ELLIPTICALLY EMBEDDED SUBGROUPS AND NILPOTENT PRODUCTS

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1. Introduction. A recent paper of Rhemtulla and Wilson [4] is concerned with elliptically embedded subgroups of groups. A subgroup H of a group G is *elliptically embedded* in G if, for each subgroup K of G, there is some integer m such that $\langle H, K \rangle = (HK)^m$. Some sufficient conditions for elliptic embedding are given in Section 2 of [4], and some consequences of the presence of this property are to be found in Theorems 1 and 2 of the same paper and in the main theorem of [5]. It is evident from all of these results that the property of being elliptically embedded is closely related to the nilpotency and subnormality of certain subgroups. One of the questions considered here is the following.

If H, K are subnormal subgroups of the group G, when is the subgroup $J = \langle H, K \rangle$ of the form $J = (HK)^m$ for some integer m?

Some further conditions are certainly necessary here, even if J is nilpotent (of class two)—this is indicated by the example in Section 2 of [4]. One might expect that finiteness of rank would be fairly decisive in this context, and Theorem 2.5 states, in effect, that most of the conditions known to be sufficient for J to be subnormal in any group G in which H and K are subnormally embedded also imply that $J = (HK)^m$ for some m. However, there is one such condition, concerning the tensor product of H and K, which does not carry this implication, and a suitable example is given in Section 3. It is clear that $J = (HK)^{m+1}$ if $[H, K] \subseteq (HK)^m$, and we shall see that if $[H, K]$ has finite rank then there does exist such an integer m.

Consideration of tensor products and the subgroup $[H, K]$ leads one quite naturally to examine "nilpotent products" of H and K. It is well known (see page 125, Vol. 1 of [7]) that $H/H' \otimes K/K'$ is isomorphic to $[H, K]$ in the second nilpotent product of H and K, and in Section 4 it is shown that this result can be partially extended to higher nilpotent products. Associated results on tensor products are also included in that section. Perhaps the main results presented here can be summarized as follows.

Let H, K be groups such that $H/H' \otimes K/K'$ has finite rank. Then, in any nilpotent product of H and K, the subgroup $[H, K]$ also has finite rank. Further, if H and K are subnormally embedded in any group G, then there exists an integer m such that $\langle H, K \rangle = (HK)^m$.

2. Conditions for a union of subnormal subgroups to be an elliptic set. We begin with a lemma which will reduce the proof of the subsequent theorem to that of the "finitely generated case".

LEMMA 2.1. *Let J be a finitely generated nilpotent group, generated by subgroups H and K, and suppose $[H, K]$ has rank r. Then $[H, K]$ is generated by $2r$ elements of the form $[h, k]$, where $h \in H$, $k \in K$.*

Proof. It suffices to prove that $[H, K]$ can be thus generated modulo $[H, K]'$, since $[H, K]$ is nilpotent. Since $[H, K]'$ is normal in J, we may factor and thus assume that $[H, K]$ is abelian. Let T be the torsion subgroup of $[H, K]$. Then, modulo T, $[H, K]$ is free abelian of rank at most r, and so there is a subgroup S of $[H, K]$, generated by r elements of the form $[h, k]$, such that $|[H, K] : ST|$ is finite and hence $|[H, K] : S|$ is finite. Writing S_0 for the core of S in J, we have that $|[H, K] : S_0|$ is finite. Modulo S_0, we need to show that $[H, K]$ is generated by r elements $[h, k]$. So we assume that $[H, K]$ is finite. Then, since J is residually finite, there is a normal subgroup N of finite index in J such that $N \cap [H, K] = 1$, and we can make the further assumption that $N = 1$. But J is then a finite nilpotent group, and so $[H, K]$ is the direct product of subgroups $[H_p, K_p]$, where, for each prime p dividing $|J|$, H_p and K_p are the Sylow p-subgroups of H and K respectively. Now each $[H_p, K_p]$ is a finite p-group of rank at most r and so, by Burnside's Basis Theorem, can be generated by r commutators of the form $[h_p, k_p]$. Thus $[H, K]$ is generated by r subgroups, each of which is generated by elements $[h_{p_1}, k_{p_1}], \ldots, [h_{p_n}, k_{p_n}]$, for different primes p_i. If Π_1, Π_2 are disjoint sets of primes and x_i, y_i are π_i-elements of J ($i = 1, 2$) then $[x_1 x_2, y_1 y_2] = [x_1, y_1][x_2, y_2]$, so by induction on the number of primes dividing $|J|$ we obtain the desired result.

In view of the fact that the nilpotency class of J plays no part in the proof of the lemma above, we have the following easy consequence.

