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IDEMPOTENT-SEPARATING EXTENSIONS OF REGULAR SEMIGROUPS WITH ABELIAN KERNEL

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Abstract

Let S be a regular semigroup and $D(S)$ its associated category as defined in Loganathan (1981). We introduce in this paper the notion of an *extension* of a $D(S)$-module A by S and show that the set $\mathrm{Ext}(S, A)$ of equivalence classes of extensions of A by S forms an abelian group under a Baer sum. We also study the functorial properties of $\mathrm{Ext}(S, A)$.

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1. Introduction

Let S be a regular semigroup and $D(S)$ its associated category (see Section 2 for the definition of $D(S)$). By a $D(S)$-module we mean a functor $A: D(S) \to Ab$ from $D(S)$ to the category of abelian groups and by a $D(S)$-homomorphism a natural transformation between such functors.

Let $\pi: T \to S$ be an idempotent-separating homomorphism from a regular semigroup T onto S. If the kernel of π is abelian then the kernel can be viewed as a $D(S)$-module $\mathrm{Ker}\,\pi: D(S) \to Ab$. This suggests the following definition: An *extension* of a $D(S)$-module A by S is a triple (T, π, i) consisting of a regular semigroup T, an idempotent-separating surjective homomorphism of regular semigroups $\pi: T \to S$ with abelian kernel and an isomorphism $i: A \to \mathrm{Ker}\,\pi$ of $D(S)$-modules. Let $\mathrm{Ext}(S, A)$ denote the set of equivalence classes of extensions of A by S. We define an addition in $\mathrm{Ext}(S, A)$ and show that it makes $\mathrm{Ext}(S, A)$ an abelian group. We also study the functorial properties of $\mathrm{Ext}(S, A)$.

The cohomological interpretation of Ext(S, A) will be given in another paper. Here we content ourselves with the following remarks. (For details see Loganathan (1981).) When G is a group the category $D(G)$ reduces to the group G itself and in this case our results are well known and classical. When S is an inverse semigroup then the concept of the $D(S)$-module is equivalent to that of the S-module as defined by Lausch (1975). Consequently, in the case of semi-lattices of groups, our results turn out to be equivalent to those obtained by Sribala (1977). For regular semigroups the category $D(S)$ is equivalent to the category $\mathfrak{D}(S)$ introduced by Leech (1975). In particular, the type of extensions considered in this paper is a special case of the $\mathcal{H}$-coextensions studied by Leech.

2. Preliminaries

Let S be a regular semigroup and $E(S)$ its set of idempotents. For $x \in S$, we denote by

$$V(x) = \{x' \in S | xx'x = x \text{ and } x'xx' = x'\}$$

the set of inverses of x in S. If $x' \in V(x)$ then (x, x') will be called a *regular pair* in S. For $e, f \in E(S)$, we denote by

$$S(e, f) = \{h \in E(S) | he = h = fh \text{ and } ehf = ef\}$$

the *sandwich set* of e and f (Nambooripad (1979)).

If ρ is a congruence on S then the *kernel* of ρ is the set of ρ-classes which contain idempotents of S. By the kernel of a homomorphism we mean the kernel of its associated congruence. If ρ is idempotent-separating then, since $\rho \subseteq \mathcal{H}$, for each $e \in E(S)$ the ρ-class N_e containing the idempotent e is a subgroup of S. The kernel of an idempotent-separating congruence ρ is called *abelian* if the groups N_e are abelian.

A set $N = \{N_e | e \in E(S)\}$, of subgroups of S, is called a *group kernel normal system* if the following hold

(i) $e \in N_e$ for each $e \in E(S)$;

(ii) $af = fa$ for each $a \in N_e$ and $f \leqslant e, f \in E(S)$;

(iii) $x'N_e x \subseteq N_{x'x}$ for each regular pair (x, x') of S with $xx' \leqslant e$.

Let $N = \{N_e | e \in E(S)\}$ be a group kernel normal system of S. We denote by ρ_N the relation on S defined by

$$\{(x, y) \in S \times S | \text{for some } x' \in V(x) \text{ and } y' \in V(y),$$
$$xx' = yy', x'x = y'y \text{ and } y'x \in N_{x'x}\}.$$

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By Lallement (1967), Theorem 3.11, ρ_N is an idempotent-separating congruence on S with N as its kernel.

