CAPITULATION IN UNRAMIFIED QUADRATIC EXTENSIONS OF REAL QUADRATIC NUMBER FIELDS

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(Received 8 March, 1993; revised 21 May, 1993)

1. Introduction. Let k be an algebraic number field and C_k its ideal class group (in the wider sense). Suppose K is a finite extension of k. Then we say that an ideal class of k *capitulates* in K if this class is in the kernel of the homomorphism

$$j : C_k \to C_K$$

induced by extension of ideals from k to K. (See Section 2 below). In [**4**], Iwasawa gives examples of real quadratic number fields, $k = \mathbb{Q}(\sqrt{p_1 p_2 p_3})$, with distinct primes $p_i \equiv 1 \pmod 4$, for which all the ideal classes of the 2-class group, $C_{k,2}$ (the 2-Sylow subgroup of C_k), capitulate in an unramified quadratic extension of k. In these examples, $C_{k,2}$ is abelian of type $(2,2)$, i.e. isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and so all four ideal classes capitulate.

In this note, we consider an arbitrary unramified quadratic extension, K/k, of a real quadratic number field, k, and determine the number of ideal classes of C_k which capitulate in C_K. As we shall see, the number of ideal classes that can capitulate is 2, 4, or 8. We give simple criteria involving the fundamental units of the three quadratic subfields of K which determine the number of ideal classes that capitulate. We then make use of the results of Cremona and Odoni [**1**, **2**] to show that there exist infinitely many extensions K/k such that $|\ker j| = 2$, 4, and 8 respectively. Examples are then provided.

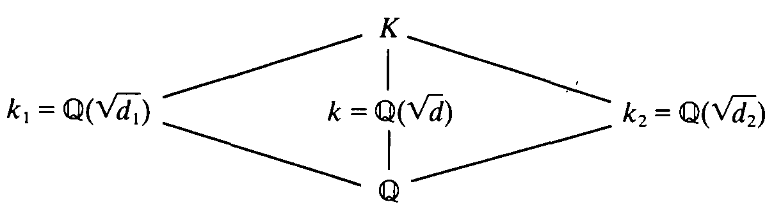
For more information on the capitulation problem, see Miyake [**7**].

2. Main Results. Let k be a real quadratic number field of even class number with discriminant

$$d_k = d = p_1^* \ldots p_t^*, \qquad p_j \text{ distinct primes.}$$

Here p^* represents the fundamental discriminant divisible only by the prime p, i.e. $p^* = (-1)^{p-1/2} p$, if p is odd, and $2^* \in \{-4, 8, -8\}$.

Since the class number of k is even, there exists at least one quadratic extension, K, of k unramified at all the primes (including the infinite ones, which means K is totally real). By genus theory, see e.g. [**5**], $K = k(\sqrt{d_1})$ for some fundamental discriminant $d_1 \mid d$ such that $d_1 > 1$ and $d_1 \neq d$. Let $d_2 = d/d_1$. Then $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ and thus $K/\mathbb{Q}$ is a Galois extension, the Galois group of which is abelian of type $(2,2)$. Consequently, K contains three real quadratic subfields: $k_0 = k = \mathbb{Q}(\sqrt{d})$, $k_1 = \mathbb{Q}(\sqrt{d_1})$ and $k_2 = \mathbb{Q}(\sqrt{d_2})$. We have the following diagram.



https://doi.org/10.1017/S0017089500031001 Published online by Cambridge University Press

We denote by $\varepsilon_0 = \varepsilon$, ε_1, and ε_2 the fundamental units (>1) of $k_0 = k$, k_1, and k_2, respectively. Also to simplify notation we let $N\kappa_i = N_{k_i/\mathbb{Q}}(\kappa_i)$ for any $\kappa_i \in k_i$ $(i = 0, 1, 2)$. Thus $N\kappa_i = \kappa_i\kappa_i'$ where κ_i' denotes the conjugate of κ_i over $\mathbb{Q}$.