COROLLARY 2.2. *Let J be a locally nilpotent group, generated by subgroups H and K, such that $[H, K]$ has finite rank r. Then, for each finitely generated subgroup F of $[H, K]$, there exist $2r$ commutators of the form $[h, k]$ which generate a subgroup containing F.*

We use Lemma 2.1 to prove the following result.

THEOREM 2.3. *Let J be a nilpotent group of class c, generated by subgroups H and K, and suppose $[H, K]$ has (finite) rank r. Then there is an integer m, depending only on c and r, such that $J = (HK)^m$.*

Proof. Since every element of J is contained in some finitely generated subgroup of the form $\langle H_0, K_0 \rangle$, where $[H_0, K_0]$ has rank at most r, we may suppose that J is finitely generated. Let d be the least integer such that $[H, K, {}_d J] = 1$. Then $d \le c - 1$. If $d = 0$, then $J = HK$, so suppose $d \ge 1$. By Lemma 2.1, $[H, K]$ is generated by elements $[h_i, k_i]$, $i = 1, \ldots, 2r$, and so, for each element σ of $[H, K]$, there are integers $\alpha_1, \ldots, \alpha_{2r}$ such that, modulo $[H, K, J]$,

$$\sigma \equiv [h_1, k_1]^{\alpha_1} \ldots [h_{2r}, k_{2r}]^{\alpha_{2r}}$$

$$\equiv [h_1, k_1^{\alpha_1}] \ldots [h_{2r}, k_{2r}^{\alpha_{2r}}] \in (HK)^{4r}.$$

If $d = 1$, then $J = HK[H, K] = (HK)^{4r+1}$ and we are done. Otherwise, we note that, modulo $[H, K, {}_2 J]$, $[H, K, J]$ is generated by $[H, K, H]$ and $[H, K, K]$, and we apply Lemma 2.1 again to deduce that $[H, K, H]$ is generated by $2r$ elements $[\sigma_i, g_i]$, where $\sigma_i \in [H, K]$ and $g_i \in H$ for each i. Then, again modulo $[H, K, {}_2 J]$, any element λ of

$[H, K, H]$ is of the form

$$\lambda \equiv [\sigma_1, g_1]^{\beta_1} \dots [\sigma_{2r}, g_{2r}]^{\beta_{2r}}$$

$$\equiv [\sigma_1, g_1^{\beta_1}] \dots [\sigma_{2r}, g_{2r}^{\beta_{2r}}]$$

and each $[\sigma_i, g_i^{\beta_i}]$ is congruent to a product of $2r$ elements $[h_{i,j}, k_{i,j}, g_i^{\beta_i}]$ $(j = 1, \dots, 2r)$, each of which belongs to $(HK)^6$. Thus there is an element λ' of $(HK)^{24r^2}$ such that $\lambda \equiv \lambda'$. A similar argument applies to the subgroup $[H, K, K]$.

Then, modulo $[H, K, {}_2J]$, we have $[H, K, J] \subseteq (HK)^{48r^2}$. If $d = 2$, we are done, otherwise we continue to consider subgroups of the J-central factors of $[H, K]$ in this manner and thus obtain the result required.

COROLLARY 2.4. *Let G be a nilpotent group of class c and let H be a subgroup of G such that $[H, G]$ has rank r. Then H is elliptically embedded in G. In fact, there is an integer $m = m(r, c)$ such that, given any subgroup K of G, $\langle H, K \rangle = (HK)^m$.*

Now Proposition 3 of [4] states that if G is any group and H is a subgroup with the property that H^G is nilpotent minimax, then H is elliptically embedded in G. An obvious question is whether "minimax" could be replaced by "finite rank" here. An (unpublished) example due to Brian Hartley shows that this is not the case (see Example 3.1 below). Next, we establish some sufficient conditions for the union of two subnormal subgroups to constitute an "elliptic set", that is, we prove the following result.

THEOREM 2.5. *Let H, K be subnormal subgroups of the group $J = \langle H, K \rangle$. Then there is an integer m such that $J = (HK)^m$ if any one of the following conditions is satisfied.*
  (i) *$J'/\gamma_3(J)$ has finite rank.*
  (ii) *$[H, K]$ has finite rank modulo each term of the lower central series of J.*
  (iii) *$H/H' \otimes K/K'$ has finite rank (where $H/H' \otimes K/K'$ denotes the tensor product of H/H' and K/K' viewed as abstract groups).*

Proof. Each of the three conditions stated in the theorem above suffices to ensure that some term of the lower central series of J is contained in the product HK (see [10], [8] and [11] respectively). Hence, factoring, we may assume J is nilpotent. But then $[H, K]$ has finite rank—in case (i) this follows from [9] and in case (iii) from Proposition 4.1 below. The result now follows from Theorem 2.3.

