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A Combinatorial Character Formula for Some Highest Weight Modules ⋆

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Abstract. We give a combinatorial formula for the weight multiplicities of some infinite-dimensional highest weight $\mathfrak{gl}(n)$-modules. Our proof, which does not rely on Kazhdan–Lusztig combinatorics, uses a reduction to finite characteristics. The character formula for the corresponding modular representations, which has been computed in a 1997 preprint by the authors, is based on a dual pair which has no obvious counterpart in characteristic zero.

Mathematics Subject Classification (1991): 17B15.

Key words: character formula, highest weight modules.

Introduction

Set $\mathfrak{g} = \mathfrak{gl}(n, K)$, where K is an algebraically closed field of characteristic zero. Although the characters of highest weight $\mathfrak{g}$-modules $L(\lambda)$ are determined by the Kazhdan–Lusztig polynomials, there are no general combinatorial formulas. For finite-dimensional $\mathfrak{g}$-modules, such formulas have been provided by the work of Littlewood and Richardson [LR], which is based on semi-standard tableaux (see also [L]). In this paper, we will show that the combinatorics of semi-standard tableaux applies as well for a certain class of infinite-dimensional highest weight representations. Indeed, the result is a corollary of some character formulas for modular representations of [MP] and is strongly connected with the combinatorics of Verlinde's formula for modular representations [GP]. Our proof uses a reduction to finite characteristics. However, we believe that there should be a natural purely characteristic zero proof, based on representation theory of the loop algebra.

Let $m \geqslant 0$. In the paper, we will determine a combinatorial formula for all highest weight modules $L(\lambda)$, where λ is m-cospecial. By definition, a non-zero weight λ is called m-cospecial if there are three integers i, s, j, with $i \leqslant s \leqslant j$, such that $\lambda = \sum_{i\leqslant l\leqslant j} a_l\omega_l$ (where ω_l is the lth fundamental weight), $a_l \geqslant 0$ for all $l \neq s$ and $j - i \leqslant m$, where $m = -\sum_{i\leqslant l\leqslant j} a_l$. Typical examples of cospecial weights are the negative multiple of the fundamental weights, i.e. the

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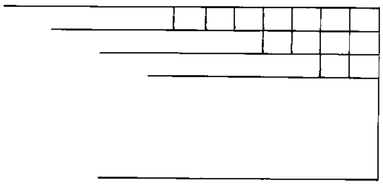
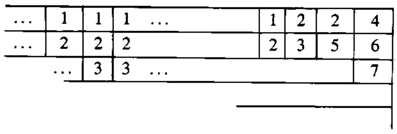
weight $-m\omega_s$ is m-cospecial. For simplicity, we will only explain the character formula for $L(-m\omega_s)$ in the introduction.

Let Y_∞ be the semi-infinite Young diagram $\mathbf{Z}_{\leqslant 0} \times \{1, \ldots, s\}$ of height s. We can draw this semi-infinite Young diagram as follows:



By convention, the first line is the top line, and the last line is the sth line which is at bottom. A semi-standard tableau of shape Y_∞ is a filling of the boxes of Y by indices running from 1 to n, with the usual convention that the indices are increasing from top to bottom, non decreasing from left to right and with the special requirement that on the ith line almost all labels are i. Here is an example of semi-standard tableau of shape Y_∞:



As the Young diagram Y_∞ is infinite, we cannot define the weight of the tableau T as usual. However, it is easy to renormalize the usual definition, which allows to define its relative weight $rw(T)$. Denote by $L_1 < L_2 \cdots < L_s$ the indices on the last column (i.e. the rightmost column) of the tableau T, and denote by $\mathcal{P}$ the set of all semi-standard tableaux such that $L_{s-m} \leqslant s$ (by convention, this condition is automatically satisfied if $s - m \leqslant 0$).

THEOREM *We have:* $\mathrm{ch}(L(-m\,\omega_s)) = \mathrm{e}^{-m\omega_s} \sum_{T \in \mathcal{P}} \mathrm{e}^{rw(T)}$.

For general m-cospecial weights, the combinatorics is slighty more complicated and it involves a pair consisting of a semi-infinite Young diagram Y_∞ and an ordinary Young diagram Y_f.

