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BOUNDARY VALUE CONTROL PROBLEMS INVOLVING THE BESSEL DIFFERENTIAL OPERATOR

K.-D. WERNER[1]

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Abstract

In this paper, we consider the hyperbolic partial differential equation $w_{tt} = w_{rr} + 1/r\, w_r - \nu^2/r^2\, w$, where $\nu \geqslant 1/2$ or $\nu = 0$ is a parameter, with the Dirichlet, Neumann and mixed boundary conditions. The boundary controllability for such problems is investigated. The main result is that all "finite energy" initial states can be steered to the zero state in time T, using a control $f \in L^2(0, T)$, provided $T > 2$. Furthermore, necessary conditions for controllability are also presented.

1. Introduction and problem formulation

In this paper, we consider controllability for the hyperbolic partial differential equation

$$\frac{\partial^2 w}{\partial t^2} + B^{\nu}_{(0,1)} w = 0, \quad (r, t) \in (0,1) \times (0, T), \tag{1.1}$$

where $B^{\nu}_{(0,1)}$ denotes the Bessel differential operator of order $\nu = 0$ or $\nu \geqslant \frac{1}{2}$ given by

$$B^{\nu}_{(0,1)} \psi \equiv -\frac{1}{r} \frac{d}{dr}\left(r \frac{d\psi}{dr}\right) + \nu^2/r^2\, \psi.$$

The control force $f \in L^2(0, T)$ for some $T > 0$ enters in the boundary condition

$$\alpha \frac{\partial w}{\partial r}(r, t) + \sigma w(r, t) = f(t), \quad t \geqslant 0, \tag{1.2}$$

[1]This paper was written during the author's stay at the School of Mathematics, University of New South Wales, N.S.W., 2033, Australia. (Present address: Institute of Geometry and Practical Mathematics, Technical University of Aachen, Templergraben 55, 5100 Aachen, Fed. Rep. of Germany)

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where $\alpha^2 + \sigma^2 \neq 0$, $\alpha\sigma \geqslant 0$. We require $\psi(0+)$ to be bounded since $B^{\nu}_{(0,1)}$ has a discontinuity at zero. We pose for (1.1) the initial conditions

$$w(r,0) = w_0(r) \in H_B, \qquad \frac{\partial w}{\partial t}(r,0) = v_0(r) \in H, \tag{1.3}$$

where H_B denotes the "energy space" to be defined in Section 2 and $H = \{x: \sqrt{r}\,x(r) \in L^2(0,1)\}$.

In this problem, $\alpha = 0$ corresponds to the Dirichlet problem; $\sigma = 0$ to the Neumann problem; and $\sigma > 0$, $\alpha = 1$ (without loss of generality) to the mixed boundary value problem. For brevity, these three problems are to be called problem 1, problem 2, and problem 3, respectively.

The term "control problem" refers to:
Let $T > 0$ be given and let $(w_0, v_0) \in H_B \times H$ be a given initial state. Then, find a control $f \in L^2(0,T)$ such that the solution of (1.1)–(1.3) also satisfies the terminal conditions

$$w(r,T) = w_T(r), \qquad \frac{\partial w}{\partial t}(r,t) = v_T(r),$$

where w_T and v_T are given elements in H_B and H, respectively.

Because of the time reversibility of (1.1), there is no loss of generality if we assume $w_T = v_T = 0$, i.e. we consider our control problem to have zero terminal conditions

$$w(r,T) = \frac{\partial w}{\partial t}(r,T) = 0. \tag{1.4}$$

Our main result is:

THEOREM 1.1. *Let $\nu = 0$ or $\nu \geqslant \frac{1}{2}$ be fixed. If $T > 2$, then the control problem is solvable with a control $f \in L^2(0,T)$ such that*

$$\|f\|^2_{L^2(0,T)} \leqslant K\left(\|w_0\|^2_{H_B} + \|v_0\|^2_H\right),$$

where K is a positive constant independent of w_0 and v_0. If $T < 2$, then the control problem is not solvable in general.

In most physical processes, control is applied at the boundary of the spatial region in which the process evolves. The equation $w_{tt} = -B^{\nu}_{(0,1)}w$ is a wave equation, which is of importance in the study of structural vibrations of a circular membrance and tubular catalytic reactions, where the wave speed depends only on the radius r. These examples are particular cases of the control problem considered in this paper.

Graham and Russell [4], Russell [11]–[16], Lagnese [5], Littmann [9], and other authors have presented certain results concerning boundary and distributed control for systems governed by linear hyperbolic partial differential equations in various spatial regions in which the process evolves. Our controllability result stated in Theorem 1.1 is of the same type as the main result on boundary control of the wave equation reported in [4] and [21]. However, the treatment of our control problem is rather different from that of [4] and [21]. More precisely, an existence and uniqueness theorem for our evolution problem is to be estimated under weaker assumptions. Note that controllability results of the same type as ours are also available in [4], [11] and [21], where the domains are, however, assumed to take specific forms. Furthermore, stronger conditions on the initial data are also required in [11]. Note that results concerning boundary control in unspecified or star-complemented regions are reported in [13] and [16] by using different approaches from ours. However, in the cases of the Neumann and mixed boundary control problems, these approaches give rise to only sufficient conditions for controllability. Our approach, however, shows that the sufficient condition is, in fact, also necessary. In the case of the Dirichlet boundary control problem, our technique needs to be slightly modified. As a result, we obtain only the same sufficient condition for controllability. However, we shall show that this sufficient condition is not necessary.

In Section 2 we rewrite the evolution problems in variational form. This enables us to define weak solutions of (1.1)–(1.3) by using the method of transposition. In Section 4 we show that each of the control problems is equivalent to a collection of trigonometric moment problems solvable by the theory of nonharmonic Fourier series developed in [11] and [12], together with certain results concerning the separation of eigenvalues of the operator $\overline{B}^{\nu}_{(0,1)}$ ($\overline{B}$ denotes the closure of B). The proof of Theorem 1.1 and the proofs of results concerning necessary conditions for controllability depend on special properties of the eigenvalues and eigenfunctions of $\overline{B}^{\nu}_{(0,1)}$ and the spectral representation of the energy spaces H_B. Some of the preparatory results reported in Section 3 (see also [20]) may also be of interest by themselves.

2. Existence, uniqueness and regularity results of the evolution problems

We shall follow the treatment given in [7, Chapter IV]. First, we introduce some notation and hypotheses.

Consider the Bessel differential operator $B^{\nu}_{(0,1)}$ of order $\nu \geqslant 0$ given by

$$B^{\nu}_{(0,1)}\psi \equiv -\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) + \nu^2/r^2\,\psi, \qquad r \in (0,1).$$

Let the domain of $B^{\nu}_{(0,1)}$ be defined by

$$D\left(B^{\nu}_{(0,1)}\right) \equiv \left\{\, x(r) | x(r) = r^{\nu}u(r), u(r) \in C^{\infty}(0,1),\right.$$
$$\left. u'(0) = \alpha u'(1) + \alpha u(1) = 0, \alpha^2 + \sigma^2 \neq 0, \alpha\sigma \geqslant 0 \right\}. \quad (2.1)$$

The requirement $u'(0) = 0$ is only necessary in the case when $\nu = 0$, since we need to show $B^{\nu}_{(0,1)}\psi \in H \equiv \{x(r)\colon \sqrt{r}\, x(r) \in L^2(0,1)\}$.