REMARK. It is easily seen (by referring to the results quoted) that the integer m is in each case bounded by the subnormal defects of H and K and the ranks involved.

The main results of [11] are concerned with a property of abelian groups which is rather weaker than finiteness of rank, namely that of being a direct product of a group of finite rank and a periodic divisible group. Thus if $H/H' \otimes K/K'$ is "*FRD*", then the "lower central series property" (etc.) holds for J. However, there is no companion result for elliptic embedding, as Example 3.2 indicates. A result that we shall need to refer to in the description of this example is a consequence of the following simple observation.

LEMMA 2.6. *Suppose H, K are subgroups of the group G and let m be a positive integer. Then each element σ of $(HK)^m$ can be expressed in the form*

$$\sigma = \sigma_1 \tau_1 \tau_2 \sigma_2 \ldots \sigma_m \tau_m hk,$$

where $h \in H$, $k \in K$ and, for each $i = 1, \ldots, m$, $\sigma_i = [a_1, b_i]$ and $\tau_i = [c_i, d_i]$, for some $a_i, d_i \in H$ and $b_i, c_i \in K$.

Proof. If $m = 1$ there is nothing to prove. Suppose $m > 1$ and that $\sigma = h_1 k_1 \ldots h_m k_m$. Then

$$\sigma = [h_1^{-1}, k_1^{-1}][k_1^{-1}, (h_1 h_2)^{-1}](h_1 h_2)(k_1 k_2)(h_3 k_3 \ldots h_m k_m),$$

which is of the form $\sigma_1 \tau_1 \lambda$, where $\lambda \in (HK)^{m-1}$. The result follows by induction on m.

If, to the hypotheses of the lemma above, we add that $G = (HK)^m$ and that $H \cap K^H = 1 = K \cap H^K$, then it is clear that every element of $[H, K]$ is a product of at most $2m$ commutators of the type indicated. In particular we note the following result.

COROLLARY 2.7. *If a nilpotent product of the groups H and K equals $(HK)^m$, for some integer m, then every element σ of $[H, K]$ can be expressed as a product of (at most) 2m elements of the form $[h, k]$ or $[k, h]$.*

We recall here that, for each positive integer r, the r-th nilpotent product of the groups H and K is the image of $F = H * K$ obtained by factoring by the subgroup $[H, K, {}_{r-1}F]$.

3. Examples. The first example in this section is of a subgroup H of a group G such that H^G is nilpotent (in fact abelian) of finite rank yet H is not elliptically embedded in G. This is in contrast to Proposition 3 of [4] (where H^G is nilpotent minimax) and Corollary 2.4 above (where G itself is nilpotent).

EXAMPLE 3.1 (Hartley). Let k be a given positive integer and let $V = V_k$ be a maximum length (n, k) code over GF(2), so that $n = 2^k - 1$. (See [1, p. 48] for details.) Then all the codewords have weight $d = 2^{k-1}$. We claim that if $v_1, \ldots, v_{k-1} \in V$, then for some value of i, all of $v_1, \ldots, v_{k-1}$ have i-th component zero. To see this, write S_k for a set with $2^k - 1$ elements. We can identify each codeword with a subset of S_k. Let Σ_k be the set of subsets so obtained. Then:

   (i) $A_1, A_2 \in \Sigma_k$ implies $A_1 * A_2 \in \Sigma_k$, where $*$ is the symmetric difference, and
   (ii) $A \in \Sigma_k$ implies $|A| = 2^{k-1}$ or zero.

LEMMA. *Conditions (i) and (ii) imply that if $A_1, \ldots, A_r \in \Sigma_k$ and $A_1 \cup \ldots \cup A_r = S_k$, then $r \geq k$.*

Proof. If $k = 2$ then $d = 2$, $n = 3$ and this is clear. Assume $k > 2$ and that the result holds for smaller values and that $A_1 \cup \ldots \cup A_r = S_k$. Let T be the complement of A_1, so that $|T| = 2^k - 1 - 2^{k-1} = 2^{k-1} - 1$. Consider

$$\Sigma' = \{A \cap T : A \in \Sigma_k\}.$$

Now if $A \in \Sigma_k$, then either $A = A_1$ and $A \cap T = \varnothing$ or $A \neq A_1$ and $|A * A_1| = 2^{k-1}$. Then $2 \cdot 2^{k-1} - 2|A \cap A_1| = 2^{k-1}$, so that $2|A \cap A_1| = 2^{k-1}$ and $|A \cap A_1| = 2^{k-2}$. Hence