1. A Semi-continuity Principle

Roughly speaking, a semi-continuity principle states that a 'finite statement' which holds in characteristic $p \gg 0$ also holds in characteristic 0. For any algebraically closed field k, set $\mathfrak{g}_k = \mathfrak{gl}(n, k)$, let P be the lattice of integral weights and let $H(k)$ be the torus of $\mathrm{GL}_n(k)$, i.e. the subgroup of diagonal matrices. Denote by $\epsilon_1, \ldots, \epsilon_n$ the natural basis of P, by $\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n$ the simple roots and by $h_1, \ldots, h_{n-1} \in \mathrm{Hom}(P, \mathbf{Z})$ the corresponding simple coroots (with

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our definition, coroots are not elements in the Cartan subalgebra). Denote by Q^+ the monoid generated by the roots α_i and set $Q^- = -Q^+$.

For any $\lambda \in P$, denote by $L_k(\lambda)$ the simple module with highest weight λ. As we are only interested by its character, we will set $L_k(\lambda) = L(\lambda)$ if the characteristic of k is zero, and we will set $L_k(\lambda) = L_p(\lambda)$ if the characteristic of the field is $p \neq 0$. When k is a field of characteristic p, $L_k(\lambda)$ is a restricted $\mathfrak{g}_k - H(k)$ module, i.e. it is a restricted $\mathfrak{g}_k$-module with a compatible action of $H(k)$.

Let $\chi(p) = \sum_{\lambda \in P} m_\lambda(p)\,\mathrm{e}^\lambda$ be a sequence of characters indexed by all prime numbers p, and let $\chi = \sum_{\lambda \in P} m_\lambda\,\mathrm{e}^\lambda$ be a character. We say that the sequence $(\chi(p))$ converges to the character χ if for all $\lambda \in P$, we have $m_\lambda(p) = m_\lambda$ for p big enough, i.e. for $p > N(\lambda)$. In such case, we set $\lim_{p\to\infty} \chi(p) = \chi$.

LEMMA 1. *Let* $\lambda \in P$.

(i) *For any prime number* p*, we have* $\mathrm{ch}(L_p(\lambda)) \leqslant \mathrm{ch}(L(\lambda))$.
(ii) *We have* $\lim_{p\to\infty} \mathrm{ch}(L_p(\lambda)) = \mathrm{ch}(L(\lambda))$.

Proof. Let k be any algebraically closed field. Let $\mathfrak{g}_k = \mathfrak{n}_k^- \oplus \mathfrak{h}_k \oplus \mathfrak{n}_k^+$ be the triangular decomposition of $\mathfrak{g}_k$. Let U_k, $U_k^\pm$, A_k be the enveloping algebras of $\mathfrak{g}_k$, $\mathfrak{n}_k^\pm$, $\mathfrak{h}_k$. Let $T\colon U_k \to A_k$ be the Harish-Chandra projector, which is uniquely defined by $T(u) = u$ if $u \in A_k$, $T(u) = 0$ if $u \in \mathfrak{n}_k^-.U_k + U_k.\,\mathfrak{n}_k^+$. For any weight $\nu \in Q^-$, one defines the Shapovalov form $B_k^\nu\colon (U_k^+)_{-\nu} \times (U_k^-)_\nu \to k$ by $B_k^\nu(u^+, u^-) = \overline{\lambda}(T(u^+.u^-))$, where $\overline{\lambda}\colon A_k \to k$ is the algebra homomorphism extending λ. As we are only interested by the rank of the Shapovalov forms, we will set $B_k^\nu = B^\nu$ if k is a field of characteristic zero and we will set $B_p^\nu = B_k^\nu$ if k is a field of characteristic $p \neq 0$.

The Shapovalov form B_k^ν is naturally defined over $\mathbf{Z}$. Hence, we have $rk(B^\nu) \geqslant rk(B_p^\nu)$ for any prime number p and $rk(B^\nu) = rk(B_p^\nu)$ for p big enough (where rk denotes the rank). As the dimension of $L_k(\lambda)_{\lambda+\nu}$ is the rank of B_k^ν, the assertions (i) and (ii) are proved. □

Let $\lambda \in P$. Define a weight $\rho_\lambda \in P$ by the following requirements:

(i) $\rho_\lambda(h_i) = 0$ if $\lambda(h_i) \geqslant 0$,
(ii) $\rho_\lambda(h_i) = 1$ if $\lambda(h_i) < 0$.

For any prime number p, set $\lambda_p = \lambda + p\,\rho_\lambda$.

COROLLARY 2.