Next we recall the definition of the category $D(S)$ introduced in Loganathan (1981).

Let S be a regular semigroup. Let $C(S)$ denote the category whose objects are the idempotents of S, where a morphism from the idempotent e to the idempotent f is a triple (e, x, x') with (x, x') a regular pair in S satisfying $xx' \leq e$, $x'x = f$. Composition is defined by $(e, x, x')(x'x, y, y') = (e, xy, y'x')$. Composition is clearly associative and (e, e, e) is the identity morphism at the object e.

On $C(S)$ define a relation ρ as follows. If (e, x, x'), (e, y, y'): $e \to f$ are morphisms from e to f then

$$(e, x, x')\rho(e, y, y') \text{ if and only if } x = y \text{ or } x' = y'.$$

The relation ρ is clearly reflexive and symmetric and compatible with composition. Hence ρ^t, the transitive closure of ρ, is a congruence on $C(S)$. We write $D(S)$ for the quotient category $C(S)/\rho^t$. If (e, x, x') is a morphism of $C(S)$, we shall write $[e, x, x']$ for the image of (e, x, x') in $D(S)$. Let θ: $S' \to S$ be a homomorphism of regular semigroups. Then the maps $e \mapsto e\theta$, $[e, x, x'] \mapsto [e\theta, x\theta, x'\theta]$, define a functor $D(\theta)$: $D(S') \to D(S)$.

DEFINITION 2.1. A map $x \mapsto x^*$: $S \to S$ is called an *inverse map* if (i) $x^* \in V(x)$ for each $x \in S$; (ii) $x^* \in H_e$ if $x \in H_e$. In particular, if $e \in E(S)$ then $e^* = e$.

Let S be a regular semigroup with an inverse map $x \mapsto x^*$. For $x, y \in S$, we denote by $K_{x,y}$ the morphism

$$[y^*y, y^*y(xy)^*xy, (xy)^*xy]: y^*y \to (xy)^*xy$$

and by $J_{x,y}$ the morphism

$$[x^*x, x^*xy, (xy)^*xh]: x^*x \to (xy)^*xy,$$

where h is any element of $S(x^*x, yy^*)$. Clearly $J_{x,y}$ does not depend on the choice of h and hence it is well-defined.

LEMMA 2.2. *For* $x, y, z \in S$, *we have*

(i) $J_{x,y}J_{xy,z} = J_{x,yz}$;

(ii) $K_{y,z}K_{x,yz} = K_{xy,z}$;

(iii) $K_{x,y}J_{xy,z} = J_{y,z}K_{x,yz}$.

PROOF. (i) and (ii) are easy to verify and (iii) also follows by observing that

$$\begin{aligned} K_{x,y}J_{xy,z} &= [y^*y, y^*y(xy)^*xyz, (xyz)^*xyh] \\ &= [y^*y, y^*yz(xyz)^*xyz, (xyz)^*xyh] \\ &= [y^*y, y^*yz(xyz)^*xyz, (xyz)^*xyk] \\ &= J_{y,z}K_{x,yz}, \end{aligned}$$

where h, k are any elements of $S((xy)^*xy, zz^*)$ and $S(y^*y, zz^*)$ respectively.

3. The group $\mathrm{Ext}(S, A)$

Let T, S be regular semigroups and let π: $T \to S$ be an idempotent-separating homomorphism from T onto S. For $e \in E(S)$, we denote by $(\mathrm{Ker}\ \pi)_e$ the group $\{t \in T \mid t\pi = e\}$. For each morphism $[h, t, t']$: $h \mapsto k$ of $D(T)$ we define

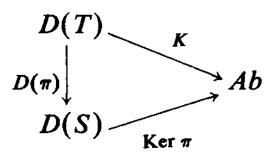
$$K([h, t, t']): (\mathrm{Ker}\ \pi)_{h\pi} \to (\mathrm{Ker}\ \pi)_{k\pi}$$

by $aK([h, t, t']) = t'at$, $a \in (\mathrm{Ker}\ \pi)_{h\pi}$. It is easy to see that $K([h, t, t'])$ is well-defined and that it is a homomorphism of groups. Therefore the maps

$$h \mapsto (\mathrm{Ker}\ \pi)_{h\pi}, \quad [h, t, t'] \mapsto K([h, t, t'])$$

define a functor K from $D(T)$ to the category of groups.

