STRUCTURE THEORY FOR MONTGOMERY-SAMELSON FIBERINGS BETWEEN MANIFOLDS. I

PETER L. ANTONELLI

1. Introduction. In (**12**), Montgomery and Samelson conjectured that an MS-fibering of polyhedra with total space an n-sphere must have a homology sphere as its singular set. Mahowald (**11**) has shown that, indeed, an orientable fibering with $n \leqq 4$ must have a Z_2-cohomology sphere as its singular set, while Conner and Dyer (**4**) have shown this for n arbitrary provided the fiber itself is a Z_2-cohomology sphere. We show that if the singular set is tame, then it is a Z-homology sphere if the fiber is also one. This result together with those of Stallings (**15**), Gluck (**7**), and Newman and Connell (**13**) are applied in the case where the singular sets are locally flat and tame. It is shown (Theorem 5.2) that MS-fiberings of spheres on spheres, with closed connected manifold fibers and singular sets, are topologically just suspensions of (Hopf) sphere bundles. In a subsequent publication, the case where the singular sets are finite shall be considered. The reader is invited to consult (**3**) and (**18**) in this case.

2. Definitions and preliminaries. In what is to follow, all manifolds will be assumed finitely triangulable, without boundary, orientable over the integers, and, for convenience, connected. A *fibering* β will be a 4-tuple $[X(\beta), f(\beta), Y(\beta), F(\beta)]$, where $f(\beta)\colon X(\beta) \to Y(\beta)$ has polyhedral covering homotopy property (PCHP) and is locally trivial (**9**).

An *MS-fibering of manifolds* β is a 6-tuple $[X(\beta), A(\beta), f(\beta), Y(\beta), B(\beta), F(\beta)]$, where

(i) $A(\beta) \subseteq X(\beta), B(\beta) \subseteq Y(\beta)$ are closed non-separating sets and $[X(\bar\beta), f(\bar\beta), Y(\bar\beta), F(\bar\beta)] = \bar\beta$ is a fibering with $X(\bar\beta) = X(\beta) - A(\beta)$, $Y(\bar\beta) = Y(\beta) - B(\beta), f(\bar\beta) = f(\beta)/X(\bar\beta)$, and $F(\bar\beta) = F(\beta)$; $\bar\beta$ is the fibering *associated* with β;

(ii) $f(\beta)$ is an open map and the restriction $f(\beta)/A\colon A(\beta) \to B(\beta)$ is a homeomorphism. $A(\beta)$ and $B(\beta)$ are the *singular sets* of β;

(iii) $X(\beta)$, $Y(\beta)$, and $F(\beta)$ are manifolds of positive dimension.

A *fibering of manifolds* α is an MS-fibering of manifolds in which $A(\alpha)$ and $B(\alpha)$ are empty. A *spine* α of an MS-fibering of manifolds β is a fibering of manifolds with

(i) $F(\alpha) = F(\beta)$;

Received August 4, 1967. The material in this paper constitutes a portion of the author's doctoral dissertation directed by Professor E. Hemmingsen and submitted to Syracuse University. This research was partially supported by a NASA Graduate Fellowship and an NSF Summer Graduate Fellowship and NSF contract # GP-5420.

(ii) there are deformation retractions, $r_x^{(\alpha)}$ and $r_y^{(\alpha)}$, for which the following *spine diagram* commutes:

$$\begin{array}{ccc} & r_x^{(\alpha)} & \\ X(\bar{\beta}) & \longrightarrow & X(\alpha) \\ f(\bar{\beta}) \downarrow & & \downarrow f(\alpha) \\ Y(\bar{\beta}) & \longrightarrow & Y(\alpha) \\ & r_y^{(\alpha)} & \end{array}$$

It is apparent that a spine α of β is necessarily unique up to homotopy equivalence. Moreover, the deformation retractions for α must actually be strong ones since both $X(\bar{\beta})$ and $Y(\bar{\beta})$ are absolute neighbourhood retracts (ANRs) (**10**).

Browder (**1**) has recently proved that if $\alpha = [S^n, f(\alpha), Y(\alpha), F(\alpha)]$ is a fibering in which $Y(\alpha)$ and $F(\alpha)$ are connected polyhedrons, then $F(\alpha)$ has the homotopy type of a 1-, 3- or 7-sphere. Furthermore, if α were a spine of an MS-fibering of manifolds β, the results cited in (**13**) on the topological Poincaré conjecture in dim $\geqq 5$ imply that $F(\alpha)$ is either a 1- or 7-sphere or a homotopy 3-sphere.