Let $j: C_k \to C_K$ be the homomorphism induced by extension of ideals from the ring of integers, O_k, of k to O_K. (Hence if $[A]$ is the ideal class of C_k containing the ideal A, then $j([A]) = [AO_K]$.) We shall be interested in determining $|\ker j|$, the number of ideal classes of C_k that capitulate in C_K. It is well-known, cf. [**3**], that the exponent of the group $\ker j$ divides $[K:k]$ (which for us is 2). Hence $\ker j \subseteq C_{k,2}$, the 2-Sylow subgroup of C_k. Moreover it is also well-known, cf. [**9**], that when K/k is cyclic and unramified at the infinite primes,

$$|\ker j| = [K:k][E_k : N_{K/k}(E_K)].$$

Hence in our case $|\ker j| = 2[E_k : N_{K/k}(E_K)]$. Here E_k, E_K represent the group of units in O_k, O_K, respectively. Notice that $E_k^2 \subseteq N_{K/k}(E_K)$. But then since $E_k = \{\pm 1\} \times \langle \varepsilon \rangle$, we see that

$$[E_k : N_{K/k}(E_K)] \leq [E_k : E_k^2] = 4.$$

Thus $[E_k : N_{K/k}(E_K)] = 1$, 2, or 4 and so $|\ker j| = 2$, 4, or 8. We now determine conditions under which each of these three possibilities can happen. To this end, we obtain more information about E_K.

By Dirichlet's unit theorem, since $[K:\mathbb{Q}] = 4$ and K is totally real, E_K possesses a system of three fundamental units, i.e. $E_K = \{\pm 1\} \times \{\mu_1\} \times \{\mu_2\} \times \{\mu_3\}$ for some $\mu_i \in E_K$ $(i = 1, 2, 3)$.

By Kubota [**6**, Satz 1], there exist the following eight possibilities for a system of fundamental units of E_K:

(i) $\varepsilon_i, \varepsilon_j, \varepsilon_k$ (ii) $\sqrt{\varepsilon_i}, \varepsilon_j, \varepsilon_k$ (iii) $\sqrt{\varepsilon_i}, \sqrt{\varepsilon_j}, \varepsilon_k$

(iv) $\sqrt{\varepsilon_i\varepsilon_j}, \varepsilon_j, \varepsilon_k$ (v) $\sqrt{\varepsilon_i\varepsilon_j}, \sqrt{\varepsilon_k}, \varepsilon_j$

(vi) $\sqrt{\varepsilon_i\varepsilon_j}, \sqrt{\varepsilon_j\varepsilon_k}, \sqrt{\varepsilon_k\varepsilon_i}$ (vii) $\sqrt{\varepsilon_i\varepsilon_j\varepsilon_k}, \varepsilon_j, \varepsilon_k$

(viii) $\sqrt{\varepsilon_i\varepsilon_j\varepsilon_k}, \varepsilon_j, \varepsilon_k$ (with $N\varepsilon_l = -1$ $(l = 0, 1, 2)$)

where $\{\varepsilon_i, \varepsilon_j, \varepsilon_k\} = \{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$. Also in cases (ii)–(vii), any ε_t that appears under a radical is assumed to have norm equal to 1.

Proposition 1. *Suppose $N\varepsilon_i = -1$ for $i = 0, 1, 2$. Then*

$$[E_k : N_{K/k}(E_K)] = \begin{cases} 1 & \text{if } \sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in K, \\ 2 & \text{otherwise.} \end{cases}$$

Remark. Using [**6**, Zusatz 1], it is easy to determine whether $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in K$. See the first example of Section 3.

Proof of the Proposition. Since $N_{K/k}(\varepsilon_1) = N_{k_1/\mathbb{Q}}(\varepsilon_1) = N\varepsilon_1 = -1$, we have $-1 \in N_{K/k}(E_K)$. If $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in K$, then $N_{K/k}(\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2}) = \pm\varepsilon_0$ (because $(N_{K/k}\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2})^2 = N_{K/k}(\varepsilon_0\varepsilon_1\varepsilon_2) = \varepsilon_0^2 N\varepsilon_1 N\varepsilon_2 = \varepsilon_0^2$). Thus $\pm\varepsilon_0 \in N_{K/k}(E_K)$ and so $[E_k : N_{K/k}(E_K)] = 1$.