Suppose now that the kernel of π is abelian, that is, the groups $(\mathrm{Ker}\ \pi)_e$, $e \in E(S)$, are abelian. Then K: $D(T) \to Ab$ induces a $D(S)$-module $\mathrm{Ker}\ \pi$: $D(S) \to Ab$ such that the diagram



is commutative. By assigning to each regular pair (x, x') in S a regular pair $((x, x')j_1, (x, x')j_2)$ in T such that $((x, x')j_1\pi, (x, x')j_2\pi) = (x, x')$, we see that $\mathrm{Ker}\ \pi$ can be described, more explicitly, as the $D(S)$-module which associates to each object e of $D(S)$ the group $(\mathrm{Ker}\ \pi)_e$ and to each morphism $[e, x, x']$: $e \to f$ the homomorphism $\mathrm{Ker}\ \pi([e, x, x'])$: $(\mathrm{Ker}\ \pi)_e \to (\mathrm{Ker}\ \pi)_f$, given by

$$(a)\ \mathrm{Ker}\ \pi([e, x, x']) = (x, x')j_2 a (x, x')j_1, \qquad a \in (\mathrm{Ker}\ \pi)_e.$$

Motivated by the above observation we introduce the following

DEFINITION 3.1. Let S be a regular semigroup and A a $D(S)$-module. An *extension of A by S* is a triple $E = (T, \pi, i)$ consisting of a regular semigroup T, an idempotent-separating surjective homomorphism π: $T \to S$ such that the kernel of π is abelian, and an isomorphism of $D(S)$-modules i: $A \to \mathrm{Ker}\ \pi$.

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Consider the commutative diagram

$$\begin{array}{ccc} T' & \overset{\pi'}{\to} & S' \\ \downarrow\mu & & \downarrow\theta \\ T & \overset{\pi}{\to} & S \end{array}$$

of homomorphisms of regular semigroups such that π' and π are idempotent-separating surjective homomorphisms with abelian kernel. Let $D(\theta)$ (Ker π) be the composite

$$D(S') \overset{D(\theta)}{\to} D(S) \overset{\text{Ker}\,\pi}{\to} Ab.$$

Then μ induces a homomorphism ϕ_μ: Ker $\pi' \to D(\theta)$ (Ker π) of $D(S')$-modules such that

$$\phi_\mu|_{(\text{Ker}\,\pi')_e} = \mu|_{(\text{Ker}\,\pi')_e}: (\text{Ker}\,\pi')_e \to D(\theta)(\text{Ker}\,\pi)_e = (\text{Ker}\,\pi)_{e\theta}.$$

Let $E' = (T', \pi', i')$ be an extension of A' by S' and $E = (T, \pi, i)$ an extension of A by S. A *morphism* Γ: $E' \to E$ *of extensions* is a triple $\Gamma = (\phi, \mu, \theta)$ consisting of homomorphisms μ: $T' \to T$, θ: $S' \to S$ of regular semigroups and a homomorphism ϕ: $A' \to D(\theta)$ A of $D(S')$-modules, such that: $\mu\pi = \pi'\theta$ and $\phi D(\theta)i = i'\phi_\mu$, where

$$D(\theta)i: D(\theta)A \to D(\theta)(\text{Ker}\,\pi)$$

is the homomorphism of $D(S')$-modules induced by i: $A \to$ Ker π. We say two extensions $E' = (T', \pi', i')$, $E = (T, \pi, i)$ of A by S are *equivalent* if there exists a homomorphism (necessarily an isomorphism) μ: $T' \to T$ such that $(\text{Id}_A, \mu, \text{Id}_S)$: $E' \to E$ is a morphism of extensions. We call μ an *equivalence of extensions*. We denote by Ext(S, A) the equivalence classes of extensions of A by S and by $[E]$ the equivalence class containing the extension $E = (T, \pi, i)$.

