SINGULAR PLEATED SURFACES AND $\mathbf{CP}^1$-STRUCTURES

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1. Introduction. Let F_g be a closed orientable surface of genus $g > 1$ and let $\mathcal{T}_g$ be the Teichmuller space of F_g, i.e., the space of marked hyperbolic structures on F_g. We shall also denote by $\mathcal{T}_{g,1}$ the space of marked hyperbolic structures on F_g with one distinguished point; by this, we mean a distinguished point on the universal cover $\tilde{F}_g$ of F_g. This space is isomorphic to the space of marked complete hyperbolic structures on a genus g surface with 1 cusp which is the usual interpretation of $\mathcal{T}_{g,1}$. Choose a decomposition of F_g into pairs of pants by a collection of non-intersecting, totally geodesic simple closed curves. The Fenchel–Nielsen coordinates for $\mathcal{T}_g$ relative to this decomposition are given by the lengths of the curves as well as twist parameters defined on each curve. Varying the length and twist parameters gives deformations of the marked hyperbolic structures.

To obtain deformations of the $\mathbf{CP}^1$-structure such that the holonomy group cannot be conjugated to lie in $PSL(2, \mathbf{R})$, one can bend along any set of non-intersecting, totally geodesic curves. These deformations, called the bending deformations were first used by Thurston [**18**] in his "Mickey mouse" examples, they have since been studied and generalized by many authors, see for example [**2**], [**9**], [**10**] and [**12**]. More generally, a hyperbolic structure on F_g together with a measured geodesic lamination gives rise to a $\mathbf{CP}^1$-structure on F_g by bending the hyperbolic structure along the lamination by the measure of the lamination. Conversely, every $\mathbf{CP}^1$-structure can be obtained in this way. This was proved by Thurston around 1976 (unpublished, see [**6**] and [**10**] for ideas of proof) which gives the following theorem:

THEOREM 1. (Thurston)

$$\mathscr{C}_g \cong \mathcal{T}_g \times \mathcal{ML}_g$$

where

$\mathscr{C}_g$ is the space of marked $\mathbf{CP}^1$-structures on F_g,
$\mathcal{ML}_g \cong \mathcal{T}_g$ is the measured lamination space of F_g.

The proof of the theorem above is based on the maximal disc approach which has been generalized to higher dimensions in [**1**], [**2**], [**10**] and [**13**] where the space studied is the deformation space of flat conformal structures. We note that the proof of Thurston's theorem does not actually shed any light on the structure of $\mathcal{ML}_g$, the structure of this space however is well-known from the use of other techniques like train-tracks, see [**18**]. The first examples of non-bending deformation in higher dimensions were discovered by Apanasov in [**2**], he called these "stamping" deformations, these were subsequently generalized by the author in [**16**] where the connection between these deformations, the bending deformations and the maximal ball approach was clarified. Roughly speaking, these deformations correspond to bending along totally geodesic hypersurfaces that intersect in a (possibly disconnected) codimension 2 totally geodesic submanifold. The underlying structure however may now have cone singularities at the codimension 2 submanifold and the bending parameters now have to satisfy some spherical polygonal conditions.

The main problem with the maximal disc approach is that it does not at the present provide us with good knowledge of the local structure of the deformation space for a hyperbolic manifold M of dimension greater than 2. This is because it is not clear what the analogue of $\mathcal{T}_g$ and $\mathcal{ML}_g$ should be in higher dimensions. Indeed, in dimension 3, apart from some examples of Kapovich [11] where he showed that infinitely many 3-manifolds obtained by Dehn surgery on a two bridge knot have locally trivial deformation spaces of flat conformal structures, very little is known about the local structure of these deformation spaces. In dimensions ≥ 4, Johnson and Millson [9] have shown that there are examples of manifolds where the deformation space is singular.

The main aim of this paper is to give an approach which when generalized to higher dimensions may give us a better understanding of the local picture of the deformation space of flat conformal structures. We concentrate in this paper on the dimension 2 case where the manifold is a surface. Here the picture is much nicer since we have a simple classification of surfaces by the genus and the deformation space of $\mathbf{CP}^1$-structures is a contractible manifold with dimension depending only on the genus by Thurston's theorem. The dimension 3 case will be dealt with in a forthcoming paper [17].

We give a brief outline of our approach. Given a closed orientable surface F_g, $g > 1$ with a hyperbolic structure, we can triangulate F_g by $6g - 3$ geodesic curves based at some point x_0 on F_g. The structure on F_g together with the position of x_0 is determined by the lengths of the curves which form the sides of the triangulation, let us denote the parameters by $\vec{l}_0$. Conversely, if we vary the lengths of the curves slightly, using new parameters $\vec{l}$, we obtain a hyperbolic structure on F_g with possibly a cone singularity at x_0. There is exactly one positive scalar λ such that $\lambda\vec{l}$ gives a smooth hyperbolic structure on F_g, thus we can think of the lengths of the curves as local projective coordinates for the space of hyperbolic structures on F_g with one distinguished point x_0.

