Note on the application of complex integration to the equation of Conduction of Heat, with special reference to Dr Peddie's problem.

By John Dougall, M.A.

1. In Dr Peddie's problem of a sphere cooling in a well-stirred liquid, the conditions to be satisfied by the temperature $v(r, t)$ are

(i) For every positive t, and every r from 0 to a, v is to be finite and one-valued, and is to possess finite derivatives $\frac{dv}{dt}, \frac{dv}{dr}, \frac{d^2v}{dr^2}$, satisfying

$$\frac{dv}{dt} = \kappa\left(\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr}\right),$$

$$\text{or} \quad \frac{d}{dt}(rv) = \kappa\frac{d^2}{dr^2}(rv).$$

(ii) For every positive t, and $r = a$,

$$\frac{dv}{dr} + \frac{pa}{\kappa}\frac{dv}{dt} = 0,$$

where $p = \frac{1}{3}$ (capacity of liquid)/(capacity of sphere).

(iii) $\underset{t=0}{\text{Limit}}\ v(r, t) = \text{a}$ given arbitrary function $f(r)$, for every r less than a.

(iv) $\underset{t=0}{\text{Limit}}\ v(a, t) = \text{initial temperature of liquid}$
$= 0$ suppose.

The classical method of Fourier, which Dr Peddie applies to the problem, is simple and beautiful so far as it goes, but it is open to the very serious objection that it leaves unverified the fundamental condition (iii).

A complete, though necessarily more tedious treatment, can be given by means of Cauchy's Theory of Residues, which I applied to some Potential problems in a paper published in last year's *Proceedings*. I have to thank Dr Peddie for kindly permitting me to illustrate this method by a discussion of his very pretty problem.

https://doi.org/10.1017/S0013091500032612 Published online by Cambridge University Press

2. In order to avoid a certain difficulty arising out of condition (iv), I suppose to begin with that $f(r)$ is zero from $r=b$ to $r=a$, where $b<a$; this restriction is afterwards removed.

Then a function satisfying (i) and (iii) is

$$V=\frac{1}{2r\sqrt{\pi\kappa t}}\int_0^b \rho f(\rho)\left[e^{-(r-\rho)^2/4\kappa t}-e^{-(r+\rho)^2/4\kappa t}\right]d\rho.$$

This is, in fact, the known expression for the temperature at time t in an *infinite* solid, when the initial temperature is $f(r)$ from 0 to b, and zero for all values of r greater than b.

We may write

$$V=\frac{2}{\pi r}\int_0^b \rho f(\rho)d\rho\,.\int_0^\infty e^{-\kappa a^2 t}\sin ar\sin a\rho\, da,$$

t being positive, or

$$V=\int_0^b \rho f(\rho)U d\rho, \text{ where}$$

$$U=\frac{2}{\pi r}\int_0^\infty e^{-\kappa a^2 t}\sin ar\sin a\rho\, da.$$

If now we can find a function U_1 satisfying (i), such that $\underset{t=0}{\text{Limit}}\, U_1=0$ for every r from 0 to a inclusive, and every ρ from 0 to b inclusive; and such that $U+U_1$ satisfies the surface condition (ii); then $v=\int_0^b \rho f(\rho)(U+U_1)d\rho$ is obviously the solution of our problem.

3. In order to obtain U_1 it is necessary to express U as a complex integral.

We have $$U=\int_{-\infty}^{+\infty}\frac{1}{i\pi r}e^{-\kappa a^2 t}\sin a\rho\, e^{iar}\,da$$

$$= \text{the complex integral}$$

$$\int \frac{1}{i\pi r}e^{-\kappa a^2 t}\sin a\rho\, e^{iar}\,da,$$

the path being the whole of the real axis from West to East.

This path we now displace into the upper half of the a plane. The precise position of the new path need not be specified, the only essential being that the ultimate directions of its ends must not make greater angles than $\pi/4$ with the real axis, since $e^{-\kappa a^2 t}$ is infinite if the phase of a lie between $\pi/4$ and $3\pi/4$.