The domain $D(B^{\nu}_{(0,1)})$ is dense in the separable Hilbert space H, because $C_0^{\infty}(0,1)$ is dense in H (see Theorem 2 of [22, page 8]).

We define the "energy space" $H_{B^{\nu}_{(0,1)}}$ for each of the three boundary-value problems as follows: $H_{B^{\nu}_{(0,1)}} = H_B \equiv$ closure of the domain (2.1) with respect to the norm

$$\|x\|_{H_B} = \left(\int_0^1 r\left(|x'(r)|^2 + |x(r)|^2 + \nu^2/r^2\, |x(r)|^2\right) dr\right)^{1/2}. \qquad (2.2)$$

The technique of the construction of the energy space outlined in [19, Section 17] leads to our definition in the cases $\alpha = 0$ or $\sigma = 0$ in the domain (2.1). For the mixed boundary condition, (i.e. $\alpha = 1$, $\sigma > 0$), the technique has to be modified, as indicated in the Appendix, in order to obtain a similar definition.

It is important to note that the space H_B depends on the parameter $\nu \geqslant 0$ and the chosen domain. However, we shall use the same symbol H_B throughout the paper.

Identifying H with its dual, and denoting by H'_B the dual of the separable Hilbert space H_B, we have

$$H_B \subset H \subset H'_B$$

and H_B is dense in H since $C_0^{\infty}(0,1) \subset H_B$ is dense in H.

With the operator $B^{\nu}_{(0,1)}$, we associate for Dirichlet and Neumann boundary conditions the symmetric bilinear form

$$\left\langle B^{\nu}_{(0,1)}\rho, \psi\right\rangle_H \equiv a(\rho,\psi) \equiv \int_0^1 \left(r\rho'(r)\psi'(r) + \nu^2/r\, \rho(r)\psi(r)\right) dr,$$
$$\rho,\, \psi \in H_B, \qquad (2.3)$$

and for the mixed boundary condition (i.e. $\sigma > 0$, $\alpha = 1$ without loss of generality),

$$\left\langle B^{\nu}_{(0,1)}\rho, \psi\right\rangle_H = a(\rho,\psi) \equiv \sigma\rho(1)\psi(1)$$
$$+ \int_0^1 \left(r\rho'(r)\psi'(r) + \nu^2/r\, \rho(r)\psi(r)\right) dr, \qquad \rho, \psi \in H_B. \quad (2.4)$$

Therefore, the operator $B^{\nu}_{(0,1)}$ is symmetric in each case and from (2.2) and (2.3) (resp. (2.4)) a Gårding-type inequality

$$a(\rho,\rho) + \|\rho\|_H^2 \geqslant \|\rho\|_{H_B}, \qquad \forall \rho \in H_B \qquad (2.5)$$

follows.

For our second-order evolution problem, we have verified that all the assumptions of Theorem 1.1 in [7, Chapter IV] are valid. Thus, we have

THEOREM 2.1. *Under the assumptions* (2.3) (*resp*. (2.4)) *and* (2.5) *there exists a unique* $y \in L^2(0, T; H_B)$ *with* $\partial y/\partial t \in L^2(0, T; H)$ *satisfying*

$$\partial^2 y/\partial t^2 + B^{\nu}_{(0,1)} y = g \in L^2(0, T; H),$$

the initial conditions

$$y(r,0) = y_0(r) \in H_B, \qquad \frac{\partial y}{\partial t}(r,0) = y_1(r) \in H$$

and the homogeneous boundary condition

$$\alpha \frac{\partial y}{\partial r}(1,t) + \sigma y(1,t) = 0, \qquad \alpha^2 + \sigma^2 \neq 0, \qquad \alpha\sigma \geqslant 0,$$

together with the requirement that $y(0+, t)$ *is bounded. Furthermore, the mapping* $\{g, y_0, y_1\} \to \{y, \partial y/\partial t\}$ *is a linear continuous map of* $L^2(0, T; H) \times H_B \times H$ *into* $L^2(0, T; H_B) \times L^2(0, T; H)$.

In [8, Chapter 3] the result is extended to

$$y \in C([0,T], H_B) = \{y\colon\ y \text{ is continuous from } [0,T] \to H_B\},$$
$$\frac{\partial y}{\partial t} \in C([0,T]; H).$$

Since we can reverse the direction of time in equation (1.1) it is clear from Theorem 2.1 that for $h \in L^2(0, T; H)$ there exists a unique $\rho \in L^2(0, T; H_B)$ satisfying $\partial \rho/\partial t \in L^2(0, T; H)$,

$$\partial^2 \rho/\partial t^2 + B^{\nu}_{(0,1)} \rho = h, \tag{2.6}$$

the zero terminal conditions

$$\rho(r,T) = \frac{\partial \rho}{\partial t}(r,T) = 0$$

and the homogeneous boundary condition.

We proceed now with the principle of transposition outlined in [1] and [7].

By use of Theorem 2.1 we construct an isomorphism that we transpose to solve the nonhomogeneous boundary value problem (1.1)–(1.3). For this purpose, we introduce the set

$$X \equiv \Big\{ \rho | \rho \in L^2(0,T; H_B),\ \frac{\partial \rho}{\partial t} \in L^2(0,T;H),$$
$$\partial^2\rho/\partial t^2 + B^{\nu}_{(0,1)}\rho = h \in L^2(0,T;H),\ \rho(r,T) = \frac{\partial \rho}{\partial t}(r,T) = 0,$$
$$\rho(0+,t) \text{ bounded},\ \alpha\frac{\partial \rho}{\partial r}(1,t) + \sigma\rho(1,t) = 0, \alpha^2 + \sigma^2 \neq 0, \alpha\sigma \geqslant 0 \Big\}.$$

Endowed with the norm $\|\rho\|_X \equiv \|\partial^2\rho/\partial t^2 + B^\nu_{(0,1)}\rho\|_{L^2(0,T;H)} = \|h\|_{L^2(0,T;H)}$, the space X is a Hilbert space and, by virtue of Theorem 2.1,

$$\rho \to \partial^2\rho/\partial t^2 + B^\nu_{(0,1)}\rho \tag{2.7}$$

is an isomorphism of X onto $L^2(0, T; H)$.

Let $\rho \to L(\rho)$ be a continuous linear form over the Hilbert space X. Application of the Riesz representation theorem combined with the isomorphism (2.7) yields the existence and uniqueness of $\tilde{h} \in L^2(0, T; H)$ such that

$$L(\rho) = \int_0^T \left\langle \tilde{h}(\cdot, t), \frac{\partial^2\rho}{\partial t^2}(\cdot, t) + B^\nu_{(0,1)}\rho(\cdot, t)\right\rangle_H dt, \qquad \forall\rho \in X. \tag{2.8}$$

Specializing to our linear form over X using Green's formula, we obtain

LEMMA 2.2. *There exists a unique function $w \in L^2(0, T; H)$ such that for $\rho \in X$,*

$$\int_0^1 \int_0^T rw\left(\partial^2\rho/\partial t^2 + B^\nu_{(0,1)}\rho\right) dt\, dr$$

$$= \int_0^1 \int_0^T rwh\, dt\, dr + \int_0^1 r\left(\rho(r,0)v_0(r) - \frac{\partial\rho}{\partial t}(r,0)w_0(r)\right) dr$$

$$+ \begin{cases} -\dfrac{1}{\sigma}\displaystyle\int_0^T f(t)\frac{\partial\rho}{\partial r}(1,t)\, dt, & \sigma \neq 0,\ \alpha = 0, \quad (2.9)\\ \dfrac{1}{\alpha}\displaystyle\int_0^T f(t)\rho(1,t)\, dt & \begin{cases} \sigma \neq 0, & \alpha = 0, \\ \text{or } \sigma > 0, & \alpha = 1, \end{cases} \quad (2.10) \end{cases}$$

where it is assumed additionally that in the case of the Dirichlet boundary condition (i.e. $\alpha = 0$)

$$\int_0^T f(t)\frac{\partial\rho}{\partial r}(1,t)\, dt$$

is defined for all $\rho \in X$.