(i) *For* p *big enough,* λ_p *is dominant and restricted.*
(ii) *We have* $\mathrm{ch}(L(\lambda)) = \lim_{p\to\infty} \mathrm{e}^{-p\rho_\lambda}\,\mathrm{ch}(L_p(\lambda_p))$.

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Proof. For p big enough, we have $|\lambda(h_i)| < p$, for all $1 \leqslant i \leqslant n$. Hence, the first assertion follows. We have $\lambda = \lambda_p - p\,\rho_\lambda$. Hence, for any field k of characteristic p, the $\mathfrak{g}_k - H(k)$-module $L_k(\lambda)$ is isomorphic to $L_k(\lambda_p) \otimes k_{-p\,\rho_\lambda}$, where $k_{-p\,\rho_\lambda}$ is the one dimensional $H(k)$-module of weight $-p\,\rho_\lambda$ with a trivial $\mathfrak{g}_k$-action. So the formula follows from Lemma 1. □

Remark. Assume that the characteristic of k is $p \gg 0$. As λ_p is restricted and dominant, Steinberg proved that $L_k(\lambda_p)$ is indeed the restriction to $\mathfrak{g}_k$ of the simple $\mathrm{GL}(n,k)$-module with highest weight λ_p [St]. Hence, the character of any module of the category $\mathcal{O}$ is the limit of characters of modular representations. This method of computing the characters of modules in the category $\mathcal{O}$ is usually not practical, because the character of a general modular representation is usually more complicated than its counterpart in the category $\mathcal{O}$ (see [So]). However, we will implicitly use a dual pair which holds in finite characteristics and which has no obvious counterpart in characteristic zero (indeed, the dual pair occurs in the proof of Theorem 3, see [MP]).

2. Some Results about Modular Theory

Let k be an algebraically closed field of characteristic p. Let Y be a Young diagram. Denote by M the number of columns of Y, and by $c_i(Y)$ the number of boxes on the ith column. For any subdiagram Y' in Y, set $c_{\mathrm{first}}(Y') = c_1(Y')$ and $c_{\mathrm{last}}(Y') = c_M(Y')$. As usual, the dominant weight λ associated to Y is the weight $\lambda = \sum_{l \leqslant M} \omega_{c_i(Y)}$. The weight λ is called M-special if and only if $c_{\mathrm{first}}(Y) - c_{\mathrm{last}}(Y) \leqslant p - M$ and $M < p$. Denote by $\mathcal{P}(\lambda)$ the set of all semi-standard tableaux T of shape Y such that $c_{\mathrm{first}}(T[l]) - c_{\mathrm{last}}(T[l]) \leqslant p - M$ for all $l \leqslant n$. For any semi-standard tableau T of shape Y, denote by $w(T)$ its weight and set $rw(T) = w(T) - \lambda$.

THEOREM 3. *Let λ be a M-special weight as before. We have*

$$\mathrm{ch}(L_p(\lambda)) = \mathrm{e}^{\lambda} \sum_{T \in \mathcal{P}(\lambda)} \mathrm{e}^{rw(T)}.$$

THEOREM 4. *Let λ be a M-special. Then $L_p(\lambda)|_{\mathrm{GL}(n-1,k)}$ is semi-simple.*

Theorem 3 is proved in [MP] (Theorem 4.3). Theorem 4 has been proved independently by Brundan, Kleshchev, Suprunenko [BKS] and the authors [MP] by very different methods. Soon after that, Brundan, Kleshchev and Suprunenko used their methods to give a new proof of Theorem 3.

3. The Character Formula for Some Highest Weight Modules

Let K be an algebraically closed field of characteristic zero. For $a \in \mathbf{Z}$, we denote by $]-\infty, a]$ the set of all integers $n \leqslant a$. Such a set is called a semi-infinite interval

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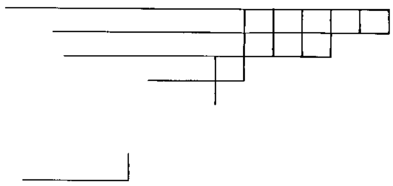
of **Z**. A semi-infinite Young diagram of heigth s is a collection of s semi-infinite intervals I_i such that $I_1 \supset I_2 \supset \cdots \supset I_s$. We draw a semi-infinite Young diagram as follows



A semi-standard tableau of shape Y is a filling of the boxes of Y by indices running from 1 to n, which is increasing from top to bottom, non decreasing from left to right and with the stabilization requirement that almost all boxes on the ith line are filled with the index i, as on the introduction.