We say that a spine α of a given MS-fibering β is a *Hopf spine* if

$$\alpha = [S^n, f(\alpha), S^p, F(\alpha)].$$

From the above comments it is clear that a Hopf spine will always have a sphere for fiber if the Poincaré conjecture is valid in dim 3. Furthermore, $(n, p) = (3, 2), (7, 4)$ or $(15, 8)$, and in the $(3, 2)$ case, the spine α, viewed as a fiber bundle with group the space of self-homeomorphisms of S^1, is actually bundle equivalent to the Hopf map $h: S^3 \to S^2$.

A submanifold M^n of N^p is *locally flat* in N^p if for each point $x \in M^n$ there is a containing open set U and a homeomorphism $h: (U, U \cap M) \to (R^p, R^n)$ onto euclidean spaces. In the smooth category, all submanifolds are locally flat in their containing manifolds. However, this is not true in the topological category.

A subset T of a polyhedron P is *tame* in P if there is a self-homeomorphism of P which carries T onto a subpolyhedron of P. If P is the standard n-sphere, this notion agrees with that of Brown (**2**).

By an MS-*fibering of polyhedra* we mean a tuple as in the definition of an MS-fibering of manifolds except that condition (iii) is replaced by

(iii)′ $X(\beta)$, $Y(\beta)$, and $F(\beta)$ are all connected finite polyhedra of positive dimension.

It is not required that $A(\beta)$ and $B(\beta)$ be polyhedra or subpolyhedra of $X(\beta)$ and $Y(\beta)$ nor is it necessary that they be connected.

We say that β is *orientable over* G if $\pi_1(Y(\bar{\beta}))$ acts simply on $H_*(F(\beta); G)$ and *co-orientable over* G if $\pi_1(Y(\bar{\beta}))$ acts simply on $H^*(F(\beta); G)$.

It is not hard to see that if β is orientable over Z, then it is co-orientable over Z_p for every prime p. If β is *orientable over* Z we merely say that it is *orientable.*

3. A tame form of the MS-conjecture.

PROPOSITION 3.1. *If β is an MS-fibering of polyhedrons co-orientable over G, the integers or a field, and $B(\beta)$ is tame in $Y(\beta)$, then the Gysin sequence*

$$\ldots H_c{}^q(Y-B) \xrightarrow{\phi^*} H_c^{q+r+1}(Y-B) \xrightarrow{f(\bar{\beta})^*} H_c^{q+r+1}(X-A) \ldots$$

of Čech groups with compact supports, is exact provided $F(\beta)$ is a G-cohomology r-sphere.

For the proof we shall need the following lemma.

LEMMA 3.2. *Let β be an MS-fibering of polyhedrons with $F(\beta)$ a (singular) cohomology r-sphere. If $U \supseteq B(\beta)$ is an open set containing $B(\beta)$ and $\pi_1(Y(\beta) - U)$ acts simply on $H^*(F(\beta); G)$, then the relative sequence*

$$\ldots H^i(Y, U) \xrightarrow{\phi^*} H^{i+r+1}(Y, U) \xrightarrow{f(\beta)^*} H^{i+r+1}(X, f^{-1}(U)) \xrightarrow{\pi^*} \ldots$$

is exact.

The proof follows from the relative Gysin sequence (**9**) and excision. Note that since the spaces (Y, U) and $(X, f^{-1}(U))$ will always be ANRs, we may replace the singular groups of the above lemma with the corresponding Čech groups (**5**).

Suppose that (X, A) is a compact T_2 pair. Define

$$H_c{}^q(X-A; G) = \operatorname*{dir\,lim}_{\mathscr{D}(A)} H^q(X, U; G)$$

for all $q \geqq 0$ and any abelian group G, where $\mathscr{D}(A)$ is the directed set of open neighbourhoods of A. This isomorphism defines the qth Čech cohomology group of $X - A$ with compact supports. The groups on the right-hand side are the usual Čech groups, but by the remark above we may suppose that they are singular when necessary.