Now if $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \notin K$, then ε_0, ε_1, ε_2 must be a system of fundamental units of E_K and thus $N_{K/k}(E_K) = \{\pm 1\} \times \{\varepsilon_0^2\}$ which is of index 2 in E_k. This establishes the proposition.

Proposition 2. *Suppose $N\varepsilon_i = 1$ for some $i = 0$, 1, 2. Furthermore suppose $N\varepsilon_1 = -1$*

or $N\varepsilon_2 = -1$. (*Without loss of generality assume* $N\varepsilon_2 = -1$). *Then*

$$[E_k : N_{K/k}(E_K)] = \begin{cases} 1 & \text{if } \sqrt{\varepsilon_0} \in K \text{ or } \sqrt{\varepsilon_0\varepsilon_1} \in K, \\ 2 & \text{otherwise.} \end{cases}$$

In particular, if $N\varepsilon_0 = -1$, *then* $[E_k : N_{K/k}(E_K)] = 2$.

REMARK. We shall see that the condition $\sqrt{\varepsilon_0}$, $\sqrt{\varepsilon_0\varepsilon_1} \in K$ is easy to check by Kubota [**6**].

Proof of the Proposition. Suppose $\sqrt{\varepsilon_0} \in K$ or $\sqrt{\varepsilon_0\varepsilon_1} \in K$. Then $N_{K/k}(\sqrt{\varepsilon_0\varepsilon_1^a}) = \pm\varepsilon_0$ ($a = 0$ or 1) (arguing as in the proof of Proposition 1). Moreover, $N_{K/k}(\varepsilon_2) = N\varepsilon_2 = -1$. Hence $E_k = N_{K/k}(E_K)$ establishing part of the proposition.

Now suppose $\sqrt{\varepsilon_0} \notin K$ and $\sqrt{\varepsilon_0\varepsilon_1} \notin K$. Then by Kubota [**6**, Satz 1] (cf. above) a system of fundamental units of E_K consists of ε_0, ε_1, ε_2 or perhaps ε_0, $\sqrt{\varepsilon_1}$, ε_2. (Note that any unit, ε_i, under a radical must have positive norm). In either case, $N_{K/k}(E_K) = \langle -1, \varepsilon_0^2 \rangle$ which is of index 2 in E_k. This establishes the proposition.

In Proposition 3 below, we consider the case that $N\varepsilon_1 = N\varepsilon_2 = 1$. As this case requires more effort we first single out a major concept.

DEFINITION. Suppose μ is a unit of a real quadratic field such that $N\mu = 1$. We define, as in [**6**], $\delta(\mu)$ as the square-free kernel of the rational integer $\mu + \mu' + 2$, i.e. if $\mu + \mu' + 2 = m^2 n$ for some integers m, n and n is square-free, then $\delta(\mu) = n$.

For convenience we isolate facts about δ found in [**6**].

LEMMA. *Let L be a noncyclic normal real quartic extension of* $\mathbb{Q}$ *containing three real quadratic fields* k_1, k_2, *and* k_3. *Suppose* $\eta_i \in E_{k_i}$ *with* $N\eta_i = 1$ *for* $i = 1, 2, 3$. *Then*

(1) $\delta(\eta_i) \mid d_{k_i}$;

(2) $\eta_1\eta_2\eta_3 \in E_L^2$ (*the squares in* E_L) *iff* $\delta(\eta_1)\delta(\eta_2)\delta(\eta_3) \in L^2$.

Also if k is any real quadratic field such that $k = \mathbb{Q}(\sqrt{\Delta})$ *with* Δ *square-free and such that* $N(\varepsilon) = 1$ *for the fundamental unit* ε, *then*

(3) $\delta(\varepsilon) \neq 1, \Delta$.

Proof. See Kubota [**6**], Hilfssätze 8, 11, and 9, respectively.