Now consider singular pleated surfaces with fixed cone singularity built up from totally geodesic triangular pieces in the same combinatorial manner as above. Together with the length parameters, we now have bending or pleating parameters along the edges of the triangles. If we allow the pleating measures to vary independently, the resulting holonomy representation lies in $\mathrm{PSL}(2, \mathbf{C})$ and may have non-trivial holonomy about the vertex of the triangles. This means that when we are developing the pleated surface to $\mathbf{H}^3$, the pieces do not necessarily close up around the vertex. The link of the vertex inherits a spherical structure (in the case when the singularity has positive cone angle) and the holonomy is trivial about the vertex precisely when the link of the vertex forms a spherical polygon. The sides of the polygons are parametrised by the interior angles of the triangles and the exterior angles of the polygon are parametrised by the bending parameters. The set of length and bending parameters on the edges of the triangulation satisfying this spherical polygonal condition gives local coordinates for an open subset of $\mathscr{C}_{g,1}$ which includes the $\mathbf{CP}^1$-structures uniformised by quasi-fuchsian groups. We prove this by showing that given

(i) a $\mathbf{CP}^1$-structure on F_g uniformised by a quasi-fuchsian group,

(ii) a distinguished point x_0 on F_g and a suitable choice of a set of curves on F_g based at x_0 and triangulating F_g, and

(iii) a fixed choice of a small cone angle θ, then there is a unique singular pleated structure on F_g with cone angle θ and comprising of totally geodesic triangles corresponding to the triangulation in (ii) such that the cone point corresponds to x_0.

Of particular interest is the limiting case when the cone point of the singular pleated structure is a cusp (i.e., has cone angle zero), in this case the link of the cusp inherits a Euclidean structure and the condition is now a euclidean polygonal condition, the sides of the polygon are now parametrised by lengths of horocyclic segments.

The basic idea that one can imbed a singular pleated surface into $\mathbf{H}^3$ corresponding to a quasi-fuchsian group can be found in [**18**]. Our construction also bears some similarity to that used in [**8**]. The idea that the pleated hypersurface may be singular and that pleating can occur in both directions for higher dimensions can be found in [**2**], [**3**] and [**16**]. The main difference between our paper and [**2**] and [**16**] is that we are now allowing singularities of codimension n. This results in the introduction of distinguished points in the spaces under consideration but on the other hand, in the dimension two case, it allows us to do away with the twisting parameters and consideration of general measured geodesic laminations. In the case of dimension $n=3$, we can show ([**17**]) that the deformation space of flat conformal structures with a distinguished point can be identified to the deformation space of Euclidean polyhedra satisfying certain conditions, this gives an intriguing connection with the results of [**19**].

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2. Preliminaries. In this section, we establish the notation and some basic facts.

Let F_g be a closed orientable surface of genus $g>1$, $\tilde{F}_g$ its universal cover and π its fundamental group. Let x_0 be a point on F_g. We can find a set of $6g-3$ curves based at x_0 that triangulate (topologically) F_g. This set of curves can be obtained as follows:

First take a standard set of curves $\{\alpha_i, \beta_j \mid 1 \leq i, j \leq g\}$ based at x_0 cutting F_g into a $4g$-gon (see Figure 1) and then triangulate the $4g$-gon by a set of $4g-3$ curves