For the sake of definiteness, however, we shall suppose the displaced path to be symmetrical about the axis of imaginaries, and to be ultimately inclined to the axis of reals at an angle $\pi/8$; this path we shall call the path *we*.

$$\text{Hence } U = \frac{1}{i\pi r}\int e^{-\kappa a^2 t} \sin a\rho \, e^{iar} \, da, \text{ path } we;$$

$$\text{or } U = \int u da, \text{ path } we;$$

$$\text{where } u = \frac{1}{i\pi r} e^{-\kappa a^2 t} \sin a\rho \, e^{iar}$$

Take $u_1 = \dfrac{A}{i\pi r} e^{-\kappa a^2 t} \sin a\rho \sin ar$, which satisfies (i), choosing the constant A, so that $u + u_1$ satisfies (ii).

$$\text{This gives} \quad A = -\frac{(pa^2a^2 - iaa + 1)e^{iaa}}{(pa^2a^2 + 1)\sin aa - aa\cos aa},$$

$$\text{and} \quad u + u_1 = \frac{1}{i\pi r} e^{-\kappa a^2 t} \sin a\rho \frac{(pa^2a^2 + 1)\sin a(a - r) - aa\cos a(a - r)}{(pa^2a^2 + 1)\sin aa - aa\cos aa}.$$

Then $\int u_1 da$, path *we*, is the function U_1 we require.

For $\int (u + u_1) da$ satisfies the surface condition; also $\underset{t=0}{\text{Limit}} \int u_1 da$ is zero for all values of ρ and r in question.

To prove the latter statement, we have

$$\int u_1 da = -\frac{1}{i\pi r}\int e^{-\kappa a^2 t} \sin a\rho \sin ar \frac{(pa^2a^2 - iaa + 1)e^{iaa}}{(pa^2a^2 + 1)\sin aa - aa\cos aa} da,$$

path *we*, and the integral at both ends of the path converges uniformly with respect to t right up to $t = 0$, provided $\rho + r < 2a$. Hence $\underset{t=0}{\text{Limit}} \int u_1 da$ may be found by putting $t = 0$ in the integrand. But this gives a zero integral, there being no singularities of the new integrand in the part of the a plane above the path *we*.

https://doi.org/10.1017/S0013091500032612 Published online by Cambridge University Press

Hence

$$U + U_1 = \frac{1}{i\pi r}\int e^{-\kappa a^2 t}\sin a\rho \frac{(pa^2a^2+1)\sin a(a-r) - aa\cos a(a-r)}{(pa^2a^2+1)\sin aa - aa\cos aa}da,$$

path *we*.

If w_1e_1 be the image of the path *we* in the real axis, this integral is the same, element for element, as the integral over the path e_1w_1 (from right to left), for the integrand is an odd function of a.

Thus $$U + U_1 = \tfrac{1}{2}\int(\text{path } e_1w_1) + \tfrac{1}{2}\int(\text{path } we).$$

These two paths are equivalent to a complete circuit, in the negative direction, embracing the real axis. The singularities within this circuit are the zeros of the function

$$(pa^2a^2+1)\sin aa - aa\cos aa,$$

which, as will be shown immediately, are all real and simple, viz., $a = 0$, $a = \pm a_1$, etc.

Hence, replacing the integral by

$$-(2\pi i)\ (\text{sum of residues at poles within circuit}),$$

we obtain

$$U + U_1$$

$$= \frac{\rho}{(p+\frac{1}{3})a^3} + 2\Sigma e^{-\kappa a^2 t}\sin a\rho \,.\, \frac{\sin ar}{r} \,.\, \frac{(pa^2a^2+1)\cos aa + aa\sin aa}{\dfrac{d}{da}\left\{(pa^2a^2+1)\sin aa - aa\cos aa\right\}},$$