PROOF. By virtue of (2.8), it remains to show that the integrals over $(0, T)$ are well defined. First, under the additional assumption for the Dirichlet problem, the chosen linear form is well defined on X and it is a continuous linear form thereon. From the eigenfunction expansion of (2.6) for the Neumann or mixed homogeneous boundary condition as described in Section 3, we obtain, for $\nu = 0$ or $\nu \geqslant \frac{1}{2}$,

$$|\rho(1,t)| \leqslant \sum_{j=1}^\infty \lambda_{\nu,j}^{-1/2}\left|\int_T^t \sin\left(\left(\sqrt{\lambda_{\nu,j}}(t-\tau)\right)h_{\nu,j}(\tau)\, d\tau\right|\left|f_{\nu,j}(1)\right|$$

$$\leqslant c\sqrt{T}\sum_{j=1}^\infty \lambda_{\nu,j}^{-1/2}\left(\int_0^T \left|h_{\nu,j}(\tau)\right|^2 d\tau\right)^{1/2} < \infty,$$

using $\sqrt{\lambda_{\nu,j}} = O(j)$ (see Lemma 3.1 in Section 3) together with Parseval's equality

$$\sum_{j=1}^{\infty} \int_0^T |h_{\nu,j}(\tau)|^2 \, d\tau = \|h(r,t)\|^2_{L^2(0,T;H)}$$

where $h_{\nu,j}(t) = \langle h(\cdot,t), f_{\nu,j}(\cdot)\rangle_H$ with $f_{\nu,j}$ in H orthonormalized eigenfunctions. (If $\nu = 0$ in the case of the Neumann problem, then $\lambda_{0,1} = 0$ and we shall adopt the convention $\sin\sqrt{\lambda_{0,1}}\,t/\sqrt{\lambda_{0,1}} = t$.) Since $f \in L^2(0,T)$, we conclude that the linear functional

$$\rho \to \int_0^T f(t)\rho(1,t)\,dt$$

is continuous on X. This completes the proof.

We summarize the above results in

THEOREM 2.3. *Let $\nu = 0$ or $\nu \geqslant \frac{1}{2}$ be fixed, $(w_0, v_0) \in H_B \times H$ and $f \in L^2(0,T_1)$ with T replaced by $T_1 > T$. Then, there exists a unique function $w \in L^2(0,T_1;H)$ such that for all solutions ρ of the equation* (2.6) *as described in Theorem* 2.1 *together with the zero terminal conditions*

$$\rho(r,T_1) = \frac{\partial\rho}{\partial t}(r,T_1) = 0,$$

the following equation is valid:

$$\begin{aligned}\int_0^1 \int_0^{T_1} w\left(\partial^2\rho/\partial t^2 + B^{\nu}_{(0,1)}\rho\right) dt\,dr &= \int_0^1 \int_0^{T_1} rwh\,dt\,dr \\ &= \int_0^1 r\left(v_0(r)\rho(r,0) - w_0(r)\frac{\partial\rho}{\partial t}(r,0)\right) dr \\ &\quad + (2.9) \text{ or } (2.10), \qquad (2.11)\end{aligned}$$

where it is assumed that in the case of the Dirichlet boundary condition

$$\rho \to \int_0^{T_1} f(t)\frac{\partial\rho}{\partial r}(1,t)\,dt$$

is a well defined continuous linear functional on X.

Note that, (2.11) is, in fact, a definition of weak solutions of problem (1.1)–(1.3).

REMARK 2.4. (i) Using Green's formula, we can verify as in [7, pages 194–195, Remark 7.2, page 320] that the function $w(r,t)$ satisfies (1.1)–(1.3) in a generalized sense. Indeed, the choice of the linear functional was motivated by that.

(ii) It is easy to derive an eigenfunction expansion for $w(r,t)$ similar to that for the Neumann problem given in [7, Chapter IV, Remark 7.3].

(iii) Smooth solutions of (1.1)–(1.3) satisfy the integral relation (2.11). This is of importance for the proof of our main result, since the function h is specialized to be in $C^\infty([0,1]\times[0,T_1])$.

As interpreted in Theorem 2.3, the solution w of (1.1)–(1.3) is an element of $L^2(0,T;H)$ and thus $w(\cdot,T)$, $\partial w/\partial t(\cdot,T)$ are not necessarily defined at the fixed T as required in the statement of the control problem. Therefore, we replace condition (1.4) by the following.

Let $T_1 > T$ and f be extended from $[0,T]$ to $[0,T_1]$ by setting $f(t)=0$, $t\in(T,T_1]$. Furthermore, let $w(r,t)$ be the solution obtained from Theorem 2.3. Then (1.4) is replaced by

$$w(r,t)\equiv 0, \qquad (r,t)\in(0,1)\times[T,T_1]. \tag{2.12}$$

Clearly, (1.4) and (2.12) are equivalent in the case of classical solutions of (1.1)–(1.3).

3. Properties of eigenfunctions and eigenvalues of $\overline{B}^{\nu}_{(0,1)}$.

This material is required for effective representation of our controllability result.

Let $\nu=0$ or $\nu\geqslant 1/2$ be arbitrary but fixed. We consider from now on the operator $\overline{B}^{\nu}_{(0,1)}$, which is the closure of $B^{\nu}_{(0,1)}$ as defined in [19, Section 17].

It is known that the spectrum of $\overline{B}^{\nu}_{(0,1)}$ consists, in each of the three cases, only of the eigenvalues λ, which, together with an associated eigenfunction f, satisfy

$$d^2f/dr^2+\frac{1}{r}\frac{df}{dr}+(\lambda-\nu^2/r^2)f=0, \tag{3.1}$$

where f is subject to the homogeneous boundary condition at $r=1$ and $f(0+)$ is bounded. Such solutions are $J_\nu(\sqrt{\lambda_{\nu,j}}\,r)$, the Bessel functions (first kind) of order ν. In [2], it is shown that the numbers λ which satisfy (3.1), (3.2) can be identified as

$$\lambda_{\nu,j}=\mu^2_{\nu,j}, \qquad j=1,2,\ldots.$$

Here $\mu_{\nu,j}$, $j=1,2,\ldots$, denote any of $p_{\nu,j}$, $y_{\nu,j}$ or $\gamma_{\nu,j}$, where these are respectively the jth nonnegative roots of the equations

$$J_\nu(p_{\nu,j})=0 \quad \text{(Dirichlet boundary condition, } \alpha=0,\sigma>0),$$
$$J'_\nu(y_{\nu,j})=0 \quad \text{(Neumann boundary condition, } \sigma=0,\alpha>0)$$

and

$$\gamma_{\nu,j}J'_\nu(\gamma_{\nu,j})+\alpha J_\nu(\gamma_{\nu,j})=0 \quad \text{(mixed boundary condition, } \alpha=1,\sigma>0).$$