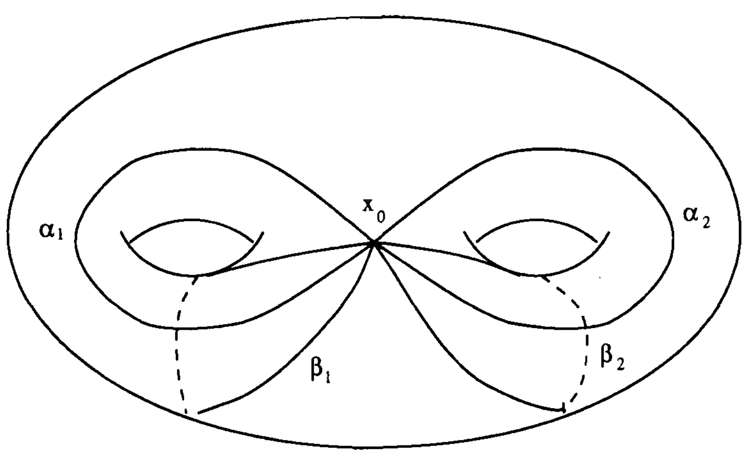

Figure 1

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$\{\gamma_k \mid 1 \leq k \leq 4g-3\}$. If F_g has a hyperbolic structure then the triangulation can be made geometric, i.e., we can find a set of curves based at x_0 that cuts F_g into hyperbolic triangles. If γ^x is a curve on F_g starting at x_0 and ending at some point x, we can get a triangulation of F_g based at x by conjugating the original curves by γ^x. We shall denote these curves by $\{\alpha_i^x, \beta_j^x, \gamma_k^x \mid 1 \leq i, j \leq g, 1 \leq k \leq 4g-3\}$. In the case when F_g has a hyperbolic structure, shifting the base point may change the geometric invariants of the triangulation even if the underlying hyperbolic structure on F_g remains the same, thus, the geometric invariants should be thought of as determining both the underlying structure and the base point. Note that if the base point is continuously deformed back to the same point on F_g by following a non-trivial loop, the resulting deformation of the geometric invariants may not return to the original invariants so that the base point should really be thought of as a point in $\tilde{F}_g$.

We now turn our attention to hyperbolic polygons. Let $P(s_1, \ldots, s_n; \theta_1, \ldots, \theta_n)$ denote a hyperbolic n-gon with side lengths s_i and interior angles θ_i all numbered cyclically (anti-clockwise) from a fixed vertex v_1. The parameters $\{s_i, \theta_i\}$ are not independent, for example when $n=3$, we have a triangle and the angles are determined by the lengths and conversely. We denote this special case by $T(s_1, s_2, s_3)$. We next define a generalized notion of similarity for hyperbolic triangles which will prove useful later.

Definition. Two hyperbolic triangles T and T' are *length similar* if the ratio of their sides is the same, we denote it by $T \sim T'$.

Clearly, length similarity is an equivalence relation.

Proposition 1. *Let $T(s_1, s_2, s_3)$ be a non-degenerate hyperbolic triangle. Then*

(i) *The equivalence class of $T(s_1, s_2, s_3)$, $[T(s_1, s_2, s_3)] = \{T(\lambda s_1, \lambda s_2, \lambda s_3) \mid \lambda > 0\}$.*

(ii) *The interior angles of $T(\lambda s_1, \lambda s_2, \lambda s_3)$ are decreasing functions of λ, each approaching zero as λ approaches ∞.*

(iii) *Area $(T(\lambda s_1, \lambda s_2, \lambda s_3))$ is an increasing function of λ and approaches π as λ approaches ∞.*

Proof. (i) is obvious. The first part of (ii) follows easily from Toponogov's theorem in the non-positive case, see [**4**]. That the angles approach zero as r approaches infinity follows from the hyperbolic cosine rule, see [**5**]. (iii) now follows from the fact that the area of a triangle is equal to the angle defect.

Remark. We can define other notions of similarity for hyperbolic triangles, for example two hyperbolic triangles are angle similar if the ratio of the interior angles is the same. Note that length similarity and angle similarity are different equivalence relations.

Instead of trying to extend the notion of length similarity directly to arbitrary polygons, we shall use triangulations of the polygons and apply length similarity to the triangulation instead.

Definition. A hyperbolic polygon is *marked* if we distinguish one of the vertices, we shall denote it by (P, v_1). Two marked hyperbolic polygons are equivalent if there is an orientation-preserving isometry taking one polygon to the other that matches the marked vertices.

DEFINITION. Let (P, v_1) and (P', v_1') be two marked hyperbolic n-gons and let Δ and Δ' be homeomorphic triangulations of P and P' relative to the marked vertices, i.e., there is a homeomorphism from P to P' mapping v to v' and Δ to Δ'. (P, v_1) is said to be *length similar to* (P', v_1') *relative to the triangulation* Δ if the triangles of Δ are length similar to the corresponding triangles of Δ'. We denote this by $(P', v_1') \sim_\Delta (P, v_1)$.

Clearly, for a fixed triangulation Δ, the above is an equivalence relation, we shall denote the equivalence class of P relative to Δ by $[P]_\Delta$. Note that a polygon in $[P]_\Delta$ may have some interior angles greater than 2π. If P has side lengths $(s_1, \ldots, s_n)$, and $P' \in [P]_\Delta$, then P' has side lengths $(\lambda s_1, \ldots, \lambda s_n)$ for some $\lambda > 0$. The converse is not true when $n > 3$ as can be seen by a simple dimension count. $[P]_\Delta$ is a one-dimensional family parametrised by λ, also, by Proposition 1, it is clear that the sum of the interior angles is a decreasing function of λ, taking values in the range $(0, n\pi)$. From this, we obtain the following lemma:

LEMMA 1. *Let P be a hyperbolic n-gon and Δ a triangulation of P. For each $0 < \theta < n\pi$ there exists exactly one polygon in $[P]_\Delta$ such that the sum of the interior angles is equal to θ.*