LEMMA 3.3. *$\mathscr{D}(B)$ contains a cofinal collection $\mathscr{C}(B) = \{N_r\}, r = 0, 1, 2 \ldots,$ for which*

(i) *$N_r \subseteq N_{r-1}$ $(r \geqq 1)$;*

(ii) *$Y - N_r$ and $Y - B$ are homotopically equivalent for every choice of $r \geqq 0$.*

This lemma (once established) together with Lemma 3.2 and the fact that direct limit of exact sequences is exact (see, for example, **6**) will yield Proposition 3.1.

Proof. Without loss of generality, we may suppose that $B(\beta)$ is connected. Since B is tame in Y, there is a homeomorphism h of $Y - B$ onto $Y - B^*$, where B^* is a subpolyhedron in some triangulation of Y. We may suppose that this triangulation (K, L) is "full" in the sense that any simplex of K having all its vertices in L is itself in L.

Let $N(r)$ denote the subcomplex of $K^{(r)}$ none of whose vertices are contained in $L^{(r)}$. $N(r)$ and $L^{(r)}$ are full in $K^{(r)}$, and $L^{(r)}$ is the largest subcomplex of $K^{(r)}$ disjoint from $N(r)$. Therefore, the carrier $|N(r)|$ will be a strong deformation retract of

$$|K^{(r)}| - |L^{(r)}| \underset{\text{homeo}}{\approx} Y - B^*.$$

Let $N_r = h^{-1}(|N(r)|')$ (the prime denotes complement) and set $\mathscr{C}(B) = \{N_r\}$, $r \geqq 0$. It is not hard to show that $\mathscr{C}(B)$ is cofinal in $\mathscr{D}(B)$ and since

$$Y - N_r \overset{\text{homeo}}{\approx} |K^{(r)}| - |N^{(r)}|' = |N^{(r)}| \underset{\text{retract}}{\overset{\text{deform}}{\rightleftarrows}} Y - B^* \overset{\text{homeo}}{\approx} Y - B,$$

the proof is complete.

The next proposition is closely related to a result of Conner and Dyer (**4**).

PROPOSITION 3.4. *If β is an MS-fibering of polyhedrons with both $F(\beta)$ and $X(\beta)$ G-cohomology spheres co-orientable over G, a field, and with $B(\beta)$ tame in $Y(\beta)$, then $B(\beta)$ is a G-cohomology sphere.*

In (**4**), no tameless condition on the singular set is assumed and the result is for Z_2-cohomology only. The central point of their argument is based on the existence of a Gysin sequence for the associated fibering $\bar{\beta}$. The proof of Proposition 3.4 above, based on ideas in (**4**, Theorem 2.1), requires untwisted coefficients in a field. This explains why we require the orientability condition of simple action of $\pi_1(Y(\bar{\beta}))$ on $H^*(F(\beta); G)$. Proposition 3.1 above provides the necessary Gysin sequence for the proof of Proposition 3.4. For details, the reader is referred to (**4**).

We now prove *the tame form of the MS-conjecture.*

THEOREM 3.5. *If β is an orientable MS-fibering of polyhedrons and both $X(\beta)$ and $F(\beta)$ are Z-homology spheres with $B(\beta)$ tame in $Y(\beta)$, then $B(\beta)$ is a Z-homology sphere.*

Proof. It is immediate that a Z-homology sphere is a mod p cohomology sphere for every choice of a prime p. Since $\pi_1(Y(\bar{\beta}))$ acts simply on $H^*(F(\beta); Z_p)$ for every prime p, Proposition 3.4 applies to yield that $B(\beta)$ is a Z_p-cohomology sphere for every prime p. Now,

$$H^q(B(\beta); Z_p) = \operatorname{Hom}(H_q(B(\beta); Z); Z_p),$$

where $H_q(B(\beta); Z)$ is a finitely generated abelian group so that the fundamental theorem yields

$$H_q(B(\beta); Z) = \begin{cases} Z & q = \dim B(\beta), 0, \\ 0 & \text{otherwise,} \end{cases}$$

which proves the theorem.

4. The existence of Hopf spines. In general, the existence or non-existence is not easy to judge. However, when suitable niceness conditions are imposed,

https://doi.org/10.4153/CJM-1969-016-4 Published online by Cambridge University Press

an existence theorem can be proved. The reader should note that in the result below, triangulability of our manifolds is not required.