the summation extending over the positive roots of the equation

$$(pa^2a^2+1)\sin aa - aa\cos aa = 0. \qquad \text{(E)}$$

The solution of the original problem is $\int_0^b \rho f(\rho)(U+U_1)d\rho$, but the series for $U+U_1$ manifestly converges uniformly with respect to ρ, *provided t is greater than zero.* Hence the integration can be performed term by term, and we have

$$v = \frac{1}{(p+\frac{1}{3})a^3}\int_0^b \rho^2 f(\rho)d\rho$$

$$+ \frac{2}{a}\Sigma e^{-\kappa a^2 t}\frac{\sin ar}{r}\frac{p^2a^4a^4+(2p+1)a^2a^2+1}{p^2a^4a^4+(3p+1)a^2a^2}\int_0^b \rho f(\rho)\sin a\rho\, d\rho,$$

where we have simplified the general term by means of the equation (E).

4. The roots of (E) are all real and simple.

(1) Suppose, if possible, that $\mu = m + in$ is a root, then $\nu = m - in$ is also a root.

$$u = e^{-\kappa\mu^2 t}\frac{\sin\mu r}{r} = e^{-\kappa\mu^2 t}\,\mathrm{U},$$

and $$v = e^{-\kappa\nu^2 t}\frac{\sin\nu r}{r} = e^{-\kappa\nu^2 t}\,\mathrm{V}$$

are solutions of the equation of conduction, and by Green's Theorem

$$4\pi a^2\left(u\frac{dv}{dr} - v\frac{du}{dr}\right)_{r=a} = \frac{1}{\kappa}\int_0^a\left(u\frac{dv}{dt} - v\frac{du}{dt}\right)4\pi r^2 dr$$

or $$pa^3\left(u\frac{dv}{dt} - v\frac{du}{dt}\right)_{r=a} + \int_0^a\left(u\frac{dv}{dt} - v\frac{du}{dt}\right)r^2 dr = 0$$

or $$(\mu^2 - \nu^2)\left\{pa^3(\mathrm{UV})_{r=a} + \int_0^a r^2\mathrm{UV}\,dr\right\} = 0.$$

This is obviously impossible if μ^2, ν^2 are different, since p is positive, and U, V are conjugate complexes, so that UV is positive.

(2) Suppose, if possible, that $\alpha = \dfrac{i\nu}{a}$ is a pure imaginary root.

Then $(p\nu^2 - 1)\sinh\nu + \nu\cosh\nu = 0.$

The derivative with respect to ν of the function on the left is $$p\nu^2\cosh\nu + (2p+1)\nu\sinh\nu,$$ which is constantly positive if ν is real and positive. Hence the function, starting from zero for $\nu = 0$, can never again become zero.

(3) If $\alpha = \dfrac{\nu}{a}$ is a repeated root, then, as we have shown, ν is real. We must have, simultaneously,

$$(p\nu^2 + 1)\sin\nu = \nu\cos\nu$$
$$(2p + 1)\sin\nu = -p\nu\cos\nu$$

Therefore $(p\nu^2 + 1)(2p + 1)\sin^2\nu + p\nu^2\cos^2\nu = 0,$

which cannot be if p is positive, unless $\nu = 0$, which is obviously excluded. A more searching investigation will show that the roots are all real as long as $p > -\frac{1}{3}$, but that if $p < -\frac{1}{3}$, there are two pure imaginary roots.

https://doi.org/10.1017/S0013091500032612 Published online by Cambridge University Press

5. If b be put equal to a in the investigation of §3, it will be seen that one point requires examination. It was proved that $\underset{t=0}{\text{Limit}}\int u_1 d\alpha = 0$, if $\rho + r < 2a$. If ρ and r each be equal to a, this condition is violated, and the limit is in fact infinite. $\underset{t=0}{\text{Limit}}\int u d\alpha$ is infinite at the same time, but it is easy to prove that $\underset{t=0}{\text{Limit}}\int (u_1 + u) d\alpha$ is finite. and the investigation stands. We shall, however, verify in another way that the function