DEFINITION 3.2. Let S be a regular semigroup and A a $D(S)$-module. The *semi-direct product* of S and A with respect to an inverse map $x \mapsto x^*$: $S \to S$ is the regular semigroup

$$S \times A = \{(x, a) | x \in S, a \in A_{x^*x}\}$$

with the multiplication given by

$$(x, a)(y, b) = (xy, aA(J_{x,y}) + bA(K_{x,y})).$$

Associativity of the multiplication follows from Lemma 2.2. The set $E(S \times A)$ of idempotents of $S \times A$ is $\{(e, 0_e) \in S \times A | e \in E(S)\}$ and if $(x, a) \in S \times A$ then

$$V((x, a) = \{(y, (-a)A([x^*x, x^*xy, y^*yx])) \in S \times A | y \in V(x)\}.$$

Suppose now that $S \times A$ is the semi-direct product of S and A with respect to an inverse map $x \mapsto x^*$. Then the projection π_0: $S \times A \to S$, defined by $(x, a)\pi_0 = x$, and the isomorphism i_0: $A \to \operatorname{Ker} \pi_0$, given by $(a)i_0 = (e, a)$, $a \in A_e$, $e \in E(S) = \operatorname{Ob}(D(S))$, of $D(S)$-modules define an extension, denoted $E_0 = (S \times A, \pi_0, i_0)$, of A by S. Also, there is a homomorphism ν_0: $S \to S \times A$, given by $(x)\nu_0 = (x, 0_{x^*x})$, which satisfies $\nu_0\pi_0 = \mathrm{Id}_S$.

We call an extension $E = (T, \pi, i)$ of A by S *split* if there exists a homomorphism ν: $S \to T$ such that $\nu\pi = \mathrm{Id}_S$; the homomorphism ν is then called a *splitting*. For example E_0 is a split extension with a splitting ν_0.

If $E = (T, \pi, i)$ is a split extension of A by S with a splitting ν then the map μ: $S \times A \to T$, given by $(x, a)\mu = (x\nu)(ai)$, is a homomorphism of regular semigroups and it is an equivalence of extensions. Conversely, if an extension E is equivalent to E_0 then, obviously, it is a split extension. In other words, an extension E of A by S is split if and only if it is equivalent to E_0. In particular, the equivalence class determined by E_0 does not depend on the particular choice of the inverse map $x \mapsto x^*$ used to define the multiplication in $S \times A$.

Let $E_r = (T_r, \pi_r, i_r)$, $r = 1, 2$, be extensions of A by S. Consider the regular subsemigroup

$$T = \{(t_1, t_2) \in T_1 \times T_2 | t_1\pi_1 = t_2\pi_2\}$$

of $T_1 \times T_2$. Since π_1 and π_2 are idempotent-separating homomorphisms, the set $N = \{N_e | e \in E(S)\}$, where $N_e = \{((a)i_1, (-a)i_2) | a \in A_e\}$, is a group kernel normal system of T. Write $T_1 + T_2 = T/\rho_N$ and denote the element of $T_1 + T_2$ containing (t_1, t_2) by $\overline{(t_1, t_2)}$. It is easy to see that the map π: $T_1 + T_2 \to S$, given by $\overline{(t_1, t_2)}\pi = t_1\pi_1 (= t_2\pi_2)$, is an idempotent-separating homomorphism from $T_1 + T_2$ onto S. Also, it is easy to see that the map i: $A \to \operatorname{Ker} \pi$, defined by $(a)i = \overline{((a)i_1, (0)i_2)}$, $a \in A_e$, $e \in E(S)$, is an isomorphism of $D(S)$-modules. Thus we obtain an extension, denoted $E_1 + E_2 = (T_1 + T_2, \pi, i)$, of A by S. We call $E_1 + E_2$ the *Baer sum* of the extensions E_1 and E_2.

THEOREM 3.3. $\operatorname{Ext}(S, A)$ *is an abelian group under the operation* $[E_1] + [E_2] = [E_1 + E_2]$. *The zero element in the abelian group* $\operatorname{Ext}(S, A)$ *is the equivalence class* $[E_0]$ *determined by the split extensions.*

PROOF. *Commutativity of* +. Let $E_r = (T_r, \pi_r, i_r)$, $r = 1, 2$, be extensions of A by S. Then the map μ: $T_1 + T_2 \to T_2 + T_1$, given by $\overline{(t_1, t_2)}\mu = \overline{(t_2, t_1)}$, is easily seen to be an equivalence of extensions so that $[E_1] + [E_2] = [E_1 + E_2] = [E_2] + [E_1]$.