PROPOSITION 3. *Suppose* $N\varepsilon_1 = N\varepsilon_2 = 1$. *Then*

$$[E_k : N_{K/k}(E_K)] = \begin{cases} 2 & \text{if } \sqrt{\varepsilon_0\varepsilon_1^a\varepsilon_2^b} \in K \text{ some } a, b \in \{0, 1\} \\ 4 & \text{otherwise.} \end{cases}$$

In particular, if $N\varepsilon_0 = -1$, *then* $[E_k : N_{K/k}(E_K)] = 4$.

Proof. We first claim that $\sqrt{\varepsilon_1}$, $\sqrt{\varepsilon_2}$, $\sqrt{\varepsilon_1\varepsilon_2} \notin K$. To this end let $k_i = \mathbb{Q}(\sqrt{\Delta_i})$ ($i = 0, 1, 2$) where $\Delta = \Delta_0$, Δ_1, Δ_2 are square-free rational integers. Also let $\delta_i = \delta(\varepsilon_i)$ for $i = 1$, 2 and $\delta_0 = \delta(\varepsilon_0)$ if $N\varepsilon_0 = 1$. Notice that $K = \mathbb{Q}(\sqrt{\Delta_1}, \sqrt{\Delta_2})$. We show that δ_1, δ_2, $\delta_1\delta_2 \notin K^2$. We do this by considering three cases according as $\Delta \equiv 1$, -1, 2 (mod 4), respectively. Also in each case p_i ($i = 1, \ldots, s$) denote distinct primes $\equiv 1$ (mod 4) and q_j($j = 1, \ldots, t$) distinct primes $\equiv -1$ (mod 4).

Case 1. Suppose $\Delta \equiv 1$ (mod 4). Then $d = \Delta$.

Let $d = p_1 \ldots p_s q_1 \ldots q_t = \Delta$ with t even. ($s = 0$ or $t = 0$ is possible.)

Also let $d_1 = p_1 \ldots p_{s_1} q_1 \ldots q_{t_1} = \Delta_1$ with t_1 even. Then $d_2 = d/d_1 = \Delta_2$. By the lemma,

$$\delta_1 \mid d_1 = \Delta_1 \quad \text{and} \quad \delta_1 \neq 1, \Delta_1,$$

$$\delta_2 \mid d_2 = \Delta_2 \quad \text{and} \quad \delta_2 \neq 1, \Delta_2.$$

Hence $\delta_1, \delta_2, \delta_1\delta_2 \notin K^2$.

Case 2. Suppose $\Delta \equiv -1 \pmod 4$. Then $d = 4\Delta$.

Let $\Delta = p_1 \ldots p_s q_1 \ldots q_t$ with t odd. Without loss of generality, let $\Delta_1 = p_1 \ldots p_{s_1} q_1 \ldots q_{t_1}$ with t_1 odd and so $d_1 = 4\Delta_1$. Then $d_2 = d/d_1 = \Delta/\Delta_1 = \Delta_2$.

By the lemma, $\delta_2 \mid d_2 = \Delta_2$ and $\delta_2 \neq 1$, Δ_2; hence $\delta_2 \notin K^2$. On the other hand, $\delta_1 \mid d_1 = 4\Delta_1$ and $\delta_1 \neq 1$, Δ_1. Thus since δ_1 is square-free $\delta_1 = 2$, $2\Delta_1$, a_1, or $2a_1$ for some $a_1 \mid \Delta_1$, $a_1 \neq 1$, Δ. If $\delta_1 = a_1$ or $2a_1$, then δ_1, $\delta_1\delta_2 \notin K^2$. If $\delta_1 = 2$ or $2\Delta_1$, then the only way $\delta_1 \in K^2$ can occur is if $\sqrt{2} \in K$. But then $k_i = \mathbb{Q}(\sqrt{2})$ for some $i = 1$, 2, which is contrary to the assumption that $N\varepsilon_1 = N\varepsilon_2 = 1$. Thus $\delta_1, \delta_2, \delta_1\delta_2 \notin K^2$.