3. Local projective coordinates for $\mathcal{T}_{g,1}$. In this section, we construct local projective coordinates for $\mathcal{T}_{g,1}$ by relating the space $\mathcal{T}_{g,1}$ to the space of hyperbolic polygons satisfying certain conditions. The relation between $\mathcal{T}_{g,1}$ and the space of hyperbolic polygons seems to be fairly well-known, but we do not know of any exact statement in the literature. We state this in the form of the following theorem:

THEOREM 2. *Let $\mathcal{P}_g(2\pi)$ be the space of marked $4g$-gons up to equivalence satisfying the following conditions:*

(1) $s_{4k+1} = s_{4k+3}$ *and* $s_{4k+2} = s_{4k+4}$ *for* $0 \le k < g$, *and*

(2) $\sum_{i=1}^{4g} \alpha_i = 2\pi$.

where $\{s_i\}$, $\{\alpha_i\}$, $1 \le i \le 4g$ *are the side lengths and interior angles numbered cyclically from the marked vertex.*
Then $\mathcal{P}_g(2\pi)$ is homeomorphic to $\mathcal{T}_{g,1}$.

Proof. Using the standard set of curves $\{\alpha_i^x, \beta_j^x \mid 1 \le i, j \le g\}$ on F_g and making them geodesic, we get a continuous, surjective map from $\mathcal{T}_{g,1}$ to $\mathcal{P}_g(2\pi)$ by cutting F_g along the curves. To see that the map is injective, we first note that if two marked hyperbolic structures on F_g have the same fundamental polygons relative to the same set of homotopic curves, then they must be isometric relative to the marking, i.e. correspond to the same point in $\mathcal{T}_g$. It therefore reduces to showing that if we fix the hyperbolic structure on F_g and choose two different base points for the same set of standard curves, we obtain different marked polygons. Let ρ be the holonomy representation and let x_0 and x be two distinct base points. Suppose the two fundamental polygons obtained from the two sets of curves $\{\alpha_i, \beta_i \mid 1 \le i \le g\}$ and $\{\alpha_i^x, \beta_i^x \mid 1 \le i \le g\}$ are the same. Since the angle subtended by α_1 at x_0 must be the same as the angle subtended by α_1^x at x, x must lie on the same invariant curve of $\rho(\alpha_1)$ as x_0. The same argument holds for the other curves α_i and β_j, which implies that $x_0 \equiv x$, a contradiction.

Combining Theorem 2 and Lemma 1, we obtain the following theorem:

THEOREM 3. *Let F_g be a closed hyperbolic surface of genus $g > 1$ and let $x \in F_g$. If $\{\alpha_i^x, \beta_j^x, \gamma_k^x \mid 1 \le i, j \le g, 1 \le k \le 4g - s\}$ is a set of geodesic curves on F_g based at x and triangulating F_g, then the lengths of the curves $[A_1, \ldots, A_g, B_1, \ldots, B_g, C_1, \ldots, C_{4g-3}] \in$ $\mathbf{RP}^{6g-3}$ gives local projective coordinates for $\mathcal{T}_{g,1}$.*

If we are looking at the space of marked singular hyperbolic structures on F_g with one cone point of positive variable curvature (equivalently, cone angle $0 < \theta < 2\pi$), denoted by $\mathcal{T}_{g,1}^+$, we have the following theorem:

THEOREM 4. *Let $\mathcal{P}_g$ be the space of marked 4g-gons up to equivalence satisfying the following conditions:*

(1) $s_{4k+1} = s_{4k+3}$ *and* $s_{4k+2} = s_{4k+4}$ *for* $0 \le k < 4g$, *and*

(2) $\sum_{i=1}^{4g} \alpha_i \in (0, 2\pi)$.

where $\{s_i\}$, $\{\alpha_i\}$, $1 \le i \le 4g$ are the side lengths and interior angles numbered cyclically from the marked vertex.
Then $\mathcal{P}_g$ is isomorphic to $\mathcal{T}_{g,1}^+$.

Thus the lengths of the curves $\{\alpha_i^x, \beta_j^x, \gamma_k^x \mid 1 \le i, j \le g, 1 \le k \le 4g - 3\}$ gives local coordinates $(A_1, \ldots, A_g, B_1, \ldots, B_g, C_1, \ldots, C_{4g-3}) \in \mathbf{R}^{6g-3}$ for $\mathcal{T}_{g,1}^+$. Note that $\mathcal{T}_{g,1}^+ \cong \mathcal{T}_{g,1} \times \mathbf{R}$.

In the limiting case when the cone angle is zero, we have a complete hyperbolic structure on F_g with a cusp, the lengths of the edges of the triangulation are all infinite and we need to construct a different set of coordinates for the space. There are various ways to construct coordinates for these structures, we describe two different sets.