Associativity of +. Let $E_r = (T_r, \pi_r, i_r)$, $r = 1, 2, 3$, be extensions of A by S. We have to show that $(E_1 + E_2) + E_3$ is equivalent to $E_1 + (E_2 + E_3)$. This follows by noting that the map μ: $(T_1 + T_2) + T_3 \to T_1 + (T_2 + T_3)$, defined by

$$\overline{(\overline{(t_1, t_2)}, t_3)}\,\mu = \overline{(t_1, \overline{(t_2, t_3)})},$$

is an equivalence of extensions.

Identity for +. Let $E = (T, \pi, i)$ be an extension of A by S. Then the map μ: $T \to T + (S \times A)$, given by $t\mu = \overline{(t, (t\pi, 0_{(t\pi)^* t\pi}))}$, is an equivalence of extensions between E and $E + E_0$. Hence $[E] + [E_0] = [E + E_0] = [E] = [E_0] + [E]$.

Inverse (relative to +). Let $E = (T, \pi, i)$ be an extension of A by S. Denote by $-E = (T, \pi, -i)$ the extension obtained from E by defining $a(-i) = (-a)i$, $a \in A$. We claim that $[-E]$ is the inverse of $[E]$. It clearly suffices to show that the extension $E + (-E)$ is split. Choose a map j: $S \to T$ such that $j\pi = \mathrm{Id}_S$. Define ν: $S \to T + T$ by $(x)\nu = \overline{(xj, xj)}$. It follows from the definition of $E + (-E)$ that the map ν is a homomorphism. Obviously $\nu\pi = \mathrm{Id}_S$. Hence $E + (-E)$ is a split extension. This completes the proof of the theorem

Next we will study the functorial properties of $\mathrm{Ext}(S, A)$.

PROPOSITION 3.4. *Let $E = (T, \pi, i)$ be an extension of A by S and let ϕ: $A \to B$ be a homomorphism of $D(S)$-modules. Then there is an extension $\phi E = (U, \pi', i')$ of B by S and a homomorphism μ: $T \to U$ such that $(\phi, \mu, \mathrm{Id}_S)$: $E \to \phi E$ is a morphism of extensions. The pair $(\phi E, \mu)$ is unique upto an equivalence.*

PROOF. Let $T \times B'$ be the semi-direct product of T and B' with respect to an inverse map $t \mapsto t^*$: $T \to T$, where B' denote the composite $D(T) \overset{D(\pi)}{\to} D(S) \overset{B}{\to} Ab$. For $(e, 0_e) \in E(T \times B')$, let

$$N_{(e, 0_e)} = \{((a)i, (-a)\phi) \in T \times B' | a \in A_{e\pi}\}.$$

In view of the bijection $(e, 0_e) \leftrightarrow e$ between $E(T \times B')$ and $E(T)$, we denote the group $N_{(e,0_e)}$ simply by N_e. We shall prove that $N = \{N_e | e \in E(T)\}$ is a group kernel normal system of $T \times B'$.

(i) Clearly $(e, 0_e) \in N_e$, for each idempotent $(e, 0_e)$ of $T \times B'$.

(ii) Suppose that $((a)i, (-a)\phi) \in N_e$ and let $(f, 0_f) \leq (e, 0_e)$. Then $f \leq e$ in T so that $(ai)f = f(ai)$ and, by Definition 2.1(ii), $((ai)f)^* = ((-a)i)f$. It follows that

$$((a)i, (-a)\phi)(f, 0_f) = (f, 0_f)((a)i, (-a)\phi).$$

(iii) Suppose that $((a)i, (-a)\phi) \in N_e$ and let

$$((t, b), (u, (-b)B'([t^*t, t^*tu, u^*ut]))$$

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be a regular pair in $T \times B'$ such that

$$(t, b)\big(u, (-b)B'([t^*t, t^*tu, u^*ut])\big) = (tu, 0_{tu}) \leqslant (e, 0_e).$$