Case 3: Suppose $\Delta \equiv 2 \pmod 4$. Then $d = 4\Delta$.

Let $\Delta = 2p_1 \ldots p_s q_1 \ldots q_t$. Without loss of generality, let $\Delta_1 = 2p_1 \ldots p_{s_1} q_1 \ldots q_{t_1}$ with $t_1 \equiv t \pmod 2$ and so $d_1 = 4\Delta_1$. Then $d_2 = d/d_1 = \Delta/\Delta_1 = \Delta_2$.

The argument of Case 2 now applies and we see once again that $\delta_1, \delta_2, \delta_1\delta_2 \notin K^2$.

Thus by the lemma we see $\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}, \sqrt{\varepsilon_1\varepsilon_2} \notin K$.

Now by Kubota [**6**, Satz 1] and by our claim we have the following possibilities for a system of fundamental units in E_K:

(i) $\varepsilon_0, \varepsilon_1, \varepsilon_2$ (ii) $\sqrt{\varepsilon_0}, \varepsilon_1, \varepsilon_2$

(iv) $\sqrt{\varepsilon_0\varepsilon_1}, \varepsilon_i, \varepsilon_2$ ($i = 0$ or 1) or $\sqrt{\varepsilon_0\varepsilon_2}, \varepsilon_1, \varepsilon_i$ ($i = 0$ or 2)

(vii) $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2}, \varepsilon_i, \varepsilon_j$ (i or $j \in \{1, 2\}$).

From this list it follows that if $\sqrt{\varepsilon_0\varepsilon_1^a\varepsilon_2^b} \in K$, then $[E_k : N_{K/k}(E_K)] = 2$, whereas if not, then we are in case (i) in which case $[E_K : N_{K/k}(E_K)] = 4$.

This establishes the proposition.

We summarize our results in the following theorem.

THEOREM. *Let K be an unramified quadratic extension of a real quadratic number field k. Then*

(1) $|\ker j| = 2 \Leftrightarrow$ (a) $N\varepsilon_i = -1$ *for* $i = 0$, 1, 2 *and* $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in K$ *or* (b) ($N\varepsilon_1 = -1$ *or* $N\varepsilon_2 = -1$) *and* $N\varepsilon_0 = 1$ *and* ($\sqrt{\varepsilon_0}$ *or* $\sqrt{\varepsilon_0\varepsilon_1}$ *or* $\sqrt{\varepsilon_0\varepsilon_2} \in K$).

(2) $|\ker j| = 8 \Leftrightarrow$ (a) $N\varepsilon_1 = N\varepsilon_2 = 1$ *and* $N\varepsilon_0 = -1$ *or* (b) $N\varepsilon_i = 1$ *for* $i = 0$, 1, 2 *and* $\sqrt{\varepsilon_0\varepsilon_1^a\varepsilon_2^b} \notin K$ *for all* $a, b \in \{0, 1\}$.

(3) $|\ker j| = 4 \Leftrightarrow$ *anything else occurs.*

PROPOSITION 4. *There exist infinitely many real quadratic fields k for which there exists an unramified quadratic extension K in which* $|\ker j| = 2$, 4, *and* 8, *respectively.*

Proof. First consider $|\ker j| = 2$. Let $k = \mathbb{Q}(\sqrt{p_1p_2})$ where $p_i \equiv 1 \pmod 4$. Then by genus theory, $C_{k,2}$ is cyclic and nontrivial. Since the $\ker j$ is a nontrivial elementary subgroup of $C_{k,2}$, it follows that $|\ker j| = 2$. Obviously there are infinitely many such fields k.