Because the semi-standard tableau T is infinite, we cannot define its weight as usual. However for any semi-infinite Young tableau there is an highest semi-standard tableau Y_h of shape Y for which the ith line is filled only with the index i. Then the difference of the weights of T and T_h is well defined, and will be called the relative weight of T (and it will be denoted by $rw(T)$). More precisely, the relative weight $rw(T)$ of T is defined as follows. For any box b on the ith line of T, we set $\alpha(b) = \epsilon_j - \epsilon_i$ if b is filled with the index j. For almost all b, we have $\alpha(b) = 0$, and $\alpha(b)$ is a negative root otherwise. We set $rw(T) = \sum_b \alpha(b)$.

Let $\omega_i = \epsilon_1 + \cdots + \epsilon_i$ be the ith fundamental weight of $\mathrm{GL}(n)$. Let $m \geqslant 0$. A *m-cospecial weight with support* $[i, j]$ *and singular node* $s \in [i, j]$ is a weight λ such that $\lambda = \sum_{i \leqslant l \leqslant j} a_l \omega_l$, and we assume:

(i) $a_l \geqslant 0$ for any $l \neq s$,
(ii) $\sum_{i \leqslant l \leqslant j} a_l = -m$ and $j - i \leqslant m$,
(iii) $a_s < 0$, or equivalently $\lambda \neq 0$.

For a m-cospecial weight λ as before, we associate a pair $(Y_\infty(\lambda), Y_f(\lambda))$ consisting of a semi-infinite Young diagram $Y_\infty(\lambda)$ and a ordinary Young diagram $Y_f(\lambda)$ defined as follows

(i) $Y_\infty(\lambda)$ is the semi-infinite Young diagram of height s defined by the semi-infinite intervals $I_1 \supset I_2 \supset \cdots I_s$ such that $I_l =]-\infty, \sum_{\max(l,i) \leqslant k \leqslant s} a_k]$,
(ii) $Y_f(\lambda)$ is the Young diagram associated with the dominant weight $\sum_{l > s} a_l \omega_l$.

Let T_∞, T_f be semi-standard standard tableaux of shape respectively $Y_\infty(\lambda)$ and $Y_f(\lambda)$. Let L be the last column of T_∞ and let F be the first column of T_f. For $l \leqslant n$, denote by $T_f[l]$ (respectively $T_\infty[l]$) the Young subdiagram of $Y_f(\lambda)$ (respectively of the semi-infinite Young diagram of $Y_\infty(\lambda)$) consisting of all boxes with label $\leqslant l$. We set $c_{\text{first}}(T_f[l]) = \mathrm{card}(T_f[l] \cap F)$, $c_{\text{last}}(T_\infty[l]) = \mathrm{card}(T_\infty[l] \cap L)$. However, when $s = j$, the Young diagram $Y_f(\lambda)$ is empty, and the definition of $c_{\text{first}}(T_f[l])$ is slighty different. In such a case, we set $c_{\text{first}}(T_f[l]) = \min(l, s)$.

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Let $\mathcal{T}$ the set of all pair (T_∞, T_f) of semi-standard tableaux of shape respectively $Y_\infty(\lambda)$ and $Y_f(\lambda)$ such that:

(i) for any $l \leqslant s$, all boxes of lth line of $Y_f(\lambda)$ are filled with the index l,
(ii) for any $l \leqslant n$, $c_{\text{first}}(T_f[l]) - c_{\text{last}}(T_\infty[l]) \leqslant m$.

THEOREM 5. *Let λ be a m-cospecial weight. We have*

$$\mathrm{ch}(L(\lambda)) = \mathrm{e}^\lambda \sum_{(T_\infty, T_f)\in\mathcal{T}} \mathrm{e}^{rw(T_\infty)+rw(T_f)}.$$

Proof. Let p be a prime number, and set $M = p - m$. For p big enough, the weight $\lambda_p = \lambda + p\omega_s$ is restricted (Corollary 2), and by definition λ_p is M-special. Let Y be the Young diagram of λ_p, and set $a = \sum_{l>s} a_l$, $b = p - m - a$. The first a columns of Y is a Young subdiagram Y_1 which is identical to Y_f. The remaining b columns of Y are identical to the Young diagram Y_2 consisting of the last b columns of the semi-infinite diagram Y_∞.