$|A \cap T| = 2^{k-1} - 2^{k-2} = 2^{k-2}$. Hence $D \in \Sigma'$ implies $|D| = 2^{k-2}$ or 0; if $D_1, D_2 \in \Sigma'$ then either $D_1 = D_2$ and $D_1 * D_2 = \Phi$, or $D_1 \neq D_2$ and $D_i = A_i \cap T$. Then

$$D_1 * D_2 = (A_1 * A_2) \cap T \in \Sigma'.$$

Thus Σ' satisfies the hypotheses with $k - 1$ for k. Clearly $T = (A_2 \cap T) \cup \ldots \cup (A_r \cap T)$ so that $r - 1 \geqslant k - 1$ and the lemma is proved.

Now writing $n = 2^k - 1$, choose n distinct primes $p_1, \ldots, p_n$. Let H_i, L_i be cyclic groups of order p_i, $M_i = H_i \times L_i$, and α_i an automorphism of order 2 such that $H_i^{\alpha_i} = L_i$. Set $N_i = M_i \langle \alpha_i \rangle$, $G_n = N_1 \times \ldots \times N_n = M \langle \alpha_1, \ldots, \alpha_n \rangle$, where $M = M_1 \times \ldots \times M_n$, and $H = \langle H_1, \ldots, H_n \rangle$. Let K be the subgroup of $\langle \alpha_1, \ldots, \alpha_n \rangle$ corresponding to V_k. Now, given $i \leqslant i \leqslant n$, there exists $v \in V$ with ith component $\neq 0$; this means $H^G = M$. But if we take any $k - 1$ elements $x_1, \ldots, x_{k-1} \in K$, then there exists i such that all of $x_1, \ldots, x_{k-1}$ have ith component 0, so $\langle H^{x_1}, \ldots, H^{x_{k-1}} \rangle \cap M_i = H_i$. Thus $L_1 \times \ldots \times L_n$ is not contained in the product of any $k - 1$ conjugates of H; hence neither is its generator, and so $L_1 \times \ldots \times L_n \nleqslant (HK)^{k-1}$.

Now write G_k for the group above and form $G_1 \times G_2 \times \ldots = G$, where these are constructed from pairwise disjoint sets of primes. This contains a locally cyclic subgroup H and a subgroup K such that H^G is of rank 2, K is elementary abelian 2-group and $G = \langle H, K \rangle \neq (HK)^l$ for all l.

Our second example concerns a (nilpotent) group G, generated by abelian subgroups H and K such that $H \otimes K$ is periodic divisible (or, rather, radicable since G is written multiplicatively) but $G \neq (HK)^m$ for any integer m. This shows that the hypothesis of having finite rank in Theorem 2.5 (iii) cannot be weakened to that of being FRD (that is, a direct sum of a group of finite rank and a periodic divisible group), thus dispensing with an obvious conjecture that arises from Theorem 2.5(iii) and the main results of [11].

EXAMPLE 3.2. *Let p be a prime. For each integer $i = 1, 2, \ldots$, let A_i be a group with presentation*

$$A_i = \langle a_{i1}, a_{i2}, \ldots : a_{i1}^p = 1, (a_{i(j+1)})^p = a_{ij}, j = 1, 2, \ldots \rangle.$$

Thus A_i is a quasicyclic group of type p^∞. Denote by H the direct product of the groups A_i and let K be a free abelian group with basis $\{x_1, x_2, \ldots\}$. Then our group G is defined to be the second nilpotent product of H and K. We note that $H \otimes K$ is a divisible p-group and, by considering the isomorphism determined by $h \otimes k \to [h, k]$, it is clear that $[A_i, \langle x_j \rangle] = [A_k, \langle x_l \rangle]$ if and only if $i = k$ and $j = l$. Also, $[H, K]$ is the direct product of the subgroups $[A_i, \langle x_j \rangle]$, $i, j = 1, 2, \ldots$. Now let n be any positive integer, and let $\sigma = [a_{11}, x_1] \ldots [a_{(n+1)1}, x_{n+1}]$.

Claim. σ cannot be written as a product of fewer than $n + 1$ commutators of the form $[h, k]$, where $h \in H$ and $k \in K$.

Before we establish this claim, let us note that it has as a consequence the fact that $G \neq (HK)^m$ for any integer m. By Corollary 2.7, if $G = (HK)^m$, then every element of $[H, K]$ can be expressed as a product of $2m$ commutators $[h, k]$ or $[k, h]$. But, since G is

nilpotent of class 2, $[k, h] = [h, k]^{-1} = [h^{-1}, k]$ and so, putting $n = 2m$ in the above gives a contradiction.