In [2], it is further shown that the complete set $\{f_{\nu,j}\}$, $j = 1, 2, \ldots,$ of normalized (in H) eigenfunctions are given by

$$f_{\nu,j}(r) = \sqrt{2}\,|J_{\nu+1}(p_{\nu,j})|^{-1} J_\nu(p_{\nu,j} r), \tag{3.3}$$

$$f_{\nu,j}(r) = \sqrt{2}\left[\left(1 - \nu^2/y_{\nu,j}^2\right) J_\nu^2(y_{\nu,j})\right]^{-1/2} J_\nu(y_{\nu,j} r), \qquad \nu \geqslant \tfrac{1}{2}, \tag{3.4}$$

$$\left(f_{0,j}(r) = \sqrt{2}\,|J_0(y_{0,j})|^{-1} J_0(y_{0,j} r)\right),$$

$$f_{\nu,j}(r) = \sqrt{2}\left[\left(1 + (\sigma^2 - \nu^2)/\gamma_{\nu,j}^2\right) J_\nu^2(\gamma_{\nu,j})\right]^{-1/2} J_\nu(\gamma_{\nu,j} r) \tag{3.5}$$

respectively.

The solution of the second-order evolution problem of Theorem 2.1 in terms of eigenfunctions is then given by

$$y(r,t) = \sum_{j=1}^{\infty} \Big\{ \langle y_0, f_{\nu,j}\rangle_H \cos \mu_{\nu,j} t + \langle y_1, f_{\nu,j}\rangle_H \sin \mu_{\nu,j} t/\mu_{\nu,j}$$
$$+ \frac{1}{\mu_{\nu,j}} \int_0^T g_{\nu,j}(\tau) \sin \mu_{\nu,j}(t - \tau)\, d\tau \Big\} f_{\nu,j}(r),$$

where

$$g_{\nu,j}(t) = \langle g(\cdot, t), f_{\nu,j}(\cdot)\rangle_H.$$

Certain results concerning the roots of the eigenvalues for each of the problems 1, 2, 3 are summarized in

LEMMA 3.1. *For each $j = 1, 2, \ldots,$ and fixed $\nu = 0$ or $\nu \geqslant \frac{1}{2}$, let $\mu_{\nu,j} = \sqrt{\lambda_{\nu,j}}$ denote any of $p_{\nu,j}$, $y_{\nu,j}$ or $\gamma_{\nu,j}$. Then, it is true that*

$$\lim_{j\to\infty} (\mu_{\nu,j+1} - \mu_{\nu,j}) = \pi, \tag{3.6}$$

$$\mu_{\nu,j} = O(j), \tag{3.7}$$

$$D \equiv \lim_{j\to\infty} j/\mu_{\nu,j} = 1/\pi, \tag{3.8}$$

$$\limsup_{y\to\infty} \limsup_{x\to\infty} \frac{\phi(x+y) - \phi(x)}{y} = \frac{1}{\pi}, \tag{3.9}$$

where $\phi(u)$ = the number of $\mu_{\nu,j} < u$. Let $\nu \geqslant \frac{1}{2}$. For all three cases of homogeneous boundary conditions (except in the case $\sigma = \frac{1}{2}$, $\alpha = 1$), the inequalities

$$\mu_{\nu,2} - \mu_{\nu,1} > \cdots > \mu_{\nu,j+1} - \mu_{\nu,j} > \cdots > \to \pi \tag{3.10}$$

are valid. If $\nu = \frac{1}{2}$, (3.10) is also true in the case of $\sigma = 0$. For the Dirichlet and mixed boundary condition with $\sigma = \frac{1}{2}$ all inequalities in (3.10) become equalities.

For $\nu = 0$,

$$\left(j - \frac{1}{4}\right)\pi \leqslant p_{0,j} \leqslant \left(j - \frac{1}{8}\right)\pi,\ p_{0,j+1} - p_{0,j} < \pi, \quad \forall j \in N, \tag{3.11}$$

$$\left(j - \frac{7}{8}\right)\pi \leqslant y_{0,j} \leqslant \left(j - \frac{3}{4}\right)\pi, \quad y_{0,j+1} - y_{0,j} > \pi, \quad y_{0,1} = 1, \tag{3.12}$$

$$\gamma_{0,j} = j\pi + q + \psi_1(j)/j, \qquad j \in N, \tag{3.13}$$

where q *is a constant and* $\psi_1(j)$ *is a suitable bounded function in* N.

PROOF. For each fixed $\nu \geqslant \frac{1}{2}$ and for the corresponding choice of $\mu_{\nu,j}$, the inequalities (3.10), the asymptotic gap statement (3.6) and the density statement (3.8) are proved in [3], Theorems 1.1, 1.4 and 1.2, respectively.

Case $\nu = 0$: a) $\mu_{0,j} = p_{0,j}$; (3.6) and (3.11) are known (see [18, page 43]).

b) $\mu_{0,j} = y_{0,j}$: see [17, page 314] for the first inequality in (3.12). Since $J_0'(r) = -J_1(r)$ implies $y_{0,j} = p_{1,j}$, the validity of (3.6) and the second inequality in (3.12) follows from part a).

c) $\mu_{0,j} = \gamma_{0,j}$: (3.13) follows from equation (57) of [10, page 406]. The expression (3.6) follows then directly from (3.13).

The first inequalities in (3.11), (3.12) and the expression (3.13) imply (3.8) in the case of $\nu = 0$. The statement under (3.9) for each ν considered follows from (3.6), since $\phi(x)$ is proportional to x/π and hence $\phi(x + y) - \phi(x) \to [y/\pi]$ for $x \to \infty$ ($[x]$ = greatest integer less than or equal to x).

Because all intervals between two successive positive roots $\mu_{\nu,j}$ are equal or greater than π, it follows that

$$\mu_{\nu,j} \geqslant (j-1)\pi, \qquad j = 1, 2, \ldots,$$

except in the cases of (3.11) and (3.13), where in the latter case such a result is not known. But $\mu_{\nu,j} \geqslant (j-1)\pi$ together with the first inequality in (3.11) (resp. (3.13)) in the exceptional cases imply that $\mu_{\nu,j} = O(j)$ for each $\nu \geqslant 1/2$ or $\nu = 0$. This completes the proof.

To establish our controllability result, we need lower bounds for the eigenfunctions $f_{\nu,j}$ and $f_{\nu,j}'$ at $r = 1$.