Fix a weight $\nu = -\sum_{1\leqslant i\leqslant n-1} m_i\,\alpha_i \in Q^-$, and set $N = \sum_{1\leqslant i\leqslant n-1} m_i$. In what follows, we will assume that p is big enough, i.e. $p > N - a_s$. All lines of Y_2 contains more than $N+1$ boxes. Denote by $\mathcal{P}_p^\nu$ (respectively $\mathcal{T}^\nu$) the set of all semi-standard tableaux $T \in \mathcal{P}(\lambda_p)$ (respectively the pairs $(T_\infty, T_f) \in \mathcal{T}$) such that $rw(T) = \nu$ (respectively $rw(T_\infty) + rw(T_f) = \nu$).

Let $T \in \mathcal{P}_p^\nu$. For any box $b \in T$ such that $\alpha(b) \neq 0$, we also have $\alpha(b') \neq 0$ for all b' in the hook of b. Thus, if the hook lenght of b is $\geqslant N+1$, we have $\alpha(b) = 0$. In particular for all boxes of the first s lines of Y_1, we have $\alpha(b) = 0$. So, the restriction of T to Y_1 determines a semi-standard tableau $T_f(T)$ of shape Y_1 and the lth line of $T_f(T)$ is only filled with the index l, for any $l \leqslant s$. Similarly, the restriction of T to Y_2 determines a semi-standard tableau of shape Y_2, and we can extend it to get a semi-standard tableau $T_\infty(T)$ of shape Y_∞ by requiring that any box on the lth line of $Y_\infty \setminus Y_2$ is filled by the index l. It is clear that the map $T \mapsto (T_f(T), T_\infty(T))$ is a bijection from $\mathcal{P}_p^\nu$ to $\mathcal{T}^\nu$. Hence the Theorem 5 follows from Corollary 2 and Theorem 3. □

Remark. Assume now that $s = j$. In such a case, $Y_f(\lambda) = \emptyset$, and an element in $\mathcal{T}$ consists essentialy in one tableau T_∞, because $T_f = \emptyset$. Let T_∞ be a semi-standard tableau of shape $Y_\infty(\lambda)$. Denote by $L_1 < \cdots < L_i$ be the indices of the boxes on the last column L of Y_∞. We have $c_{\text{first}}(T_f[l]) = \min(l, s)$ and $c_{\text{last}}(T_\infty[l]) \leqslant c_{\text{last}}(T_\infty[l+1]) \leqslant c_{\text{last}}(T_\infty[l]) + 1$. Hence, the function $l \mapsto c_{\text{first}}(T_f[l]) - c_{\text{last}}(T_\infty[l])$ takes its maximal value at s. The condition $s - c_{\text{last}}(T_\infty[s]) \leqslant m$ is equivalent to $L_{s-m} \leqslant s$ (and is automatically satisfied if $s \leqslant m$). Hence T_∞ belongs to $\mathcal{T}$ if and only if $L_{s-m} \leqslant s$ or $s \leqslant m$. This proves the theorem stated in the introduction.

The combinatorics presented here is strongly connected with the Verlinde's formula for $\mathrm{GL}(n, \overline{\mathbf{F}}_p)$, see [GM] and [MP] for the details. Using Theorem 4 and another version of the semi-continuity principle, we can also deduce that the restriction of $L(\lambda)$ to $\mathfrak{gl}(n-1, K)$ is semi-simple. However, this can be deduced directly. Denote by Λ the set of all weights μ such that there exists a pair $(T_f, T_\infty) \in \mathcal{T}$ such that $\mu = \lambda + \sum_{b \in Y_f \setminus T_f[n-1]} \alpha(b) + \sum_{b \in Y_\infty \setminus T_\infty[n-1]} \alpha(b)$. For $\mu \in \Lambda$, denote by $l(\mu)$ the $\mathfrak{gl}(n-1, k)$-module with highest weight μ.

COROLLARY 6. *As a* $\mathfrak{gl}(n-1, K)$*-module, we have:* $L(\lambda) = \oplus_{\mu \in \Lambda} l(\mu)$.

Proof. The proof is identical to the proof in [MP], so it will be only sketched. Using Theorem 5, one proves that $L(\lambda)$ and $\oplus_{\mu \in \Lambda} L(\mu)$ have the same character. It follows that the multiplicity of any simple module occuring in a Jordan–Holder series of the $\mathfrak{gl}(n-1, K)$-module $L(\lambda)$ is one. It follows that $L(\lambda)$ is semi-simple. □

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