Now assume that the claim is false and, with σ as defined, write

$$B = \langle A_i : i = 1, \ldots, n+1 \rangle, \qquad C = \langle x_i : i = 1, \ldots, n+1 \rangle.$$

Then, in view of our remarks concerning the structure of $[H, K]$, it follows that σ can be written as a product of (at most) n commutators of the form $[b, c]$, where $b \in B$, $c \in C$. Further, since each A_i is locally cyclic, it is not difficult to see that σ can be expressed in the form

$$\sigma = [a_1^{\alpha_{11}} \ldots a_{n+1}^{\alpha_{(n+1)1}}, x_1^{m_{11}} \ldots x_{n+1}^{m_{(n+1)1}}] \ldots [a_1^{\alpha_{1n}} \ldots a_{n+1}^{\alpha_{(n+1)n}}, x_1^{m_{1n}} \ldots x_{n+1}^{m_{(n+1)n}}],$$

where, for each $i = 1, \ldots, n+1$, a_i is an element of A_i, and α_{ij} and m_{ij} are integers $(j = 1, \ldots, n)$. Further, we may assume that the order of each a_i is the same, say p^{k+1}, where $k \geq 0$. So, again for $i = 1, \ldots, n+1$, there is an integer β_i $(0 < \beta_i < p)$ such that $a_i^{p^k \beta_i} = a_{i1}$. (We could arrange for each β_i to equal 1 here, but it is convenient to be less specific). From the two expressions for σ, we deduce the following system of congruences modulo p^{k+1}:

$$S_{0,i} : \sum_{j=1}^{n} \alpha_{ij} m_{ij} \equiv \beta_i p^k, \, p \nmid \beta_i \, (i = 1, \ldots, n+1),$$

$$S_{r,i} : \sum_{j=1}^{n} \alpha_{(i+r)j} m_{ij} \equiv 0, \, i = 1, \ldots, n+1, r = 1, \ldots, n,$$

where $(i + r)$ is in each case reduced modulo $n + 1$ if it is greater than $n + 1$.

We show, by induction on n, that this system of congruences has no solution (in integers). Clearly (for a given n) we may assume that not every α_{ij} is divisible by p (else we could divide through by p and replace k by $k - 1$). By relabelling if necessary, we assume that p does not divide $\alpha_{(n+1)n}$. If $n = 1$, we obtain the congruences

$$\alpha_{11} m_{11} \equiv \beta_1 p^k, \qquad \alpha_{21} m_{21} \equiv \beta_2 p^k, \qquad \alpha_{21} m_{11} \equiv 0, \qquad \alpha_{11} m_{21} \equiv 0.$$

But the first and third of these relations imply that $\alpha_{21} \beta_1 p^k \equiv 0$ and hence that p divides α_{21}, contrary to our hypothesis. Suppose then that $n \geq 2$ and that the non-existence of solutions has been established for the case $(n - 1)$. Viewing the system of congruences as a system in the (integer) unknowns m_{ij} (for all i, j) we note that the congruences $S_{0,1}, \ldots, S_{0,n+1}$ may be represented by the $(n + 1) \times n$ coefficient matrix $P = (\alpha_{ij})$ (augmented by the column vector $p^k(\beta_1, \ldots, \beta_{n+1})^t$) and that, for each $r = 1, \ldots, n$, the system of congruences $S_{r,1}, \ldots, S_{r,n+1}$ is represented by a matrix P_r obtained from P by a cyclic permutation of the rows. We replace the subsystem $\{S_{0,1}, \ldots, S_{0,n+1}\}$ by the set of congruences $T_{0,1}, \ldots, T_{0,n}$ in the unknowns m_{ij} $(1 \leq i \leq n, 1 \leq j \leq n - 1)$ obtained from the original system by setting $T_{0,i} = \alpha_{(n+1)n} S_{0,i} - \alpha_{in} S_{(n+1-i)i}$; that is,

$$T_{0,1} : \sum_{j=1}^{n-1} (\alpha_{(n+1)n} \alpha_{1j} - \alpha_{1n} \alpha_{(n+1)j}) m_{1j} \equiv \alpha_{(n+1)n} \beta_1 p^k$$

$$\vdots$$

$$T_{0,n} : \sum_{j=1}^{n-1} (\alpha_{(n+1)n} \alpha_{nj} - \alpha_{nn} \alpha_{(n+1)j}) m_{nj} \equiv \alpha_{(n+1)n} \beta^n p^k.$$

Writing Q for the $n \times (n-1)$ coefficient matrix appearing here and recalling that p does not divide $a_{(n+1)n}\beta_i$ for any i, we see that the inductive hypothesis can be applied, provided we can deduce $(n-1)$ "homogeneous" systems of congruences in the m_{ij} $(1 \le i \le n, 1 \le j \le n-1)$ whose coefficients matrices are (all) those matrices obtainable from Q by cyclic permutation of the rows. But, with the aid of the following observations, it is not difficult to see that such systems can in fact be obtained by applying appropriate "row operations."