Since $(u(a)i)^*u(a)i\mathcal{L}u^*u\mathcal{L}tu,\ tu \in S((u(a)i)^*u(a)i, tt^*)$. Therefore,

$$\begin{aligned}\big(u, (-b)B'([t^*t, t^*tu, u^*ut])\big)&((a)i, (-a)\phi)(t, b)\\ &= \big(u(ai)t, (-b)B'([t^*t, t^*(ai)t, u(u(ai))^*ut])\\ &\qquad + (-a)\phi B'([e, t, u]) + bB'([t^*t, t^*t, ut])\big)\\ &= \big(u(ai)t, (-a)\phi B'([e, t, u])\big),\end{aligned}$$

since

$$B'([t^*t, t^*(ai)t, u(u(ai))^*ut]) = B'([t^*t, t^*t, ut]).$$

Now, by putting $a' = aA([e\pi, t\pi, u\pi])$, we see that

$$\big(u(ai)t, (-a)\phi B'([e, t, u])\big) = ((a')i, (-a')\phi) \in N_{ut}.$$

Hence $N = \{N_e \mid e \in E(T)\}$ is a group kernel normal system of $T \times B'$.

Write $U = T \times B'/\rho_N$. The composite $T \times B' \xrightarrow{\pi_0} T \xrightarrow{\pi} S$ induces a surjective homomorphism $\pi'\colon U \to S$, which is idempotent-separating. If we define $i'\colon B \to \operatorname{Ker} \pi'$ by $(b)i' = (e, b)\rho_N$, $b \in B'_e$, then, clearly, i' is a monomorphism. It is also an epimorphism. For, if $((t, b)\rho_N)\pi' = ((e, 0_e)\rho_N)\pi'$, for some $(e, 0_e)\rho_N \in E(U)$, then $t\pi = e\pi$ so that we can find an element $a \in A_{e\pi}$ such that $(a)i = t$. Hence $(t, b)\rho_N = (e, (a)\phi - b)\rho_N = ((a)\phi - b)i'$. Therefore i' is an isomorphism of $D(S)$-modules. Consequently, (U, π', i') is an extension of B by S which we denote by ϕE. Finally, we define $\mu\colon T \to U$ by $(t)\mu = (t, 0_{t^*t})\rho_N$. It is easily seen that $(\phi, \mu, \mathrm{Id}_S)\colon E \to \phi E$ is a morphism of extensions.

Suppose that $((V, \pi'', i''), \mu')$ is any other such pair. Then the map $\mu''\colon T \times B' \to V$, defined by $(t, a)\mu'' = (t)\mu'(a)i''$, is easily seen to be a homomorphism of regular semigroups. If $((a)i, (-a)\phi) \in N_e$ then $((a)i, (-a)\phi)\mu'' = (ai)\mu'((-a)\phi)i'' = (a\phi)i''((-a)\phi)i'' = 0$. Therefore μ'' induces a homomorphism $\bar{\mu}\colon U \to V$. Clearly $\bar{\mu}$ is an equivalence of extensions and satisfies the equation $\mu\bar{\mu} = \mu'$. This completes the proof of the proposition.

Proposition 3.5. *Let $E = (T, \pi, i)$ be an extension of A by S and let $\theta\colon S' \to S$ be a homomorphism of regular semigroups. Let $D(\theta)A$ denote the composite $D(S') \xrightarrow{D(\theta)} D(S) \xrightarrow{A} Ab$. Then there is an extension $E\theta = (T', \pi', i')$ of $D(\theta)A$ by S' and a homomorphism $\mu\colon T' \to T$ of regular semigroups such that $(\mathrm{Id}_{D(\theta)A}, \mu, \theta)\colon E\theta \to E$ is a morphism of extensions. The pair $(E\theta, \mu)$ is unique up to an equivalence.*

PROOF. Let $T' = \{(x, t) \in S' \times T | x\theta = t\pi\}$ be the regular subsemigroup of $S' \times T$. The projection π': $T' \to S'$, defined by $(x, t)\pi' = x$, and the homomorphism i': $D(\theta)A \to \text{Ker}\, \pi'$ of $D(S')$-modules, given by $(a)i' = (e, (a)i)$, $e \in E(S')$, $a \in (D(\theta)A)_e = A_{e\theta}$, define an extension $E\theta = (T', \pi', i')$. If we define μ: $T' \to T$ by $(x, t)\mu = t$, $(x, t) \in T'$, then, clearly $(\text{Id}_{D(\theta)A}, \mu, \theta)$: $E\theta \to E$ is a morphism of extensions.