Next consider $|\ker j| = 4$. Let $k = \mathbb{Q}(\sqrt{p_1p_2p_3})$, where $p_i \equiv 1 \pmod 4$. Then by genus theory, $C_{k,2}$ has 2-rank equalling 2 and so $|\ker j| = 2$ or 4 since $\ker j$ is nontrivial and

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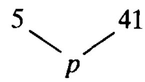
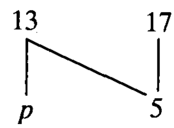
elementary. Hence by our theorem, if we can choose p_1, p_2, p_3 such that $N\varepsilon_0 = -1$ and $N\varepsilon_1 = 1$, then $|\ker j| = 4$. To this end choose $k = \mathbb{Q}(\sqrt{(5)(41)p}$ where $p \equiv 3 \pmod{205}$ and $p \equiv 1 \pmod 4$. Then we claim if $K = k(\sqrt{p})$, then $|\ker j| = 4$. For let $k_1 = \mathbb{Q}(\sqrt{205})$. Then $N\varepsilon_1 = 1$. Moreover the graph $\gamma(5, 41)$ is



since for $p \equiv 3 \pmod{205}$, $\left(\frac{p}{5}\right) = \left(\frac{p}{41}\right) = -\left(\frac{41}{5}\right) = -1$. (See [**1**] for the relevant definitions about graphs.) Thus, since $\gamma(5, 41, p)$ is odd, Proposition 1.1 of [**1**] implies $N\varepsilon_0 = -1$. There are obviously infinitely many such k.

Finally consider $|\ker j| = 8$. Let $k = \mathbb{Q}(\sqrt{p_1p_2p_3p_4})$, $p_i \equiv 1 \pmod 4$. If we are able to choose $K = \mathbb{Q}(\sqrt{p_1p_2}, \sqrt{p_3p_4})$ with $k_1 = \mathbb{Q}(\sqrt{p_1p_2})$ and $k_2 = \mathbb{Q}(\sqrt{p_3p_4})$ such that $N\varepsilon_0 = -1$ and $N\varepsilon_1 = N\varepsilon_2 = 1$, then our theorem implies that $|\ker j| = 8$. We begin by letting $p_1 = 13$, $p_2 = 17$, $p_3 = 5$, $p_4 = p$ such that $p \equiv 2 \pmod{13 \times 17}$ and $p \equiv 1 \pmod 5$. Then the graph $\gamma(13, 17, 5, p)$ is



since $\left(\frac{13}{17}\right) = \left(\frac{p}{5}\right) = \left(\frac{p}{17}\right) = -\left(\frac{p}{13}\right) = -\left(\frac{13}{5}\right) = 1$. Notice that this graph is odd and thus $N\varepsilon_0 = -1$. Moreover, $N\varepsilon_1 = 1$. We now need to put additional restrictions on p to insure that $N\varepsilon_2 = 1$. To this end, write $p = \pi\bar{\pi}$ in $\mathbb{Z}[i]$ with π, $\bar{\pi}$ prime and π primary, i.e. $\pi \equiv 1 (\mathrm{mod}(1+i)^3)$. Choose such a prime π in $\mathbb{Z}[i]$ such that

$$\pi \equiv 1\ (\mathrm{mod}(1+i)^3),\ \pi \equiv i\ (\mathrm{mod}\ 1+2i),\ \pi \equiv i\ (\mathrm{mod}\ 1-2i),$$

$$\pi \equiv 1 + i \pmod{13},\ \pi \equiv 1 + i \pmod{17}.$$

The last two congruences imply $p \equiv 2 \pmod{13 \times 17}$ whereas the first three show that the biquadratic residues

$$\left(\frac{\pi}{1+2i}\right)_4 = \left(\frac{\pi}{1-2i}\right)_4 = i.$$

By (2.2) of [**8**], this implies that $N\varepsilon_2 = 1$. Moreover since the righthand sides of the above congruences determine a ray class modulo the ideal $(1+i)(2210)$ in $\mathbb{Z}[i]$, we conclude by class field theory that there are infinitely many primes π of residue class degree one over $\mathbb{Q}$ satisfying the congruences. Hence there are infinitely many p such that $p \equiv 2 \pmod{13 \times 17}$, $p \equiv 1 \pmod 5$ and such that $N\varepsilon_2 = 1$.

This proves the proposition.

3. Examples. In this section we present examples of K/k in which 2, 8, and 4 ideal classes capitulate, respectively. We follow the format of our theorem.