(1) For each $i = 1, \ldots, n$, the $(n+1)$st row of P appears as the i-th row of some P_r $(1 \le r \le n)$.

(2) For each pair of distinct integers i, i' in the set $\{1, \ldots, n\}$, the i-th row of P appears as the i'-th row of some P_r $(1 \le r \le n)$.

The inductive hypothesis leads to the required contradiction, so our earlier "Claim" is established, and $G \ne (HK)^m$ for any m.

4. Tensor products and nilpotent products. The following result was utilized in the proof of Theorem 2.5.

PROPOSITION 4.1. *Suppose H and K are (nilpotent) groups such that the tensor product $H/H' \otimes K/K'$ has finite rank, and let J be a nilpotent group generated by (isomorphic copies of) H and K. Then $[H, K]$ has finite rank in J.*

We shall establish this proposition and some related results by considering the "J-central factors" of $[H, K]$ in the case where J is a nilpotent product of H and K. As remarked in the introduction, it is well known that $H/H' \otimes K/K'$ is isomorphic to $[H, K]$ in the second nilpotent product. The proof of the following lemma is similar to that presented in [6], and is omitted here. We denote by $\bar{X}$ the abelianisation of a group X, and by $\bar{x}$ the element xX'.

LEMMA 4.2. *Let J be the $(k+2)$-nd nilpotent product of the groups H_1 and H_2, where k is some non-negative integer. Then, for each k-tuple $(H_{i_1}, \ldots, H_{i_k})$ of groups with $H_{i_j} \in \{H_1, H_2\}$ for all j, there is a homomorphism θ from $\bar{H}_1 \otimes \bar{H}_2 \otimes \bar{H}_{i_1} \otimes \ldots \otimes \bar{H}_{i_k}$ onto the subgroup $L = [H_1, H_2, H_{i_1}, \ldots, H_{i_k}]$ of J.*

Now, with the notation of the lemma above, the subgroup $[H_1, H_{2,k}J]$ is generated by a finite number of subgroups having the form of L. We then have the following result.

PROPOSITION 4.3. *Let $\mathfrak{X}$ be a class of groups such that*
   (i) *$X \otimes Y \in \mathfrak{X}$ implies that $X \otimes Y \otimes Y \in \mathfrak{X}$ (for any groups X and Y).*
   (ii) *Every homomorphic image of an $\mathfrak{X}$-group is in $\mathfrak{X}$.*
   (iii) *If G is an abelian group generated by the $\mathfrak{X}$-groups A and B, then G is in $\mathfrak{X}$.*
Let F be the free product of the groups H_1 and H_2. If $\bar{H}_1 \otimes \bar{H}_2$ belongs to $\mathfrak{X}$, then so does $[H_1, H_2, {}_kF]/[H_1, H_2, {}_{k+1}F]$ for every $k = 0, 1, 2 \ldots$.

Proof. In view of Lemma 4.2 and the subsequent remark, it need only be shown that, with the hypotheses of the Proposition satisfied, $\bar{H}_1 \otimes \bar{H}_2 \otimes \bar{H}_{i_1} \otimes \ldots \otimes \bar{H}_{i_k}$ also belongs to $\mathfrak{X}$ (where, as before, $H_{i_j} = H_1$ or H_2 for all j). But this is readily established by

induction on k, owing to the commutativity of the tensor product operation and the presence of hypothesis (i).

COROLLARY 4.4. *If, in addition to the hypotheses of Proposition* 4.3, *we also have*
(iv) $\mathfrak{X}$ *is closed with respect to forming extensions* (*that is*, $\mathfrak{X} = \text{P}\mathfrak{X}$)
then $[H_1, H_2]/[H_1, H_2, {}_{k+1}F]$ *is in* $\mathfrak{X}$ ($k = 0, 1, 2, \dots$).

Hypothesis (iii) is of course satisfied by every N_0-closed class of groups and is implied by (ii) and (iv) above.