The first set of coordinates are related to the coordinates of R. Penner's decorated Teichmuller space, see **[15]**. Fixing the triangulation above, choose a small horocycle about the cusp and remove a neighbourhood of the cusp up to the horocycle, we obtain a compact hyperbolic manifold with horocyclic boundary. Now the edges of the original triangulation become finite geodesics and the $6g - 3$ lengths provide coordinates for the point in the decorated Teichmuller space which is the hyperbolic structure together with the choice of the horocycle. Changing the horocycle used adds or subtracts the same number to each coordinate so there are really only $6g - 4$ independent coordinates for the hyperbolic structure with cusp if we "forget" the horocycle. Note that this makes sense even when some of the coordinates are negative which occurs when the horocycle chosen is large.

Alternatively, fixing the triangulation and the horocycle as above, the horocycle is divided into $12g - 6$ segments by the edges of the triangulation and the lengths of the segments determine the hyperbolic structure as well as the horocycle chosen. It is not difficult to see by tracing through the link of the cusp that in fact there are only $6g-3$ independent length coordinates for the horocyclic segments so that the point in the decorated Teichmuller space is actually determined by these $6g-3$ parameters (which are all positive). Changing the horocycle used corresponds to multiplying the lengths of the horocyclic segments by a fixed positive real constant. The $6g-3$ parameters therefore gives local projective coordinates for the space of hyperbolic structures on F_g with one cusp.

4. Coordinates for $\mathbf{CP}^1$-structures. In this section, we show the relation between an open subset of the space of marked flat conformal structures with a distinguished point and the space of singular pleated structures with pleating along the sides of a triangulation of F_g. We first show how to obtain a singular pleated structure on F_g from a $\mathbf{CP}^1$-structure uniformised by a quasi-fuchsian group by fixing a base point, a standard set of triangulating curves and a constant $\epsilon \geq 0$.

Fix a $\mathbf{CP}^1$-structure on F_g uniformised by a quasi-Fuchsian group $\Gamma \subset \mathrm{PSL}(2, \mathbf{C})$, i.e. $\rho(\pi) = \Gamma$ and $F_g \cong \Omega_0/\Gamma$ where $\Omega_0 \subset \mathbf{CP}^1$ is one of the components of the domain of discontinuity of Γ acting on $\mathbf{CP}^1$. Let Ω_1 be the other component of the domain of discontinuity and let Λ_Γ be the limit set of Γ, we have $\mathbf{CP}^1 = \Omega_0 \cup \Lambda_\Gamma \cup \Omega_1$. $\mathbf{CP}^1$ under the action of $\mathrm{PSL}(2, \mathbf{C})$ can be thought of as the ideal boundary of hyperbolic 3-space $\mathbf{H}^3$. Let $\mathscr{C}(\Lambda_\Gamma)$ be the convex hull of Λ_Γ in $\mathbf{H}^3$, $\mathscr{C}(\Lambda_\Gamma)$ has two boundary components B_0 and B_1 which are complete pleated hyperbolic surfaces in $\mathbf{H}^3$ invariant under the action of Γ. Because of convexity, there are canonical "nearest point" maps

$$p_i: \Omega_i \rightarrow B_i, \qquad i = 0, 1$$

Thurston's parameters for Ω_0/Γ are obtained by taking B_0/Γ which is a pleated hyperbolic structure on F_g and thus gives a point in $\mathscr{T}_g \times \mathscr{ML}_g$.

Now fix a point $x_0 \in F_g \cong \Omega_0/\Gamma$ and a standard set of curves $\{\alpha_i, \beta_j, \gamma_k \mid 1 \leq i, j \leq g, 1 \leq k \leq 4g-3\}$ based at x_0 which triangulate the surface, as in the previous section. For $\epsilon \geq 0$, let B_ϵ be the hypersurface in $\mathbf{H}^3$ between Ω_0 and B_0 which is ϵ-distant from B_0. The triangulation of F_g lifts to a triangulation of $\tilde{F}_g$ and hence also Ω_0, B_0 and B_ϵ. The vertices of the triangulations are fixed for a fixed choice of x_0 as are the homotopy classes of the edges of the triangulations relative to the vertices. (Note that each vertex on B_ϵ lies on a geodesic joining a vertex on Ω_0 and a vertex on B_0.) We can straighten the edges of the triangulation on B_ϵ so that they are all geodesic in $\mathbf{H}^3$. The vertices of the triangulation still lie on B_ϵ but the edges do not necessarily lie on B_ϵ anymore. This gives a geodesic graph in $\mathbf{H}^3$ which spans a singular pleated surface B'_ϵ in $\mathbf{H}^3$ invariant under the action of Γ. B'_ϵ/Γ is the required singular pleated hyperbolic structure on F_g.