1. $|\ker j| = 2$.

(a) Let $k = \mathbb{Q}(\sqrt{d})$ where $d = 65 = (5)(13)$. Let $K = k(\sqrt{5}) = \mathbb{Q}(\sqrt{5}, \sqrt{13})$. Let $k_i = \mathbb{Q}(\sqrt{d_i})$ $(i = 1, 2)$ with $d_1 = 5$ and $d_2 = 13$. Then

$$\varepsilon = \varepsilon_0 = 8 + \sqrt{65}, \qquad N\varepsilon_0 = -1,$$

$$\varepsilon_1 = \frac{1 + \sqrt{5}}{2}, \qquad N\varepsilon_1 = -1,$$

$$\varepsilon_2 = \frac{3 + \sqrt{13}}{2}, \qquad N\varepsilon_2 = -1.$$

Set

$$c_0 = Tr_{K/\mathbb{Q}}(\varepsilon_0\varepsilon_1\varepsilon_2 + \varepsilon_0 + \varepsilon_1 - \varepsilon_2) = 117,$$

$$c_1 = Tr_{K/\mathbb{Q}}(\varepsilon_0\varepsilon_1\varepsilon_2 + \varepsilon_0 - \varepsilon_1 + \varepsilon_2) = 125,$$

$$c_2 = Tr_{K/\mathbb{Q}}(\varepsilon_0\varepsilon_1\varepsilon_2 - \varepsilon_0 + \varepsilon_1 + \varepsilon_2) = 65,$$

$$c_3 = Tr_{K/\mathbb{Q}}(\varepsilon_0\varepsilon_1\varepsilon_2 - \varepsilon_0 - \varepsilon_1 - \varepsilon_2) = 49.$$

Since $\sqrt{c_j} \in K$ for $j = 0, \ldots, 3$, we have $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in K$ by [**6**, Zusatz 1].

(b) Let $k = \mathbb{Q}(\sqrt{d})$ where $d = 105 = (5)(3)(7)$. Let $K = k(\sqrt{5}) = \mathbb{Q}(\sqrt{5}, \sqrt{21})$. Let $k_i = \mathbb{Q}(\sqrt{d_i})$ $(i = 1, 2)$ with $d_1 = 5$ and $d_2 = 21$. Then

$$\varepsilon = \varepsilon_0 = 41 + 4\sqrt{105}, \qquad N\varepsilon_0 = 1,$$

$$\varepsilon_1 = \frac{1 + \sqrt{5}}{2}, \qquad N\varepsilon_1 = -1,$$

$$\varepsilon_2 = \frac{5 + \sqrt{21}}{2}, \qquad N\varepsilon_2 = 1.$$

Moreover $\varepsilon + \varepsilon' + 2 = 84 = (2^2)(21)$. Thus $\delta(\varepsilon) = 21 \in K^2$.

REMARK. In these two examples, $C_{k,2}$ is cyclic and thus since $\ker j$ is elementary, we see independently that $|\ker j| = 2$.

2. $|\ker j| = 8$.

(a) Let $k = \mathbb{Q}(\sqrt{d})$ with $d = 77285 = (5)(13)(29)(41)$. Let $K = k(\sqrt{205}) = \mathbb{Q}(\sqrt{(5)(41)}, \sqrt{(13)(29)})$. Let $k_i = \mathbb{Q}(\sqrt{d_i})(i = 1, 2)$ with $d_1 = 205 = (5)(41)$ and $d_2 = 377 = (13)(29)$. Then

$$\varepsilon = \varepsilon_0 = 278 + \sqrt{77285}, \qquad N\varepsilon_0 = -1,$$

$$\varepsilon_1 = \frac{43 + 3\sqrt{205}}{2}, \qquad N\varepsilon_1 = 1,$$

$$\varepsilon_2 = 233 + 12\sqrt{377}, \qquad N\varepsilon_2 = 1.$$

(b) Let $k = \mathbb{Q}(\sqrt{d})$, with $d = 23205 = (3)(7)(5)(13)(17)$. Let $K = k(\sqrt{105}) = \mathbb{Q}(\sqrt{(3)(7)(5)}, \sqrt{(13)(17)})$. Let $k_i = \mathbb{Q}(\sqrt{(d_i)}$ $(i = 1, 2)$, with $d_1 = 105 = (3)(7)(5)$ and