We now turn to the question of finding some classes $\mathfrak{X}$ which satisfy the conditions above. This involves analysis of the structure of tensor products along the lines of that carried out in Section 2 of [11]. In fact, we shall adopt the required notation from [11] as follows (see Chapter XI of [2] and Chapter X of [3] for the relevant background).

For any abelian group A, denote by A_0 the torsion subgroup of A. Let $A_\infty = A/A_0$ and let $r(A_\infty)$ be the rank of A_∞. Also, for any prime p, let A_p denote the p-component of A_0.

(1) Now let X, Y be (additive) abelian groups and $T = X \otimes Y$, $V = T \otimes Y$. For each p, let r_p, s_p, u_p be the ranks of (fixed) basic subgroups of X_p, Y_p, T_p respectively, and let x_p, y_p, t_p denote the ranks of X_∞/pX_∞, Y_∞/pY_∞, T_∞/pT_∞. (Each of the ranks here is allowed to be infinite.)

With the notation as in (1), Williams [11] proves the following result.

(*) *T is the direct sum of a group of finite rank and a periodic divisible group if and only if there is an integer d such that $r(X_\infty) \cdot r(Y_\infty)$, $r_p \cdot s_p$, $r_p \cdot y_p$, $x_p \cdot s_p$ and $x_p \cdot y_p$ are all at most d, for all primes p.*

We use this to prove the next result.

LEMMA 4.5. *If $T = X \otimes Y$ is FRD, then so is $T \otimes Y$.*

Proof. By Lemma 2.3(ii) of [11], we have that $u_p = r_p \cdot s_p + x_p \cdot s_p + y_p \cdot r_p$ and so, since T is FRD, $u_p \le 3d$ for some d as in (*). Further, $r(T_\infty) = r(X_\infty) \cdot r(Y_\infty) \le d$, and so $t_p \le d$ also. Again by (*), we need only find an integer d_1 such that $r(T_\infty) \cdot r(Y_\infty)$, $u_p \cdot s_p$, $u_p \cdot y_p$, $t_p \cdot s_p$ and $t_p \cdot y_p$ are all at most d_1, for all primes p. We deal with each of these in turn:

$r(T_\infty) \cdot r(Y_\infty)$:   If $r(Y_\infty) = \infty$,   then $r(T_\infty) = 0$ (since $r(T_\infty) = r(X_\infty) \cdot r(Y_\infty) \le d$). Otherwise, $r(Y_\infty) = k_1$, say, and in either case $r(T_\infty) \cdot r(Y_\infty) \le dk_1$.

$u_p \cdot s_p, u_p \cdot y_p$: If $s_p \le d$, then $u_p \cdot s_p \le 3d^2$, while if $s_p > d$, then (by (*)) $r_p = 0$, $x_p = 0$ and so $u_p = 0$. Similarly for $u_p \cdot y_p$.

$t_p \cdot s_p$: If $s_p \le d$, then $t_p \cdot s_p \le d^2$. If $s_p > d$, then $x_p = 0$, that is $X_\infty = pX_\infty$. It follows easily that $T_\infty = pT_\infty$ and so $t_p = 0$.

$t_p \cdot y_p$: If $y_d \le d$, then $t_p \cdot y_p \le d^2$, while if $y_p > d$, then $r(Y_\infty) > d$ and so $r(X_\infty) = 0$, $r(T_\infty) = 0$, and hence $t_p = 0$.
Writing $d_1 = \max\{dk_1, 3d^2\}$, the proof of the lemma is complete.

Again, for each p, let m_p, n_p be the ranks of the divisible components of X_p, Y_p respectively. If T has finite rank then there is an integer d such that, in addition to the

inequalities indicated in (*), we also have $y_p \cdot m_p \leq d$ and $x_p \cdot n_p \leq d$, for all p. Indeed, these inequalities together give a sufficient condition for T to have finite rank, and (with the usual notation), we shall prove our next result.

LEMMA 4.6. *If* $T = X \otimes Y$ *has finite rank, then* $T \otimes Y$ *has finite rank.*

Proof. By the remarks above, it suffices to prove that $t_p \cdot n_p$ and $y_p \cdot$ (the rank of the divisible component of T_p) are each bounded by some integer d_2 (independent of p). So we must show that $t_p \cdot n_p$, $y_p \cdot x^p \cdot n_p$ and $y^2p \cdot m_p$ are bounded.

$t_p \cdot n_p$: If $n_p \leq d$, then $t_p \cdot n_p \leq d^2$ (recall that $t_p \leq r(T_\infty) \leq d$). If $n_p > d$, then $x_p = 0$ and (as before) $t_p = 0$.