REMARKS:

1. For a fixed $\mathbf{CP}^1$-structure on F_g uniformised by a quasi-fuchsian group, the actual singular pleated structure obtained depends on the choice of the base point, triangulation and the constant ϵ.

2. When the $\mathbf{CP}^1$-structure on F_g is actually uniformised by a fuchsian group and we choose $\epsilon = 0$, we are back to the regular hyperbolic structure. If the $\mathbf{CP}^1$-structure on F_g is obtained by bending along a totally geodesic curve and x_0 is chosen to lie on the curve

and $\epsilon = 0$, the pleated singular structure obtained is just the usual pleated structure obtained by bending along the curve. Again in this case, there are no singularities.

3. Bending may occur in both positive and negative directions relative to a fixed orientation of the surface.

4. As ϵ increases, the curvature of the singular point increases (equivalently, the cone angle decreases). There is a well-defined limit when $\epsilon = \infty$, when we obtain a singular hyperbolic structure where the base hyperbolic structure on F_g has a cusp and the triangles of the triangulation are all ideal triangles.

5. Given a singular pleated structure on F_g such that the pleating occurs only in one direction (note that this implies that any singular point must have non-negative curvature), $\tilde{F}_g$ develops to a locally convex surface in $\mathbf{H}^3$ and by considering the map p^{-1} which is the inverse of the nearest point map, we obtain a corresponding $\mathbf{CP}^1$-structure which decomposes into three types of pieces, hyperbolic pieces which are the inverse images of the flat parts, $\mathbf{H}^1 \times S^1$ pieces which are the inverse images of the pleating edges and S^2 pieces which are inverse images of the singular vertices. Note that the corresponding $\mathbf{CP}^1$-structure need not necessarily be uniformised by a quasi-fuchsian group.

We now count the dimension of the space of such singular pleated surfaces where pleating occurs on the edges of a triangulation of the surface with one vertex x_0. From the previous section, we see that the space of singular hyperbolic structures on F_g with cone singularity at x_0 with positive curvature has local coordinates given by the lengths of the edges. This gives $6g-3$ real parameters for the underlying singular hyperbolic structure. For the pleating coordinates, there are also $6g-3$ coordinates corresponding to the pleating measures on the edges of the triangulation but they are not independent, there are some relations. We can see what happens by taking an ϵ-sphere (ϵ sufficiently small) about a singular point of the developing image in $\mathbf{H}^3$ and looking at the intersection of the developing image with the sphere. For the holonomy representation to be trivial about a small loop about the singular point, the developing map has to close up, i.e., the intersection of the developing image with the sphere is a closed spherical polygon (not necessarily convex). The lengths of the sides of the polygons are proportional to the angles at the corresponding vertices of the triangles and the exterior angles are the pleating measures. There are three independent equations, two to ensure that the spherical polygon closes up and one to ensure that the angle condition is satisfied. There are therefore only $6g-6$ independent pleating coordinates for each fixed singular hyperbolic structure.

From this, we obtain $12g-9$ degrees of freedom for the space of singular pleated hyperbolic structures on F_g with one singular point where the cone angle at the singular point is not fixed. This corresponds to an open subset of $\mathscr{C}_{g,1} \times \mathbf{R}^+$. If we fix the cone angle, (this is equivalent to fixing the perimeter of the spherical polygon) there are $12g-10$ degrees of freedom and we obtain local coordinates for $\mathscr{C}_{g,1}$ in terms of the lengths and pleating coordinate relative to the triangulation. As observed in Remark 3 above, fixing the cone angle is equivalent to removing the dependence on ϵ. This can be done by considering the length coordinates as projective coordinates.

Let us examine again the limiting case when $\epsilon = \infty$. We have now a pleated, cusped hyperbolic structure on F_g since all the triangles in the corresponding singular pleated structure are now ideal triangles so that the cone angle at the singular point is zero.

Recall from the previous section that a horocyclic link of the cusp is divided into $12g-6$ segments by the edges of the triangulation and that the lengths of $6g-4$ of these segments determine the lengths of the other segments, up to multiplication by a positive constant. To obtain a non-singular $\mathbf{CP}^1$-structure on F_g, we need to bend along the edges of the triangulation in such a way that when we consider a horosphere about the cusp in $\mathbf{H}^3$, the intersection of the edges of the triangulation with the horosphere forms the vertices of a euclidean polygon with $12g-6$ sides. The sides of the polygons are given by the lengths of the geodesic segments above, the exterior angle at each vertex is given by the bending angle of the corresponding edge of the original triangulation of F_g. Since there are $6g-3$ edges, there are $6g-3$ bending coordinates but they are not independent, the requirement that the polygon closes up and that the sum of the exterior angles is 2π gives three independent equations so that there are only $6g-6$ independent bending coordinates. Together with the independent side coordinates, we again get $12g-10$ independent coordinates as expected.