Suppose that $(E' = (T'', \pi'', i''), \mu'')$ is any other such pair. Then the homomorphism μ': $T'' \to T'$, given by $(t)\mu' = (t\pi'', t\mu'')$, is an equivalence of extensions and satisfies the equation $\mu'\mu = \mu''$. Hence $(E\theta, \mu)$ is unique up to an equivalence.

The next result is immediate from Propositions 3.4 and 3.5.

PROPOSITION 3.6. *Let θ: $S' \to S$ be a homomorphism of regular semigroups and let ϕ: $A \to B$ be a homomorphism of $D(S)$-modules. If $E = (T, \pi, i)$ is an extension of A by S then the extension $(D(\theta)\phi)(E\theta)$ is equivalent to the extension $(\phi E)\theta$. Here $D(\theta)\phi$ denote the homomorphism $D(\theta)A \to D(\theta)B$ of $D(S')$-modules.*

We shall write $\mathcal{C}$ for the following category: an object of $\mathcal{C}$ is a pair (S, A) with S a regular semigroup and A a $D(S)$-module; a morphism (θ, ϕ): $(S, A) \to (T, B)$ in $\mathcal{C}$ consists of a homomorphism θ: $S \to T$ of regular semigroups and a homomorphism ϕ: $D(\theta)B \to A$ of $D(S)$-modules.

THEOREM 3.7. *The mapping which associates to each object (S, A) of $\mathcal{C}$, the abelian group* $\text{Ext}(S, A)$ *and, to each morphism (θ, ϕ): $(S, A) \to (T, B)$, the homomorphism* $\text{Ext}(\theta, \phi)$: $\text{Ext}(T, B) \to \text{Ext}(S, A)$ *given by the composite* $\text{Ext}(T, B) \to \text{Ext}(S, D(\theta)B) \to \text{Ext}(S, A)$ *is a contravariant functor from $\mathcal{C}$ to the category of abelian groups.*

PROOF. If θ: $S \to T$ is a homomorphism of regular semigroups and B is a $D(T)$-module then, by Proposition 3.5, the map $[E] \mapsto [E\theta]$: $\text{Ext}(T, B) \to \text{Ext}(S, D(\theta)B)$ is a homomorphism of abelian groups. Similarly, if S is a regular semigroup, then by Proposition 3.4, $\text{Ext}(S, -)$ is a functor from the category of $D(S)$-modules to the category of abelian groups. In particular, if ϕ: $A \to A'$ is a homomorphism of $D(S)$-modules then $[E] \mapsto [\phi E]$: $\text{Ext}(S, A) \to \text{Ext}(S, A')$ is a homomorphism of abelian groups. Therefore if (θ, ϕ): $(S, A) \to (T, B)$ is a morphism of $\mathcal{C}$ then $\text{Ext}(\theta, \phi)$ is a homomorphism of abelian groups from $\text{Ext}(T, B)$ to $\text{Ext}(S, A)$. It remains to show that $\text{Ext}(-, -)$ is contravariant. Let

(θ, ϕ): $(S, A) \to (T, B)$ and (μ, ψ): $(T, B) \to (U, C)$ be morphisms of $\mathcal{C}$. Let $[E] \in \mathrm{Ext}(U, C)$ then

$$\begin{aligned}([E])\,\mathrm{Ext}(\mu, \psi)\,\mathrm{Ext}(\theta, \phi) &= [\phi\{(\psi(E\mu))\theta\}] \\ &= [\phi\{(D(\theta)\psi)((E\mu)\theta)\}], \quad \text{by Proposition 3.6} \\ &= ([E])\,\mathrm{Ext}(\theta\mu, (D(\theta)\psi)\phi) \\ &= ([E])\,\mathrm{Ext}((\theta, \phi)(\mu, \psi)).\end{aligned}$$

Therefore, $\mathrm{Ext}(\mu, \psi)\,\mathrm{Ext}(\theta, \phi) = \mathrm{Ext}((\theta, \phi)(\mu, \psi))$. Hence $\mathrm{Ext}(-, -)$ is a contravariant functor.

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