THEOREM 4.1. *Let β be an MS-fibering of manifolds in which $B(\beta)$ and $A(\beta)$ are locally flat p-spheres, $p \geqq 1$; $X^n(\beta)$ and $Y^m(\beta)$ are* 1*-connected with vanishing integral cohomology groups in* dim p, $p + 1$.

(A) *If $m - p \geqq 3$, then β admits a Hopf spine α with* $\operatorname{codim}(\beta; \alpha) = p + 1$ *if and only if $X^n(\beta)$ and $Y^m(\beta)$ are topological spheres.*

(B) *If $m - p \leqq 2$, then β does not admit a Hopf spine. We note that if the Poincaré conjecture is not true in* dim 3, 4, *we would require that $m - p \geqq 5$ in* (A) *instead of $m - p \geqq 3$.*

We now quote some results from knot theory needed in the ensuing arguments. Recall that a *knot* is a pair (S^m, X^p), where S^m is the standard m-sphere and X^p is a subset of S^m homeomorphic to the standard p-sphere. A *locally flat knot* is one in which X^p is locally flat in S^m. A knot is *trivial* if there is a homeomorphism $h\colon (S^m, X^p) \to (S^m, S^p)$.

RESULT 1 (Stallings). *Every locally flat knot (S^m, X^p) with $m - p \geqq 3$ and $m \geqq 5$ is trivial* (**15**).

RESULT 2 (Gluck). *Every locally flat knot (S^4, X^1) is trivial* (**7**).

Singular cohomology theory is used throughout this section. For any closed subset $B \subseteq Y$, the following notions will be helpful. Define

$$\bar{H}^q(B; G) = \operatorname*{dir\,lim}_{\mathscr{D}(B)} H^q(U; G), \tag{1}$$

where $\mathscr{D}(B)$ is the collection of all open neighbourhoods of B directed downward by inclusion. If the natural map

$$i^*\colon \bar{H}^q(B; G) \to H^q(B; G), \tag{2}$$

induced from the inclusions $i_u\colon B \subseteq U$, is an isomorphism for every q and abelian group G, the set B is *taut* (with respect to singular theory) in Y. There are several conditions under which B is taut. For example, if B is tame in Y, it is taut in Y; any embedding of a manifold as a closed subset of another manifold must also be taut. For tautly embedded sets we have the following Lefschetz duality theorem (**14**).

DUALITY. *If Y is an m-manifold and B is taut in Y, then there is an isomorphism*

$$H_q(Y - B; G) = H^{m-q}(Y, B; G)$$

for every choice of q and G.

We are now in a position to begin the proof of Theorem 4.1.

Proof of part (A). Suppose that β admits a Hopf spine α with $\operatorname{codim}(\beta; \alpha) = p + 1$. Clearly, $X(\alpha)$ and $Y(\alpha)$ must be spheres of codimension $p + 1$, and deformation retracts of $X(\beta)$ and $Y(\beta)$, respectively. Because of the results of Newman and Connell on the topological Poincaré conjecture in dimension greater than 4 (**13**) and our assumption of its validity in dim 3, 4, it will suffice to show that both $X(\beta)$ and $Y(\beta)$ are homotopy spheres. We give the argument only for $Y(\beta)$, the other case being similar.

Since $B(\beta)$ is an embedding of S^p as a closed subset of the manifold $Y(\beta)$, it is taut, and therefore the duality yields

$$(3) \qquad H_i(Y(\bar{\beta}); Z) = H^{m-i}(Y(\beta), B(\beta); Z)$$

for all $i \geqq 0$. However, $Y(\bar{\beta})$ has the homotopy type of the $(m - p - 1)$-sphere $Y(\alpha)$, so that

$$(4) \qquad H^q(Y(\beta), B(\beta); Z) = \begin{cases} Z & q = m, p + 1, \\ 0 & \text{otherwise,} \end{cases}$$

for $q > 0$. Now consider the cohomology sequence for the pair $(Y(\beta), B(\beta))$ with coefficients in Z. Exactness and (4) then yield

$$(5) \qquad H^q(Y(\beta); Z) = 0 \text{ for } \begin{cases} p + 1 < q < m - 1, \\ 0 < q < p. \end{cases}$$

The only two groups of dimension between zero and m not shown to vanish in (5) are those of dim p and $p + 1$. But these are zero by hypothesis. Therefore, by the connectivity and orientability of $Y^m(\beta)$, the manifold $Y^m(\beta)$ must be an integral homology m-sphere. Applying 1-connectivity and the Hurewicz isomorphism theorem, the desired result is obtained using results from (**8**, p. 357), thus concluding the proof of necessity.