LEMMA 3.2. a) *For the eigenfunctions* $f_{\nu,j}$ *in* (3.3), *it is true that*:

$$\left|f_{\nu,j}'(1)\right| = \sqrt{2}\, p_{\nu,j}.$$

b) $|f_{0,j}(1)| = \sqrt{2}$, $|f_{\nu,j}(1)| \geqslant \sqrt{2}/(1 + \nu/y_{\nu,1}) \equiv c_{\nu,2}$ $(\nu \geqslant \frac{1}{2})$ *where* $f_{\nu,j}$, $j = 1, 2, \ldots,$ *are the eigenfunctions of* (3.4).

c) *In* (3.5),

$$\left|f_{\nu,j}(1)\right| \geqslant \sqrt{2}/(1+\sigma/\gamma_{\nu,1}) \equiv c_{\nu,3}.$$

PROOF. a) From the recurrence relation, $J_\nu'(x) = \nu/x\, J_\nu(x) - J_{\nu+1}(x)$, which is valid for $\nu \geqslant 0$ and $x \in \mathbf{R}$, we obtain

$$f_{\nu,j}'(r) = \sqrt{2}\left|J_{\nu+1}(p_{\nu,j})\right|^{-1}\left\{p_{\nu,j}\left[-J_{\nu+1}(p_{\nu,j}r) + \nu/p_{\nu,j}r\, J_\nu(p_{\nu,j}r)\right]\right\}.$$

Hence, since $J_\nu(p_{\nu,j}) = 0$, we get $|f_{\nu,j}'(1)| = \sqrt{2}\,p_{\nu,j}$.

b) $\nu = 0$, a direct computation gives $|f_{0,j}(1)| = \sqrt{2}$.

Since $y_{\nu,j} \geqslant y_{\nu,1}$ for all $j \in \mathbf{N}$ and for each $\nu \geqslant \frac{1}{2}$, we obtain

$$\left|f_{\nu,j}(1)\right| \geqslant \sqrt{2}\left(1-\nu^2/y_{\nu,j}^2\right)^{-1/2} \geqslant \sqrt{2}\,(1+\nu/y_{\nu,1})^{-1} \equiv c_{\nu,2}.$$

c)

$$\begin{aligned}\left|f_{\nu,j}(1)\right| &= \sqrt{2}\,\gamma_{\nu,j}\left(\gamma_{\nu,j}^2+\sigma^2-\nu^2\right)^{-1/2} \geqslant \sqrt{2}\left(1+\sigma^2/\gamma_{\nu,j}^2\right)^{-1/2}\\ &\geqslant \sqrt{2}\,(1+\sigma/\gamma_{\nu,1})^{-1} \equiv c_{\nu,3}\end{aligned}$$

since $\gamma_{\nu,j} \geqslant \gamma_{\nu,1}$ for all $j \in \mathbf{N}$.

The proof is complete.

The next two lemmas are important for the characterization of the controllable states.

LEMMA 3.3. *Let A be a positive definite self adjoint operator with pure point spectrum and let H_A denote the energy space. Let $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ denote the (multiple) eigenvalues of A and $\{x_j\}$, $j = 1, 2, \ldots,$ be the corresponding system of orthonormalized eigenvectors in the Hilbert space H. Then*

$$H_A = \left\{ x \middle| x \in H,\ \sum_{j=1}^{\infty} \lambda_j \left|\langle x, x_j\rangle_H\right|^2 < \infty \right\}.$$

Furthermore, $\{x_j\lambda_j^{-1/2}\}$, $j = 1, 2, \ldots,$ is a complete orthonormal system in H_A and the operator $A^{1/2}$ defines a unitary mapping from H_A onto H, i.e. $\|A^{1/2}x\|_H = \|x\|_{H_A}$ and $D(A^{1/2}) = H_A$, $R(A^{1/2}) = H$.

PROOF. Theorem 21.2 (a) of [19].

Note that $\lambda_{0,1} = y_{0,1}^2 = 0$ is an eigenvalue of $\overline{B}_{(0,1)}^0$ with Neumann boundary condition. Thus, $\overline{B}_{(0,1)}^0$ is not positive definite and hence we cannot apply Lemma 3.3. Nevertheless, the following result is valid.

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Lemma 3.4. *Let $f_{0,j}$, $j = 1, 2, \ldots$, be the eigenfunctions defined in* (3.4) *and $y_{0,j}^2$ the corresponding eigenvalues. Then,*

$$w \in H_{B^0_{(0,1)}} \text{ is equivalent to } \sum_{j=1}^{\infty} y_{0,j}^2 \left|\langle w_0, f_{0,j}\rangle\right|^2 < \infty.$$

The proof of the Lemma is given in the Appendix

4. Proof of Theorem 1.1

Using the definition of weak solutions of the boundary value problem (1.1)–(1.3) given in Theorem 2.3, we transform the control problems into a sequence of equivalent moment problems. These are solved using the theory of nonharmonic Fourier series. In that process, we obtain our controllability result reported in Theorem 1.1.

Let $w(r,t)$ be the weak solution of (1.1)–(1.3). Then, the expansion of the initial data $(w_0, v_0) \in H_B \times H$ in H is

$$\begin{Bmatrix} w_0(r) \\ v_0(r) \end{Bmatrix} = \sum_{j=1}^{\infty} \begin{Bmatrix} w_{\nu,j} \\ v_{\nu,j} \end{Bmatrix} f_{\nu,j}(r),$$

where

$$\begin{Bmatrix} w_{\nu,j} \\ v_{\nu,j} \end{Bmatrix} = \int_0^1 \begin{Bmatrix} w_0(r) \\ v_0(r) \end{Bmatrix} r f_{\nu,j}(r)\, dr.$$

Let $T_1 > T$ and let $g = g(t) \in C^\infty[0,\infty)$ with compact support in (T, T_1) (cf. end of Section 2). Define a function $g_{\nu,j} \in C^\infty((0,1)\times[0,\infty))$ by

$$g_{\nu,j}(r,t) = g(t) f_{\nu,j}(r), \qquad j = 1, 2, \ldots. \tag{4.1}$$

Consider the equation

$$\partial^2\rho/\partial t^2 + B^\nu_{(0,1)}\rho = g_{\nu,j}, \qquad (r,t) \in (0,1)\times[T, T_1],$$

with zero terminal data $\rho(r, T_1) = \partial\rho/\partial t(r, T_1) = 0$ together with the homogeneous boundary condition (1.2) and the boundedness condition $\rho(0+, t) < \infty$ for $t \geqslant 0$. Its unique solution is given by

$$\rho(r,t) = \rho_{\nu,j}(r,t) = \left(\alpha_1 \cos\mu_{\nu,j}t + \alpha_2 \sin\mu_{\nu,j}t/\mu_{\nu,j}\right) f_{\nu,j}(r),$$

$$\alpha_1 = \int_T^{T_1} \frac{1}{\mu_{\nu,j}} \sin(\mu_{\nu,j}\tau) g(\tau)\, d\tau, \qquad \alpha_2 = -\int_T^{T_1} \cos(\mu_{\nu,j}\tau) g(\tau)\, d\tau.$$

Here, we again adopt the convention that $\sin y_{0,1}t/y_{0,1} = t$, since $\lambda_{0,1} = y_{0,1}^2 = 0$. Thus $\rho \in C^\infty((0,1)\times[0, T_1])$ satisfies the hypotheses of Theorem 2.3. The additional assumption needed in the Dirichlet problem is also satisfied for that ρ.

