{0\cdots 0\overset{(j)}{k}0\cdots 0} \neq 0$ $(j = 1, \cdots, n+2)$. Therefore we see $\{p \in V \mid R_{j_1}(p) = \cdots = R_{j_{s-1}}(p) = 0\} = \emptyset$ for every $\{j_1, \cdots, j_{s-1}\} \subset \{1, \cdots, s\}$ by means of $\{p \in V \mid R_{i_1}(p) = \cdots = R_{i_{n+1}}(p) = 0\} = \emptyset$ for every $\{i_1, \cdots, i_{n+1}\} \subset \{1, \cdots, n+2\}$ and $s \geqq n+2$. This completes the proof of Lemma 2.

§ 5. Proof of Theorem

By the definition of divisors $\{D_j\}$, there exist holomorphic sections $\tilde{\phi}_j \in \Gamma(V, \mathcal{O}(L^{\ell_j}))$ such that $D_j = (\tilde{\phi}_j)$ and $|\tilde{\phi}_j| \leqq 1$ for $j = 1, \cdots, n+2$. Let $\ell_0 = l.c.m.\,(\ell_1, \cdots, \ell_{n+2})$ and $\ell = N\ell_0$ for some $N \in \boldsymbol{Z}^+$ so that the line bundle L^ℓ becomes very ample. We set $\phi_j = \tilde{\phi}_j^{\ell/\ell_j}$. Then ϕ_j belongs to $\Gamma(V, \mathcal{O}(L^\ell))$ $(j = 1, \cdots, n+2)$, and $\{\phi_j/\phi_i\}$ are global meromorphic functions on V. Since V has a transcendence degree n, there exists a relation R of an irreducible homogeneous polynomial in $\phi_1, \cdots, \phi_{n+2}$. We write

$$R(\phi_1, \cdots, \phi_{n+2}) \equiv \sum_{j=1}^{s} R_j \equiv 0\,. \tag{7}$$

Then for every $\{j_1, \cdots, j_{s-1}\} \subset \{1, \cdots, s\}$, $(R_{j_1}, \cdots, R_{j_{s-1}})$ has no common zero points by Lemma 2 (say, $\{R_1, \cdots, R_{s-1}\}$), since L^ℓ is a very ample line bundle over V and $\operatorname{supp}((\phi_j)) = \operatorname{supp}((\tilde{\phi}_j))$. Furthermore, it is clear that $R_j \in \Gamma(V, \mathcal{O}(L^d))$ for some $d \in \boldsymbol{Z}^+$. We set $h = \sum_{j=1}^{s-1} |R_j|^2$. Then h is a positive C^∞ function with $h_\alpha = |f_{\alpha\beta}|^2 h_\beta$, where $L^d = \{f_{\alpha\beta}\}$. Thus h is a metric in the line bundle $L^d \to V$. We note that from (*) and the definition of $T_f(L, r)$,

$$T_f(L^d, r) = d \cdot T_f(L, r) + O(1) \tag{8}$$

for any choice of a metric h in L^d. From (1) and (8), we have

$$T_f(L^d, r) = \int_{\partial B(r)} \log\,(f^*h/|f^*R_j|^2)\sigma + N(f^*(R_j), r) + O(1)\,, \tag{9}$$

where (R_j) denotes the divisor in V given by the zeros of R_j, $f^*(R_j)$ denotes the pull back divisor of (R_j) in $\boldsymbol{C}^m$ and f^*R_j is the pull back of the section R_j under f.

Now we consider a holomorphic mapping from $\boldsymbol{C}^m$ into $\boldsymbol{P}^{s-2}(\boldsymbol{C})$ with the representation $F = (f^*R_1, \cdots, f^*R_{s-1})\colon \boldsymbol{C}^m \to \boldsymbol{P}^{s-2}(\boldsymbol{C})$. Let H be the hyperplane bundle over $\boldsymbol{P}^{s-2}(\boldsymbol{C})$. Taking the Fubini-Study metric in H, we see from Theorem A

$$(10) \qquad T_F(H, r) = \int_{\partial B(r)} \log\Big(\sum_{j=1}^{s-1} |f^*R_j/f^*R_i|^2\Big)\sigma + N(f^*(R_i), r) + O(1) .$$

Hence from (9) and (10), we have

$$T_F(H, r) = T_f(H^d, r) + O(1) .$$

We now consider the following s hyperplanes $H_1, \cdots, H_s$ in $P^{s-2}(C)$ in general position; for a homogeneous coordinate system $t = (t_1, \cdots, t_{s-1})$ in $P^{s-2}(C)$, $H_j = \{t \in P^{s-2}(C) | t_j = 0\}$ $(j = 1, \cdots, s-1)$ and $H_s = \{t \in P^{s-2}(C) | \sum_{j=1}^{s-1} t_j = 0\}$. The hypothesis $\delta(D_j) = 1 - \limsup_{r\to\infty} N(f^*D_j, r)/T_f(L^{t_j}, r) = 1$ implies that

$$N(F^*H_j, r) = O\Big(\sum_{i=1}^{n+2} N(f^*D_i, r)\Big) = o\Big(\sum_{i=1}^{n+2} T_f(L^{t_i}, r)\Big) = o(T_F(H, r))$$

for $j = 1, \cdots, s-1$ and

$$N(F^*H_s, r) = N(f^*(R_s), r) = o(T_F(H, r)) .$$

Suppose first that F is rational. Note that F is rational if and only if $T_F(H, r) = O(\log r)$. Then $N(F^*H_j, r) = o(T_F(H, r))$ implies that $F(C^m) \cap H_j = \emptyset$ $(j = 1, \cdots, s)$. Thus $f^*R_j/f^*R_i \neq 0$ and is rational on C^m, and hence it is constant on C^m. Thus $f^*R_j - cf^*R_i = 0$ for some constant c, that is, $f(C^m)$ lies in the hypersurfaces $R_j - cR_i = 0$ in V for $i, j = 1, \cdots, s$.

Finally, we assume that F is transcendental. Suppose that F is not linearly degenerate. Using Theorem B with $s = q$ and $n = s-2$, we have

$$T_F(H, r) \leqq o(T(H, r)) + O(\log (r \cdot T_F(H, r)))$$

for $r \to \infty$ outside a set of finite Lebesgue measure. This is absurd. Thus F is linearly degenerate, that is, there exist constants $(c_1, \cdots, c_{s-1}) \in C^{s-1} - \{0\}$ such that

$$c_1 f^*R_1 + \cdots + c_{s-1} f^*R_{s-1} \equiv 0 .$$

Hence the image $f(C^m)$ lies in the hypersurface given by

$$c_1 R_1 + \cdots + c_{s-1} R_{s-1} \equiv 0 .$$

Therefore f is algebraically degenerate. This completes the proof of the theorem.

Remark. The theorem holds for a meromorphic mapping of C^m into a smooth projective algebraic variety V.

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