Now, suppose that both $X(\beta)$ and $Y(\beta)$ are spheres. By hypothesis, the singular set $B(\beta)$ is a locally flat p-sphere in $Y(\beta) - S^m$, $m - p \geqq 3$. Now, by Results 1 and 2 quoted above, there is a homeomorphism

$$h_B: (S^m, B(\beta)) \xrightarrow{\approx} (S^m, S^p),$$

and similarly there is a homeomorphism h_A for $(S^n, A(\beta))$. Thinking of S^m as the $(p + 1)$-fold suspension of S^{m-p-1} and S^p as its set of suspension points, it is easy to see that S^{m-p-1} is a strong deformation retract of $S^m - S^p$, letting $N^{m-p-1} = h_B^{-1}(S^{m-p-1})$, N is a deformation retract of $Y(\bar{\beta})$; setting $X(\alpha) = f(\alpha)^{-1}(Y(\alpha))$, $Y(\alpha) = N$, $f(\alpha) = f(\beta)/X(\alpha)$, and $F(\alpha) = F(\beta)$, we obtain a fibering of manifolds α. We claim that α is a Hopf spine of β with $\operatorname{codim}(\beta; \alpha) = p + 1$.

Suppose for the moment that it has been shown that α is a spine of β. Now, $Y(\alpha)$ is already a sphere of codim $p + 1$, therefore it is enough to show that $X(\alpha)$ is a sphere of codim $p + 1$ in $X(\beta)$.

Now, $X(\alpha)$ is a deformation retract of $X(\bar{\beta})$, the map h_A defines a homeomorphism of $X(\bar{\beta})$ onto $S^n - S^p$, and $S^n - S^p$ is deformation retractable onto S^{n-p-1}. Composing these maps we see that $X(\alpha)$ has the homotopy type of an $(n - p - 1)$-sphere. Therefore, by the Poincaré conjecture in dim $\geqq 5$ and our assumption of its validity in dim 3, 4, $X(\alpha)$ must be a sphere of codim $p + 1$. It remains to show that α is a spine of β.

We first convert β into a fiber bundle by using the group of all self-homeomorphisms of the fiber $F(\beta)$ in the compact-open topology as structural group. Then, noting that $Y(\bar{\beta})$ is paracompact and that $Y(\alpha)$ is a compact deformation retract of $Y(\bar{\beta})$, direct application of the first covering homotopy theorem (**16**) guarantees the existence of a deformation retract r_x such that the spine diagram commutes. The proof of sufficiency is therefore complete.

Proof of part (B). Suppose that β admits a spine α. We show that this must lead to a contradiction. Therefore, in particular, β does not admit a Hopf spine. As in the necessity portion of the proof of part (A) we have that both $X^n(\beta)$ and $Y^m(\beta)$ are topological spheres. Furthermore, Alexander duality and the fact that $B(\beta)$ is a p-sphere yields

$$H_q(Y(\alpha)) = \begin{cases} Z & q = m - p - 1, 0, \\ 0 & \text{otherwise}, \end{cases} \tag{6A}$$

and similarly,

$$H_q(X(\alpha)) = \begin{cases} Z & q = n - p - 1, 0, \\ 0 & \text{otherwise}. \end{cases} \tag{6B}$$

Suppose now that $m - p = 2$. By (6A) and the fact that $Y(\alpha)$ is a manifold, the dimension of $Y(\alpha)$ must be one. Thus, $Y(\alpha) = S^1$ and by a theorem of Whyburn (**19**), $f(\alpha)$ induces an epimorphism

$$f(\alpha)_*\colon H_1(X(\alpha); Q) \to H_1(Y(\alpha); Q)$$

on rational homology. Since $H_1(S^1; Q) \neq 0$, we know that $H_1(X(\alpha); Q) \neq 0$.

From (6B) and the fact that $\dim F(\alpha) \geqq 1$ implies $\dim X(\alpha) \geqq 2$, we must have that $H_1(X(\alpha); Z) = 0$, and via the universal-coefficient theorem, we then have that $H_1(X(\alpha); Q) = 0$, a contradiction.