$y_p \cdot x_p \cdot n_p$, $y_p^2 \cdot m_p$: If $y_p \leq d$, then each of these is bounded by d^2, while if $y_p > d$, we have $x_p = 0 = m_p$. The lemma is thus proved.

We can deduce some further results of this nature.

LEMMA 4.7. *Condition* (i) *of Proposition* 4.3 *is satisfied by each of the following classes* $\mathfrak{X}$ *of groups (in addition to groups which are* FRD *and those that have finite rank):*
(a) *torsion groups,* π-*groups (where* π *is any set of primes), groups of finite exponent,*
(b) *groups with the maximal or minimal condition, and minimax groups,*
(c) *finite groups,*
(d) *the class of (abelian) groups having finite* p-*ranks for all* $p = 0$ *or a prime.*

Proof. (a) is immediate, (c) is a consequence of (a) and Lemma 4.6, and (d) can be proved by examining the proofs of the previous two lemmas and noting that the relevant bounds need no longer apply for all primes at once. Abelian groups with the minimal condition are precisely those that are π-groups of finite rank for some finite set π of primes and so (a) and Lemma 4.6 apply. If $X \otimes Y$ has the maximal condition then there are finitely generated subgroups X_1 and Y_1 of X and Y respectively such that $X \otimes Y \cong X_1 \otimes Y_1$, and then

$$(X \otimes Y) \otimes Y \cong (X_1 \otimes Y_1) \otimes Y \cong Y_1 \otimes (X_1 \otimes Y) \cong Y_1 \otimes X_1 \otimes Y_1,$$

which is finitely generated.

Finally, suppose that $X \otimes Y$ is a minimax. From the structure of abelian minimax groups (see Lemma 10.31 of [7]), we infer that there are finitely generated subgroups X_1, Y_1 of X, Y respectively such that, modulo $U = \langle x_1 \otimes y_1 : x_1 \in X_1, y_1 \in Y_1 \rangle$, $X \otimes Y$ satisfies min, that is, $X/X_1 \otimes Y/Y_1$ satisfies min (see [2, p. 252]), as (therefore) does $Z = X/X_1 \otimes Y/Y_1 \otimes Y/Y_1$. But (similarly) $Z \cong (X \otimes Y \otimes Y)/V$, where V is generated by all elements $u \otimes y_2$ ($u \in U$, $y_2 \in Y_1$). Since V is a homomorphic image of $U \otimes Y_1$ and U is an image of $X_1 \otimes Y_1$, V satisfies max. The result is now immediate.

Now it is clear that all of the classes listed in Lemma 4.7 satisfy (ii) and (iii) of Proposition 4.3 and that those given in (a), (b) and (c) are in addition P-closed. If we denote by $\mathfrak{F}_r^*$ the class of groups whose abelian sections have finite p-ranks for all $p = 0$ or a prime, then we have established the following result.

LEMMA 4.8. *The classes* (*a*), (*b*), (*c*) *of Lemma* 4.7, *as well as the classes* $\mathfrak{F}_r^*$ *and* $\mathfrak{F}_r$ (*of groups having finite rank*) *satisfy the hypotheses* (*i*)–(*iv*) *of Proposition* 4.3 *and Corollary* 4.4. *The class* FRD *satisfies the first three of these hypotheses.*

We remark that the class FRD is certainly not P-closed (unlike its intersection with the class of abelian groups), as may be seen from the following easy example.

Let p be a fixed prime and, for each $i = 1, 2, \dots$, let A_i be as defined in Example 3.2. Let B_i be defined similarly, and set $A = \underset{i=1}{\overset{\infty}{\mathrm{Dr}}}\, A_i$, $B = \underset{i=1}{\overset{\infty}{\mathrm{Dr}}}\, B_i$, $C = A \times B$. Define $\alpha \in \mathrm{Aut}\, C$ via the assignment $a_{ij} \to a_{ij}b_{ij}$, $b_{ij} \to b_{ij}$ for all $i,j = 1, 2, \dots$ (so α^{-1} maps a_{ij} to $a_{ij}b_{ij}^{-1}$). Write $G = C]\langle \alpha \rangle$.

Then C is of course divisible and G/C is infinite cyclic, but G is not FRD, for if $G = D \times R$, where D is periodic divisible and R has finite rank, we have D central in G, and hence $G/Z(G)$ of finite rank, contradicting the fact that $A \cap Z(G) = 1$.

The question as to whether the conclusion of Corollary 4.4 nevertheless holds for the class FRD is not considered here.

Proof of Proposition 4.1. Let H, K be as defined. Then any nilpotent group generated by H and K is a homomorphic image of some nilpotent product of H and K. The result follows from Lemma 4.8 and Corollary 4.4.

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