Substituting ρ, $\partial\rho/\partial t$ together with the expansions of w_0, v_0 into (2.11), it follows that

$$\int_0^1 \int_T^{T_1} rw(r,t) g_{\nu,j}(r,t)\, dt\, dr = 0$$

for all functions $g_{\nu,j}$ as described in (4.1), and hence (2.12) is satisfied if and only if the functions $f \in L^2(0,T)$ solve the corresponding moment problems, which consist of the equations

$$\int_0^T f(t) \exp(\pm p_{\nu,j} t)\, dt = \sigma\left(v_{\nu,j} \mp i p_{\nu,j} w_{\nu,j}\right)/f'_{\nu,j}(1) \text{ (Dirichlet)}, \quad (4.2a)$$

$$\int_0^T f(t) \exp(\pm y_{\nu,j} t)\, dt = \alpha\left(-v_{\nu,j} \mp i y_{\nu,j} w_{\nu,j}\right)/f_{\nu,j}(1) \text{ (Neumann)} \quad (4.2b)$$

$$\left(\nu = 0,\ j = 1\colon \int_0^T f(t)\, dt = -\alpha v_{0,1}/f_{0,1}(1),\ \int_0^T t f(t)\, dt = \alpha w_{0,1}/f_{0,1}(1)\right)$$

and

$$\int_0^T f(t) \exp(\pm \gamma_{\nu,j} t)\, dt = \alpha\left(-v_{\nu,j} \pm i \gamma_{\nu,j} w_{\nu,j}\right)/f_{\nu,j}(1) \text{ (mixed)}. \quad (4.2c)$$

For abbreviation, we denote $\sigma(v_{\nu,j} - i p_{\nu,j} w_{\nu,j})/f'_{\nu,j}(1)$ and $\sigma(v_{\nu,j} + i p_{\nu,j} w_{\nu,j})/f'_{\nu,j}(1)$ by $c^{(1)}_{\nu,j}$ and $d^{(1)}_{\nu,j}$ respectively. Here the "+ sign" in $\exp(\pm p_{\nu,j} t)$ refers to $c^{(1)}_{\nu,j}$ and the "–sign" in $\exp(\pm p_{\nu,j} t)$ refers to $d^{(1)}_{\nu,j}$. Similarly, $c^{(2)}_{\nu,j}$, $d^{(2)}_{\nu,j}$ and $c^{(3)}_{\nu,j}$, $d^{(3)}_{\nu,j}$ are used, respectively, to denote the right-hand sides of the moment problems (4.2b) and (4.2c).

The solvability of such a moment problem depends on the value of T and properties of $\mu_{\nu,j}$ listed in Lemma 3.1.

LEMMA 4.1. *If $T < 2$, each of the moment problems* (4.2a)–(4.2c) *has no solution in general.*

PROOF. This is a special case of a result proved by Levinson [6, page 3] where $T < 2\pi D$. Since $D = 1/\pi$ (cf. (3.8)) in our case, the assertion follows from this result.

LEMMA 4.2. *Let $T > 2$ and $\nu = 0$ or $\nu \geqslant \frac{1}{2}$ be fixed. Let the sequences $\{c^{(i)}_{\nu,j}\}$, $\{d^{(i)}_{\nu,j}\}$, $j = 1, 2, \ldots$, $i = 1, 2, 3$, be elements in the Hilbert space l_2. Then, each of the moment problems* (4.2a)–(4.2c) *has a solution $f \in L^2(0,T)$ such that*

$$\tilde{K}_i \sum_{j=1}^{\infty} \left(\left|c^{(i)}_{\nu,j}\right|^2 + \left|d^{(i)}_{\nu,j}\right|^2\right) \leqslant \|f\|^2_{L^2(0,T)} \leqslant \hat{K}_i \sum_{j=1}^{\infty} \left(\left|c^{(i)}_{\nu,j}\right|^2 + \left|d^{(i)}_{\nu,j}\right|^2\right), \quad (4.3)$$

where the positive constants $\tilde{K}_i$, $\hat{K}_i$ *are determined by the asymptotic gap* π (*cf.* (3.6)) *and the positive number* $T-2$. ($\tilde{K}_i$, $\hat{K}_i$ *are independent of the particular sequences* $\{c_{\nu,j}^{(i)}\}$, $\{d_{\nu,j}^{(i)}\}$ *for a fixed* $\nu = 0$ *or* $\nu \geqslant \frac{1}{2}$).

Proof. [11, pages 549–555] under the assumption that the properties (3.6)–(3.10) of Lemma 3.1 are satisfied.

As a consequence of that lemma, we can prove the following theorem concerning solutions of the moment problems (4.2a)–(4.2c).

Theorem 4.3. *If for* $T > 2$ *and for* $\nu = 0$ *or* $\nu \geqslant \frac{1}{2}$ *fixed*

$$\sum_{j=1}^{\infty} |v_{\nu,j}|^2 \equiv a^2 < \infty, \tag{4.4}$$

$$\sum_{j=1}^{\infty} |\mu_{\nu,j} w_{\nu,j}|^2 \equiv b^2 < \infty \tag{4.5}$$

then each of the moment problems (4.2a)–(4.2c) *has a solution* $f \in L^2(0,T)$. *Furthermore*,

$$\|f\|^2_{L^2(0,T)} \leqslant K_i(a^2+b^2),$$

where the positive constant K_i, $i = 1,2,3$, *is independent of the coefficients* $v_{\nu,j}$, $w_{\nu,j}$.

Proof. Let $f \in L^2(0,T)$ be the solution of the corresponding moment problem (4.2a), (4.2b) or (4.2c), which exists by Lemma 4.2 since the sequences $\{c_{\nu,j}^{(i)}\}$, $\{d_{\nu,j}^{(i)}\}$, $j = 1,2,\ldots$, $i = 1,2,3$, are elements in l_2 by virtue of Lemma 3.1. Combining (4.3) and the right-hand sides of each of the moment problems (4.2a)–(4.2c) we have, respectively,

$$\|f\|^2_{L^2(0,T)} \leqslant \hat{K}_1 \sum_{j=1}^{\infty} \left(\left|c_{\nu,j}^{(1)}\right|^2 + \left|d_{\nu,j}^{(1)}\right|^2\right) \leqslant 4\hat{K}_1 \sum_{j=1}^{\infty} \left|f'_{\nu,j}(1)\right|^{-1}\left(|v_{\nu,j}|^2 + |p_{\nu,j} w_{\nu,j}|^2\right),$$

$$\|f\|^2_{L^2(0,T)} \leqslant \hat{K}_2 \sum_{j=1}^{\infty} \left(\left|c_{\nu,j}^{(2)}\right|^2 + \left|d_{\nu,j}^{(2)}\right|^2\right) \leqslant 4\hat{K}_2 \sum_{j=1}^{\infty} \left|f_{\nu,j}(1)\right|^{-1}\left(|v_{\nu,j}|^2 + |y_{\nu,j} w_{\nu,j}|^2\right),$$

$$\|f\|^2_{L^2(0,T)} \leqslant \hat{K}_3 \sum_{j=1}^{\infty} \left(\left|c_{\nu,j}^{(3)}\right|^2 + \left|d_{\nu,j}^{(3)}\right|^2\right) \leqslant 4\hat{K}_3 \sum_{j=1}^{\infty} \left|f_{\nu,j}(1)\right|^{-1}\left(|v_{\nu,j}|^2 + |\gamma_{\nu,j} w_{\nu,j}|^2\right).$$

Thus, from the uniform lower bounds for the numbers $f'_{\nu,j}(1)$ and $f_{\nu,j}(1)$ as described in Lemma 3.2, we obtain the inequalities

$$\|f\|^2_{L^2(0,T)} \leqslant K_i(a^2+b^2), \qquad i = 1,2,3,$$

where K_t is independent of $w_{\nu,j}$, $v_{\nu,j}$. Here, we recall that the Fourier-Bessel coefficients $w_{\nu,j}$, $v_{\nu,j}$ of w_0, v_0 (and hence $\{c_{\nu,j}^{(i)}\}$, $\{d_{\nu,j}^{(i)}\}$) depend on $f_{\nu,j}$ which, in turn, vary with the considered boundary conditions.