Suppose that $m - p = 1$. By (6A),

$$H_q(Y^m(\alpha)) = 0 \quad \text{for all } q > 0.$$

But, our manifolds are of dimension greater than zero, closed, connected, and orientable, so that we must have that

$$H_m(Y^m(\alpha)) \neq 0,$$

a contradiction. This completes the proof of part (B) of Theorem 4.1.

The following proposition is closely related to (B) of Theorem 4.1 and can be proved with similar techniques.

PROPOSITION 4.2. *If $f\colon S^n \to S^m$ is the projection map of an MS-fibering of manifolds β with singular set a tame p-sphere ($p \geqq 1$) in S^m, then $m - p \geqq 3$.*

5. A structure theorem.

PROPOSITION 5.1. *If $f\colon S^n \to S^m$ is the projection map of an orientable MS-fibering of manifolds β in which $A(\beta)$ and $B(\beta)$ are locally flat and tame p-spheres*

($p \geqq 1$), then β admits a Hopf spine α, and β is topologically the ($p+1$)-fold suspension of α.

Proof. By Proposition 4.2, $m - p \geqq 3$, so that the results of Stallings and Gluck apply. Therefore, we might just as well suppose that both $A(\beta)$ and $B(\beta)$ are equatorial spheres in S^n and S^m, respectively.

By Borsuk's *antipodensatz*, one can prove that any map of S^p onto S^p of odd degree carries at least one pair of antipodal points to antipodal points. In our case then, there is at least one pair of antipodal points (p_1, q_1) in A^p which is carried into a pair of antipodal points (p_1', q_1') in B^p by the restriction of f to the equatorial sphere A^p. Both these pairs will be antipodal in S^n and S^m as well. Therefore, their removal leaves a singular fibering for which there is a pair of commuting (strong) deformation retractions $r_x^{(1)}, r_y^{(1)}$ onto an MS-fibering of manifolds

$$\beta_1 = [S^{n-1}, A^{p-1}, f^{(1)}, S^{m-1}, B^{p-1}, F],$$

where $f^{(1)}$ is the restriction of f to S^{n-1}, and A^{p-1} and B^{p-1} are again equatorial spheres in S^{n-1} and S^{m-1}, respectively. Clearly, $S(\beta_1) = \beta$.

Proceeding inductively with $1 \leqq t \leqq p+1$, and with $\beta_0 = \beta$, we obtain the MS-fibering of manifolds

$$\beta_t = [S^{n-t+1}, A^{p-t}, f(t), S^{m-t+1}, B^{p-t}, F],$$

where $f(t)$ is just restriction of $f(\beta)$ to S^{n-t+1}, and $\beta_t = S(\beta_{t+1})$. Furthermore, there are (strong) deformation retractions $r_x^{\,t}$ and $r_y^{\,t}$ which commute in the diagram

$$\text{(A)} \qquad \begin{array}{ccc} S^{n-t+1} - (p_t, q_t) & \xrightarrow{r_x^{\,t}} & S^{n-t} \\ {\scriptstyle f(\beta_{t+1})}\big\downarrow & & \big\downarrow{\scriptstyle f(\beta_t)} \\ S^{m-t+1} - (p_t', q_t') & \xrightarrow{r_y^{\,t}} & S^{m-t} \end{array}$$

Choosing $t = p+1$ we obtain $S(\beta_{p+2}) = \beta_{p+1}$, where

$$\text{(B)} \qquad \beta_{p+2} = [S^{n-p-1}, f^{(p+1)}, S^{m-p-1}, F]$$

is a fibering of manifolds, for which we have that

$$S^{p+1}(\beta_{p+2}) = \beta_0 = \beta.$$

Composing the $r_x^{\,t}$ and the $r_y^{\,t}$ of (A), we obtain deformation retractions r_x and r_y which make the spine diagram

$$\begin{array}{ccc} S^n - S^p & \xrightarrow{r_x} & S^{n-p-1} \\ {\scriptstyle f(\beta)}\big\downarrow & & \big\downarrow{\scriptstyle f(\beta_{p+2})} \\ S^m - S^p & \xrightarrow[r_y]{} & S^{m-p-1} \end{array}$$

https://doi.org/10.4153/CJM-1969-016-4 Published online by Cambridge University Press

commute. Therefore, β_{p+2} is a spine of β and since $X(\beta_{p+2}) = S^{n-p-1}$ and $Y(\beta_{p+2}) = S^{m-p-1}$, β must be a Hopf spine and the proof is complete.