5. Dual structures. When the pleats of the singular pleated hyperbolic structure all occur in one direction relative to a fixed orientation of the surface, we have some kind of local convexity property for the developing image of the singular pleated hyperbolic structure and from this we can obtain some dual structures to the singular pleated hyperbolic structures.

Fix a triangulation of F_g with one vertex; label the vertex v, the edges $\{e_i \mid 1 \le i \le 6g-3\}$ and the triangles $\{t_j \mid 1 \le j \le 4g-2\}$. Fix a locally convex singular pleated hyperbolic structure on F_g relative to this triangulation, i.e., the structure is non-singular on the interior of the triangles and pleating occurs only in one direction with respect to a fixed orientation of the surface. Let $\{s_i \mid 1 \le i \le 6g-3\}$ and $\{b_i \mid 1 \le i \le 6g-3\}$ be the length and bending measures at the edges $\{e_i \mid 1 \le i \le 6g-3\}$ and $\alpha_{j,1}$, $\alpha_{j,2}$ and $\alpha_{j,3}$ be the interior angles of the triangle t_j, $1 \le j \le 4g-2$. The structure is determined by the s_i's and b_i's, it is also determined by the b_i's and the $\alpha_{j,k}$'s, where $1 \le i \le 6g-3$, $1 \le j \le 4g-2$, and $1 \le k \le 3$. Consider the decomposition of F_g dual to this triangulation. It has $4g-2$ vertices each of valence 3, $6g-3$ edges and 1 polygonal cell with $12g-6$ sides. Denote the vertices of the dual decomposition by V_j, $1 \le j \le 4g-2$ where V_j is dual to t_j, the edges by E_i, $1 \le i \le 6g-3$ where E_i is dual to e_i and the polygonal cell by P.

Using the singular pleated structure, develop $\tilde{F}_g$ into $\mathbf{H}^3$ to obtain an immersed singular pleated surface in $\mathbf{H}^3$. At a small ϵ neighbourhood U of a cone point x of this singular pleated surface, the surface separates U into a convex part U_c and a non-convex part U_n. Consider the subset of the tangent bundle at x consisting of vectors pointing away from U_c whose perpendicular hyperplane is disjoint from U_c. This set has a spherical structure and is a spherical polygon, it can be identified naturally to the polygon P of the dual decomposition of F_g. This gives a dual singular spherical structure on F_g with (generically) $4g-3$ singularities. If the bending measure is zero on some of the edges of the original triangulation, then some singularities may coalesce so the dual spherical structure may have less than $4g-3$ singularities.

PROPOSITION 2. *Let u_i be the length of the side E_i, $1 \le i \le 6g-3$ and $\beta_{j,k}$ be the interior angle of P dual to the angle $\alpha_{j,k}$ of the original triangulation. We have the following*:

(i) *$u_i = b_i$ for $1 \le i \le 6g-3$ i.e., the length of an edge E_i of the dual structure is equal to the bending measure of the original structure at e_i, the edge dual to E_i.*

(ii) *$\beta_{j,k} = \pi - \alpha_{j,k}$.*

(iii) *If V_j is a singularity of the dual singular spherical structure with cone angle θ_j, then $\theta_j > 2\pi$.*

(iv) *If the singularity of the original singular pleated hyperbolic structure has cone angle greater than zero, than the map from the structure to its dual is one–one.*

(v) *If the singularity of the original pleated hyperbolic structure has cone angle α, then the volume of the dual structure is equal to $2\pi - \alpha$.*

The proof is easy and left to the reader. By Proposition 2(v), above, fixing the cone angle of the original pleated surface corresponds to fixing the volume of the dual singular spherical structure on F_g, we therefore get a local homoeomorphism from the subset of $\mathscr{C}_{g,1}$ corresponding to locally convex singular pleated hyperbolic surfaces to the space of singular spherical structures on $\mathscr{F}_g$ with fixed volume and (generically) $4g-3$ cone singularities all with negative curvature. In the limiting case when the cone angle of the pleated hyperbolic structure is zero, the dual structure has as a fundamental domain a polygon which is isometric to the hemisphere, the volume of the dual structure is 2π and the singularities all have cone angles which are multiples of π. However, in this case, the map from the space of cusped pleated hyperbolic structures to the space of dual singular spherical structures is no longer one–one since different cusped pleated hyperbolic structures with the same bending parameters give the same dual structure. The space of singular spherical structures of this type has dimension $6g-4$. The fibre of the map is $6g-6$-dimensional. There are two fibrations of the total space of cusped pleated hyperbolic structures on F_g, the first by considering the fibres of the map to the base cusped hyperbolic structure (forgetting the pleating measures) and the second by considering the fibres of the map to the dual structures. Both the fibrations have fibres of dimension $6g-6$.