This completes the proof.

PROOF OF THEOREM 1.1. Condition (4.4) is equivalent to $v_0 \in H$. Lemma 3.3, or Lemma 3.4 in the Neumann problem with parameter $\nu = 0$, indicates that $w_0 \in H_B$ implies condition (4.5). Thus, Theorem 4.3 shows that the moment problems (4.2a)–(4.2c) are solvable for $f \in L^2(0,T)$, provided $T > 2$. But this is equivalent to controllability, as we have seen in the derivation of the moment problems.

The proof is complete.

REMARK 4.4. Theorem 1.1 is all one can obtain, in general, from the use of the inequalities (4.4) and (4.5). Indeed, Lemma 3.3 (Lemma 3.4 in the case of $\nu = 0$ and Neumann boundary condition) shows that if the inequality (4.5) is satisfied, then w_0 as given by its Fourier-Bessel expansion lies in H_B. We note that (4.4) is equivalent to $v_0 \in H$.

The next lemma shows that for our control problem with Dirichlet boundary condition and zero initial data, there exists a control $f \in L^2(0,T)$ such that the terminal state $(w(r,T),\ \partial w/\partial t(r,T))$ does not lie in $H_B \times H$. Due to the time reversibility in (1.1), this is a direct consequence of the following Lemma.

LEMMA 4.5. *Consider the Dirichlet boundary control problem. Let $\nu = 0$ or $\nu \geqslant \frac{1}{2}$ be fixed. Then, there exist initial states $(w_0, v_0) \notin H_B \times H$ which can be steered to the terminal state $w(r,T) = \partial w/\partial t(r,T) = 0$ by a control $f \in L^2(0,T)$, provided $T > 2$.*

PROOF. Consider the initial state given by $w_0(r) \equiv 0$ and

$$v_0(r) = \sum_{j=1}^{\infty} \alpha_{\nu,j} f'_{\nu,j}(1) f_{\nu,j}(r), \tag{4.6}$$

where $\alpha_{\nu,j} = j^{-(1+\varepsilon)/2}$ and $\varepsilon > 0$. Then, the expansion coefficients $w_{\nu,j}$ are all zero and those of v_0 are

$$v_{\nu,j} = \alpha_{\nu,j} f'_{\nu,j}(1) = O\left(j^{(1-\varepsilon)/2}\right) \tag{4.7}$$

by using Lemma 3.2(a). Thus, for the moment problem (4.2a), we have $c_{\nu,j} = \sigma\alpha_{\nu,j}$. Hence for the control f to lie in $L^2(0,T)$ with $T > 2$, we see from Lemma 4.2

that it is sufficient to have

$$\sum_{j=1}^{\infty} (\alpha_{\nu,j})^2 = \sum_{j=1}^{\infty} j^{-1-\varepsilon} < \infty,$$

which is true for any $\varepsilon > 0$. Since the coefficients in the expansion (4.6) have the asymptotic relation (4.7), it follows that for some constant $K_\nu > 0$,

$$\sum_{j=1}^{\infty} \left(\alpha_{\nu,j} f'_{\nu,j}(1)\right)^2 \geqslant K_\nu \sum_{j=1}^{\infty} j^{1-\varepsilon} = \infty,$$

if $\varepsilon \in (0, 2]$. For such a choice of ε, (4.6) and $w_0(r) \equiv 0$ represent a state (w_0, v_0) for which $v_0 \notin H$. But this state is controllable in the sense that the moment problem (4.2a) has a solution $f \in L^2(0, T)$. This completes the proof.

The main point in Lemma 4.5 is that, in the Dirichlet boundary control problem, the inequalities (4.4) and (4.5) are more restrictive than the requirements $\{c^{(1)}_{\nu,j}\}$, $\{d^{(1)}_{\nu,j}\} \in l_2$ given in Lemma 4.2. By contrast, for Neumann and mixed boundary control problems, the order of convergence of the series (4.4) and (4.5) is the same as in (4.3). This is a consequence of the uniform lower bounds for the corresponding eigenfunctions $f_{\nu,j}$ at $r = 1$ given in Lemma 3.2. Therefore, by noting that the condition for controllability in these two cases is the square summability of the coefficients appearing on the right-hand side of (4.2b) and (4.2c) respectively, we see that for the control f to lie in $L^2(0, T)$ with $T > 2$, we must have

$$\sum_{j=1}^{\infty} |v_{\nu,j}|^2 < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} |y_{\nu,j} w_{\nu,j}|^2 < \infty \quad \text{or} \quad \sum_{j=1}^{\infty} |\gamma_{\nu,j} w_{\nu,j}|^2 < \infty$$

respectively. This implies that $v_0 \in H$ and $w_0 \in H_B$, by virtue of Lemma 3.3 and 3.4. Combining these with Theorem 1.1, we have the following conclusion.

THEOREM 4.6. *Consider the Neumann (resp. mixed) boundary control problem. $(w_0, v_0) \in H_B \times H$ is a necessary and sufficient condition for the existence of a control $f \in L^2(0, T)$, $T > 2$, such that each control problem has a solution.*

LEMMA 4.7. *In the Dirichlet boundary control problem, the conditions $w_0 \in H$ and $v_0 \in H'_B$ (= dual of H_B) are necessary for the existence of a solution of the control problem with a control $f \in L^2(0, T)$, $T > 2$.*

PROOF. The controllability condition

$$\sum_{j=1}^{\infty} \left|f'_{\nu,j}(1)\right|^{-1} \left(|v_{\nu,j}|^2 + |p_{\nu,j} w_{\nu,j}|^2\right) < \infty$$

implies by virtue of the inequality in Lemma 3.2(a)

$$\sum_{j=1}^{\infty}\left(\left|v_{\nu,j}/p_{\nu,j}\right|^2+\left|w_{\nu,j}\right|^2\right)<\infty.$$

Hence $w_0 \in H$ and $v_0 \in H'_B$, which completes the proof.

5. Appendix

We start with the modification of the energy space in the case of $\alpha = 1$, $\sigma > 0$ in the domain $D(B^{\nu}_{(0,1)})$. For $x,\ y \in D(B^{\nu}_{(0,1)})$, we have associated with $B^{\nu}_{(0,1)}$ the bilinear form (2.4). Hence, we define for $x,\ y \in D(B^{\nu}_{(0,1)})$

$$\|x\|_1^2 \equiv \sigma x^2(1) + \int_0^1 r\left(|x'(r)|^2 + |x(r)|^2 + \nu^2/r^2|x(r)|^2\right)dr,$$

$$\|x\|^2 \equiv \int_0^1 r\left(|x'(r)|^2 + |x(r)|^2 + \nu^2/r^2|x(r)|^2\right)dr.$$

We assert that these two norms are equivalent. To show this, we note that for $0 < a < 1$ fixed $L^2(a,1) = L^2(a,1;\sqrt{r})$ (with equivalent norms) implies $H^1(a,1) = H^1(a,1;\sqrt{r})$ (norms are equivalent). Then, by Lemma 23.2a) of [19], there exists for $x(t) \in H^1(a,b)$ $(-\infty < a < b < \infty)$ a constant $c > 0$ such that

$$\sup_{t\in[a,b]} |x(t)| \leqslant c\|x\|_{H^1(a,b)}.$$

Therefore, with $x(t) \in H^1(a,1;\sqrt{r})$, $a > 0$, we have

$$\sup_{t\in[a,1]} |x(t)| \leqslant c_1\|x\|_{H^1(a,1;\sqrt{r})} \tag{5.1}$$

which yields

$$\sigma|x(1)|^2 \leqslant c_2\|x\|^2_{H^1(a,1;\sqrt{r})} \leqslant c_2\|x\|^2_{H^1(0,1;\sqrt{r})}.$$