$d_2 = 221 = (13)(17)$. Then

$$\varepsilon = \varepsilon_0 = \frac{457 + 3\sqrt{23205}}{2}, \qquad N\varepsilon_0 = 1,$$

$$\varepsilon_1 = 41 + 4\sqrt{105}, \qquad N\varepsilon_1 = 1,$$

$$\varepsilon_2 = \frac{15 + \sqrt{221}}{2}, \qquad N\varepsilon_2 = 1.$$

Moreover

$$\varepsilon + \varepsilon' + 2 = 459 = (3^2)(3)(17), \text{ implying } \delta_0 = \delta(\varepsilon) = (3)(17),$$

$$\varepsilon_1 + \varepsilon_1' + 2 = 84 = (2^2)(3)(7), \text{ implying } \delta_1 = \delta(\varepsilon_1) = (3)(7),$$

$$\varepsilon_2 + \varepsilon_2' + 2 = 17, \text{ implying } \delta_2 = \delta(\varepsilon_2) = 17.$$

Notice that $\delta_0, \delta_0\delta_1, \delta_0\delta_2, \delta_0\delta_1\delta_2 \notin K^2$.

3. $|\ker j| = 4$.

(i) Let $k = \mathbb{Q}(\sqrt{d})$, with $d = 77285 = (5)(13)(29)(41)$ (as in 2.a). Let $K = k(\sqrt{1885}) = \mathbb{Q}(\sqrt{(5)(13)(29)}, \sqrt{41})$. Let $k_i = \mathbb{Q}(\sqrt{d_i})$ $(i = 1, 2)$, with $d_1 = 1885 = (5)(13)(29)$ and $d_2 = 41$. Then

$$\varepsilon = \varepsilon_0 = 278 + \sqrt{77285}, \qquad N\varepsilon_0 = -1,$$

$$\varepsilon_1 = 521 + 12\sqrt{1885}, \qquad N\varepsilon_1 = 1,$$

$$\varepsilon_2 = 32 + 5\sqrt{41}, \qquad N\varepsilon_2 = -1.$$

By Proposition 2, $|\ker j| = 4$.

(ii) Let $k = \mathbb{Q}(\sqrt{d})$, with $d = 4641 = (3)(7)(13)(17)$. Let $K = k(\sqrt{21}) = \mathbb{Q}(\sqrt{(3)(7)}, \sqrt{(13)(17)})$. Let $k_i = \mathbb{Q}(\sqrt{d_i})$ $(i = 1, 2)$, with $d_1 = 21$ and $d_2 = 221 = (13)(17)$. Then

$$\varepsilon = \varepsilon_0 = 545 + 8\sqrt{4641}, \qquad N\varepsilon_0 = 1,$$

$$\varepsilon_1 = \frac{5 + \sqrt{21}}{2}, \qquad N\varepsilon_1 = 1,$$

$$\varepsilon_2 = \frac{15 + \sqrt{221}}{2}, \qquad N\varepsilon_2 = 1.$$

Moreover $\varepsilon + \varepsilon' + 2 = 1092 = (2^2)(3)(7)(13)$, implying $\delta_0 = \delta(\varepsilon) = (3)(7)(13)$,

$$\varepsilon_1 + \varepsilon_1' + 2 = 7, \text{ implying } \delta_1 = \delta(\varepsilon_1) = 7,$$

$$\varepsilon_2 + \varepsilon_2' + 2 = 17, \text{ implying } \delta_2 = \delta(\varepsilon_2) = 17.$$

Since $\delta_0\delta_2 \in K^2$, Proposition 3 shows $|\ker j| = 4$.

Acknowledgements. The authors would like to thank Professor R. W. K. Odoni for suggesting the results in Proposition 4.

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