Finally, we conclude by discussing what happens when the singular pleated surface is not locally convex, that is, pleating occurs in both positive and negative directions. A good description of what can happen is given in [**18**]. In this case, if we look at a small ϵ-neighbourhood of the singular point in $\mathbf{H}^3$, the intersection of the ϵ sphere with the surface is a non-convex spherical polygon and most of the above discussion on the dimension of the deformation space goes through. We still obtain deformations of the representation of the fundamental group π of F_g into $\mathrm{PSL}(2,\mathbf{C})$ by varying the parameters. However, because the resulting developing surface in $\mathbf{H}^3$ is not locally convex, there is no canonical nearest point map from the sphere at infinity to the pleated surface so we do not get a $\mathbf{CP}^1$-structure on F_g associated with the pleated hyperbolic structure by considering the inverse of the nearest point map. However, if the holonomy representation is quasi-fuchsian, the limit set of the holonomy group is a quasi-circle and we still have a natural $\mathbf{CP}^1$-structure associated to the singular pleated surface.

6. Example. In this section we work out a specific example. Consider the hyperbolic structure on a genus two surface obtained by identifying the sides of a regular hyperbolic octagon with interior angles $\pi/4$ in the usual fashion. The sides of the polygon

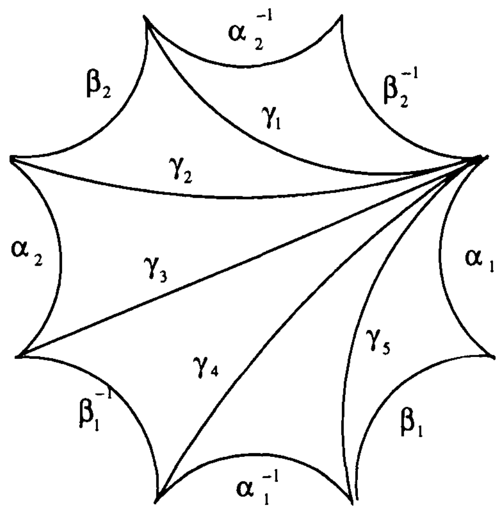

Figure 2

correspond to the standard curves α_1, β_1, α_2 and β_2 on F_g. Triangulate the surface by introducing the geodesic curves $\gamma_1, \gamma_2, \ldots, \gamma_5$ as in Figure 2. We shall consider this as a $\mathbf{CP}^1$-structure on F_g, so we shall think of $\mathbf{H}^2$ as a totally geodesic hypersurface in $\mathbf{H}^3$, the limit set Λ of the holonomy group is a circle on the sphere at infinity and the $\mathbf{CP}^1$-structure on F_g develops onto one of the components of $S^2 - \Lambda$ where $S^2 \equiv \mathbf{CP}^1$ is the sphere at infinity of $\mathbf{H}^3$. The developing map of the $\mathbf{CP}^1$-structure is obtained by composing the developing map of the hyperbolic structure with the inverse of the nearest point map. Choose some $\epsilon > 0$ and consider the (constant curvature) hypersurface B_ϵ in $\mathbf{H}^3$ ϵ-distant from $\mathbf{H}^2$. As described in § 4, this gives a singular pleated surface B'_ϵ in $\mathbf{H}^3$ invariant under the holonomy group thus giving us the singular pleated structure on F_g. The cone angle θ_ϵ at the singularity is a decreasing function of ϵ, where $\theta_0 = 2\pi$, the pleating measures on the γ_i's are all equal to zero since the vertices of the corresponding polygon on B_ϵ all lie on the same totally geodesic hypersurface in $\mathbf{H}^3$. The pleating measures on the curves α_1, α_2, β_1 and β_2 are equal by symmetry, we denote it by b. We have

$$8b = 2\pi - \theta_\epsilon$$

Since the pleating measures are all non-negative in this case, we have a dual singular spherical structure on F_g whose fundamental polygon is a regular spherical octagon of volume $2\pi - \theta_\epsilon$. In the limiting case when $\epsilon = \infty$, the euclidean polygon corresponding to the link of the cusp is a regular euclidean octagon. It is really more accurate to think of this as an 18-gon since some of the sides of the octagon really consists of several sides joined by vertices with exterior angle zero because the bending measures on the γ_i's were zero. Perturbing the point in $\mathscr{C}_{g,1}$ is equivalent to perturbing this polygon subject to the constraints mentioned at the end of § 4, we see that it is possible that the perturbed polygon is not convex, i.e. pleating may occur in both positive and negative directions in a small neighbourhood of this structure.

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