This, in turn, implies that

$$\|x\|_1^2 \leqslant \|x\|^2 + c_2\|x\|^2 = (1+c_2)\|x\|^2.$$

Since $\|x\|_1 \geqslant \|x\|$ the norms are equivalent for $x \in D(B^{\nu}_{(0,1)})$. Thus, a limit process shows that $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent for arbitrary

$$x \in H_B \equiv \text{closure of } D\left(B^{\nu}_{(0,1)}\right) \text{ in the norm } \|\cdot\|.$$

REMARK 5.1. The construction of the energy space outlines in [19, Section 17] would lead to the closure of $D(B^{\nu}_{(0,1)})$ in the norm $\|\cdot\|_1$.

REMARK 5.2. Before we return to the proof of Lemma 3.4, we need to show that the functions

$$\frac{1}{y_{0,j}}f'_{0,j}(1) = -\left(2/J_0^2(y_{0,j})\right)^{1/2}J_1(y_{0,j}r), \qquad j = 2,3,\ldots, \tag{5.2}$$

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form an orthonormal system in H. For brevity, we suppress the index "0" and write y_f, f_j instead of $y_{0,j}$, $f_{0,j}$. From expression (48) of [2, 7.10.4], we recall that

$$\int_0^1 rJ_1(y_jr)J_1(y_kr)\,dr = \begin{cases} 0, & j \neq k, \\ \frac{1}{2}J_2^2(y_j), & j = k \neq 1. \end{cases} \tag{5.3}$$

Thus, since $f_j'(1) = 0$, the recurrence relation $J_{\nu-1}(r) + J_{\nu+1}(r) = 2\nu/rJ_\nu(r)$, $\nu \in \mathbf{N}$, $r \in \mathbf{R}$, implies $J_0(y_j) + J_2(y_j) = 2/y_jJ_1(y_j) = 0$, $j = 2, 3, \ldots$. Therefore, $J_0^2(y_j) = J_2^2(y_j)$ and the assertion follows by substituting (5.2) into (5.3).

PROOF OF LEMMA 3.4. (a) $w_0 \in H_{B^0_{(0,1)}} \Rightarrow \sum_{j=1}^\infty y_j^2 |\langle w_0, f_j\rangle_H|^2 < \infty$. Since $H_B \subset H$, we can expand $w_0(r)$ in terms of eigenfunctions as follows:

$$w_0(r) = \sum_{j=1}^\infty h_j f_j(r).$$

Now,

$$h_j = \int_0^1 rw_0(r)f_j(r)\,dr = -y_j^{-2}\int_0^1 rw_0(r)\left(\frac{1}{r}\frac{d}{dr}\left(rdf_j(r)/dr\right)\right)dr, \qquad j = 2, 3, \ldots, \tag{5.4}$$

The eigenfunctions $f_j(r)$ are analytic in $\mathbf{R}$ and $B^0_{(0,1)}f_j \in H$. Thus, the Cauchy-Schwarz inequality shows that the integrals

$$\int_0^1 r\left[w_0(r)\frac{1}{r}\frac{d}{dr}\left(rdf_j(r)/dr\right) + w_0'(r)f_j'(r)\right]dr = \int_0^1 \frac{d}{dr}\left(rw_0(r)f_j'(r)\right)dr$$

are finite. In order to show that

$$\int_0^1 \frac{d}{dr}\left(rw_0(r)f_j'(r)\right)dr = rw_0(r)f_j'(r)\Big|_{r=0}^1 = w_0(1)f_j'(1) = 0, \tag{5.5}$$

we proceed as follows:

(i) By Lemma 23.2 of [19], $w_0(r) \in H_B$ is continuous in $[a, 1]$, $a > 0$ (after a possible modification in a set of measure zero). Therefore, $w_0(1)$ exists (cf. expression (5.1)). Thus, by using the fact that $f_j'(1) = 0$ we obtain $\lim_{r\to 1}(rw_0(r)f_j'(r)) = 0$.

(ii) $w_0 \in H_B$ implies that w_0 cannot have a singularity of type $r^{-\beta}$, $\beta \geqslant 1$ at $r = 0$. Hence, $f_j'(0) = 0$ implies $\lim_{r\to 0}(rw_0(r)f_j'(r)) = 0$.

Combining (5.4) and (5.5), we obtain

$$h_j = y_j^{-2}\int_0^1 rw_0'(r)f_j'(r)\,dr, \qquad j = 2, 3, \ldots.$$

By using Remark 5.2 and Bessel's inequality, it follows that the Fourier-Bessel coefficients of $w_0'(r)$ with respect to $\{f_j'(r)/y_i\}$ are square summable. These coefficients are

$$a_j \equiv \left\langle w_0'(r), y_j^{-1}f_j'(r)\right\rangle_H = y_jh_j,$$

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Hence,

$$\sum_{j=1}^{\infty} y_j^2 \left| \left\langle w_0, f_j \right\rangle_H \right|^2 < \infty.$$

(b) In order to prove that $\sum_{j=1}^{\infty} y_j^2 |\langle w_0, f_j \rangle_H|^2 < \infty \Rightarrow w_0 \in H_{B^0_{(0,1)}}$, we introduce the spaces

$$\tilde{H} \equiv H - \{1\} \quad \text{and} \quad \tilde{H}_{B^0_{(0,1)}} \equiv H_{B^0_{(0,1)}} \cap \tilde{H}.$$

We also define

$$\left(\tilde{B}^0_{(0,1)} x \right)(r) \equiv \left(B^0_{(0,1)} x \right)(r) \quad \text{for } x \in D\left(\tilde{B}^0_{(0,1)} \right) = \tilde{H} \cap D\left(B^0_{(0,1)} \right).$$

Then, the operator $\tilde{B}^0_{(0,1)}$ is positive definite and we can apply Lemma 3.3. Thus $(\tilde{B}^0_{(0,1)})^{1/2}$ is a unitary operator from $\tilde{H}_{B^0_{(0,1)}} = D((\tilde{B}^0_{(0,1)})^{1/2})$ onto $\tilde{H}$ and

$$\tilde{H}_{B^0_{(0,1)}} = \left\{ x | x \in \tilde{H}\colon \sum_{j=1}^{\infty} y_j^2 \left| \left\langle x, f_j \right\rangle_H \right|^2 < \infty \right\}.$$

Therefore, $w_0 \in \tilde{H} \subset H$ and $\sum_{j=1}^{\infty} y_j^2 |\langle w_0, f_j \rangle|^2 < \infty$ implies that $w_0 \in \tilde{H}_{B^0_{(0,1)}} \subset H_{B^0_{(0,1)}}$. If $w_0 \equiv$ constant, then it is obvious that $w_0 \in H_{B^0_{(0,1)}}$ and $\sum_{j=1}^{\infty} y_j^2 |\langle \text{const}, f_j \rangle_H|^2 < \infty$. Thus, the proof is complete.

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