$$v(r,\, t) = \frac{1}{(p+\frac{1}{3})a^3}\int_0^a \rho^2 f(\rho) d\rho$$

$$+\frac{2}{a}\Sigma e^{-\kappa\alpha^2 t}\frac{\sin\alpha r}{r}\,\frac{p^2\alpha^4 a^4 + (2p+1)\alpha^2 a^2 + 1}{p^2\alpha^4 a^4 + (3p+1)\alpha^2 a^2}\int_0^a \rho f(\rho) \sin\alpha\rho\, d\rho$$

satisfies the condition $\underset{t=0}{\text{Limit}}\; v(a,\, t) = 0$.

For if $t > 0$, v obviously satisfies (i) and (ii).

Integrating (ii) over the surface, we have

$$\iint \frac{dv}{dr} dS + \frac{4\pi a^3 p}{\kappa}\left(\frac{dv}{dt}\right)_a = 0, \text{ or,}$$

by Green's Theorem and (i),

$$\int_0^a 4\pi r^2 . \frac{1}{\kappa}\,\frac{dv}{dt} dr + \frac{4\pi a^3 p}{\kappa}\,\frac{d}{dt} v_a = 0,$$

so that $$\frac{d}{dt}\left[\int_0^a r^2 v dr + p a^3 v_a\right] = 0$$

and $\int_0^a r^2 v dr + p a^3 v_a$ is constant for all positive values of t.

Hence the *limits* of $\int_0^a r^2 v dr + p a^3 v_a$ for $t = 0$ and for $t = \infty$ are the same. The limit for $t = \infty$ is

$$\frac{1}{(p+\frac{1}{3})a^3}\int_0^a \rho^2 f(\rho) d\rho \times \left(\int_0^a r^2 dr + p a^3\right)$$
$$= \int_0^a \rho^2 f(\rho) d\rho.$$

The limit for $t = 0$ is

$$\int_0^a \rho^2 f(\rho) d\rho + p a^3 \underset{t=0}{\text{Limit}}\; v_a.$$

Hence $$\underset{t=0}{\text{Limit}}\; v_a = 0.$$

Another point may be noted in connection with condition (iv), as it explains the form of Dr Peddie's eliminating factor.

If the initial temperature is

$$\frac{\sin ar}{r} - \frac{\sin aa}{a}, \ a \text{ being a root of (E)},$$

then the temperature at time t is

$$e^{-\kappa a^2 t}\frac{\sin ar}{r} - \frac{\sin aa}{a},$$

for this satisfies all the conditions (i) to (iv).

Comparing this with the general solution, we see that the coefficients of all the terms must vanish, except that corresponding to the root a.

Hence $$\int_0^a\left(\sin a\rho - \frac{\rho}{a}\sin aa\right)\sin\beta\rho\, d\rho = 0,$$

if β is a different root from a.

6. If the solution given in § 5 is to be verified *a posteriori*, the only condition causing any trouble is that of (iii).

Writing the series for $v(r, t)$ in § 5 in the contracted form

$$\sum_{n=0} e^{-\kappa a_n^2 t} v_n(r),$$

we have to prove that as t continuously approaches zero, the series continuously approaches $f(r)$, or more shortly that

$$\operatorname*{Limit}_{t=0} \Sigma e^{-\kappa a_n^2 t} v_n(r) = f(r) \quad - \quad - \quad - \quad \text{(A)}.$$

It is well enough known, but it seems desirable to recall, that this is not quite the same thing as to prove

$$\Sigma v_n(r) = f(r) \quad - \quad - \quad - \quad - \quad \text{(B)}$$

although it is the latter theorem which is usually attacked by writers on the subject, from Fourier onwards.

We may, it is true, deduce (A) from (B), with the help of a theorem analogous to the celebrated theorem of Abel's on the continuity of a power series, but even then we lose something; in fact not only is (B) more difficult to prove than (A), but it cannot, so far as I know, be proved at all for functions with an unlimited number of turning points, though, so far as (A) is concerned, these present no difficulty whatever.