We can now prove the following theorem modulo the Poincaré conjecture in dim 3, 4.

THEOREM 5.2. *Let $f: S^n \to S^m$ be the projection map of an orientable MS-fibering β in which $A(\beta)$ and $B(\beta)$ are locally flat and tame* 1-*connected p-manifolds* ($p \geqq 1$). *The following statements are then equivalent:*
(1) *$F(\beta)$ is an r-sphere, $r = 1, 3$ or* 7;
(2) *$B(\beta)$ is a p-sphere;*
(3) *β admits a Hopf spine α of* codim $p + 1$;
(4) *β is topologically the* $(p + 1)$-*fold suspension of the spine α.*

Proof. Statement (1) implies (2) via Theorem 3.5, while (2) implies (3) via Proposition 5.1. Statement (3) implies (1), and (4) implies (1) (both trivial) and (2) implies (4) via Theorem 4.5 and Proposition 4.2.

The following is an easy corollary of Theorem 5.2 and the usual facts about suspensions (see, for example, **6**, p. 4).

COROLLARY 5.3. *If $f: S^n \to S^m$ is as above and either* (1), (2), (3) *or* (4) *holds, then f is essential and admits no cross sections.*

Remark. Via local arguments, Timourian (unpublished) has proved that condition (1) of Theorem 5.2 is always satisfied, even if the tameness requirement is dropped from the hypothesis. In fact, Theorem 5.2 holds without any tameness whatsoever, as spectral sequence arguments show.

REFERENCES

1. W. Browder, *Higher torsion in H-spaces*, Trans. Amer. Math. Soc. *108* (1963), 353–375.
2. M. Brown, *Locally flat imbeddings of topological manifolds*, Ann. of Math. (2) *75* (1962), 331–341.
3. P. E. Conner, *On the impossibility of fibering certain manifolds by compact fibre*, Michigan Math. J. *5* (1957), 249–255.
4. P. E. Conner and E. Dyer, *On singular fiberings by spheres*, Michigan Math. J. *6* (1959), 303–311.
5. J. Dugundjie, *Remark on homotopy inverses*, Portugal. Math. *14* (1955), 39–41.
6. S. Eilenberg and N. Steenrod, *Foundations of algebraic topology* (Princeton Univ. Press, Princeton, N.J., 1952).
7. H. Gluck, *Unknotting s^1 in s^4*, Bull. Amer. Math. Soc. *69* (1963), 91–94.
8. S. Hu, *Mappings of a normal space into an absolute neighborhood retract*, Trans. Amer. Math. Soc. *64* (1948), 336–358.
9. ——— *Homotopy theory* (Academic Press, New York, 1959).
10. ——— *Theory of retracts* (Wayne State Univ. Press, Detroit, 1965).
11. M. Mahowald, *On singular fiberings*, Can. J. Math. *15* (1963), 214–219.
12. D. Montgomery and H. Samelson, *Fiberings with singularities*, Duke Math. J. *13* (1946), 51–56.
13. M. H. A. Newman, *The Engulfing theorem for locally tame sets*, Bull. Amer. Math. Soc. *72* (1966), 861–862.

14. E. Spanier, *Algebraic topology* (McGraw-Hill, New York, 1966).
15. J. Stallings, *On topologically unknotted spheres*, Ann. of Math. (2) *77* (1963), 490–503.
16. N. Steenrod, *The topology of fibre bundles* (Princeton Univ. Press, Princeton, N.J., 1950).
17. ——— *Cohomology operations*, Ann. Math. Studies, No. 50 (Princeton Univ. Press, Princeton, N.J., 1962).
18. J. G. Timourian, *Singular fiberings of manifolds*, unpublished Ph.D. thesis, Syracuse University, 1967.
19. H. Whyburn, *The mapping of Betti groups under interior transformations*, Duke Math. J. *4* (1938), 1–8.

Syracuse University,
Syracuse, New York;
The University of Tennessee,
Knoxville, Tennessee