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MAXIMAL HOMOTOPY LIE SUBGROUPS OF MAXIMAL RANK

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Introduction. Let G be a compact connected Lie group with H a connected subgroup of maximal rank. Suppose there exists a compact connected Lie subgroup K with $H \subset K \subset G$. Then there exists a smooth fiber bundle $G/H \rightarrow G/K$ with K/H as the fiber. (See for example [**13**].) This can be incorporated into a diagram involving the classifying spaces as follows:

$$(1)\qquad \begin{tikzcd} K/H \arrow[d] & K/H \arrow[d] & \\ G/H \arrow[r] \arrow[d, "\pi"] & BH \arrow[r, "\phi"] \arrow[d, "\phi_1"] & BG \arrow[d] \\ G/K \arrow[r] & BK \arrow[r, "\phi_2"] & BG \end{tikzcd}$$

Here π, ϕ, ϕ_1, and ϕ_2 denote fibrations. We also know that the homogeneous spaces and the Lie groups, which are homotopy equivalent to the loop spaces of their respective classifying spaces, are homotopy equivalent to connected finite complexes.

Now suppose H is a maximal subgroup. Can there still exist spaces, which we will call BK, K/H, and G/K, and fibrations so that diagram (1) is still valid? This paper will show that in many cases either G/K or K/H will be homotopically trivial.

For some cases an answer has already been found. It has been shown for some instances of G and H that there are no nontrivial fibrations $F \rightarrow E \rightarrow B$ with E homotopy equivalent to G/H and F and B homotopy equivalent to finite complexes. See for example [**1**, **7**, **12**]. This suggests the following hypothesis.

CONJECTURE. *Let G be a compact connected Lie group with H a connected maximal subgroup of maximal rank. Consider the fibration $BH \rightarrow BG$ corresponding to the inclusion $H \subset G$. Suppose there exists a space BK such that the following hold*:

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1) *ΩBK is homotopy equivalent to a finite connected CW complex K.*

2) *There exists a sequence of fibrations*

$$BH \xrightarrow{\phi_1} BK \xrightarrow{\phi_2} BG$$

with $\phi_2\phi_1$ homotopic to ϕ.

3) *The fibrations above all have fibers which are homotopy equivalent to connected finite complexes.*

Then K is homotopy equivalent to either H or G.

In this paper it will be shown that the conjecture is true for a large number of cases.

Let $T \subset H$ be a maximal torus and let

$$BT \xrightarrow{\psi} BH$$

denote the fibration corresponding to the inclusion. Then, given the conditions of the conjecture, we can conclude that the fiber of $BT \to BK$ is also homotopy equivalent to a finite complex. (See [**11**].)

For the sake of convenience, the fibers of $BT \to BK$, $BH \to BK$, and $BK \to BG$ will be denoted K/T, K/H, and G/K respectively.

If a space BK satisfies the conditions of the conjecture, then we can construct a diagram similar to (1) and thus we get the expected fibration of the "homogeneous spaces". In this paper we will use the following strategy which was set down in [**12**].

a) Since G/T is a simply connected Poincaré-Wall complex with positive Euler characteristic, then we know from [**10**, **12**] that K/T, G/K, and K/H also have these properties. (See also [**11**].)

b) To prove the conjecture, we will try to show that $H^+(K/H, k) = 0$ (respectively $H^+(G/K, k) = 0$) for some field k. Then it follows from (a) above and [**16**] that K/H (resp. G/K) is contractible and the conjecture follows from the homotopy exact sequence of

$$H \xrightarrow{\Omega\phi_1} K \to K/H$$

(resp. $K \xrightarrow{\Omega\phi_2} G \to G/K$).

1. The main theorem. It might be prudent to recall here some properties of Lie groups and Weyl groups. Let G be a compact connected Lie group of rank n and let T denote a maximal torus of G. Then the Weyl group of G is $W_G = N(T)/T$ where $N(T)$ is the normalizer of T. W_G can be represented as a group of symmetries of $\mathbf{R}^n$ generated by reflections. (See for example [**6**].) W_G also acts on $H^*(BT, \mathbf{Z}_p)$ as a pseudo-reflection group, the pseudo-reflections corresponding to the reflections in $\mathbf{R}^n$. Here $\mathbf{Z}_p$ denotes the integers modulo a prime p. Furthermore, given the fibration

$$G/T \to BT \xrightarrow{i} BG,$$

if G has no p torsion we know (see **[5]**)

$$i^*: H^*(BG, \mathbf{Z}_p) \approx H^*(BT, \mathbf{Z}_p)^{W_G}.$$

Let BK be a space which satisfies the conditions of the conjecture. Since K is homotopy equivalent to an H-space $H^*(K, \mathbf{Z}_p)$ is a Hopf algebra. Hence there exists an m such that, for p large enough,

$$H^*(K, \mathbf{Z}_p) = \wedge(x_{2r_1-1}, x_{2r_2-1}, \ldots, x_{2r_m-1})$$

where $\dim x_{2r_i-1} = 2r_i - 1$. Consequently,

$$H^*(BK, \mathbf{Z}_p) = \mathbf{Z}_p[y_{2r_1}, y_{2r_2}, \ldots, y_{2r_m}]$$

where, without loss of generality, y_{2r_i} is the image of x_{2r_i-1} by transgression. A similar statement is true for cohomology with rational coefficients. Here m is called the rank of K. By **[11]** we have

$$\operatorname{rank} H \leqq \operatorname{rank} K \leqq \operatorname{rank} G.$$

Therefore, $m = n$ where n is the dimension of the maximal torus T. This enables us to use Theorem 2.2 of **[11]** to see that, for all p large enough (at least $r_1 r_2 \ldots r_m$),

$$H^*(BK, k) \to H^*(BT, k)$$

is monic for $k = \mathbf{Z}_p$ or $\mathbf{Q}$ and $H^*(K/T, k)$ is evenly graded. From **[17]** we have

$$(\phi_1\psi)^* H^*(BK, \mathbf{Z}_p) \approx H^*(BT, \mathbf{Z}_p)^{W_{K^p}}$$

where W_{K^p} is a pseudo-reflection group acting on $H^2(BT, \mathbf{Z}_p)$. Call W_{K^p} the (mod p) "formal Weyl group" of K.

Consider the sequence of fibrations

$$BT \to BH \to BK.$$

Let W_H be the Weyl group of H. Since W_H acting on $H^*(BT, \mathbf{Z}_p)$ leaves the image of ψ^* fixed, it leaves the image of $H^*(BK, \mathbf{Z}_p)$ invariant. Define W in $\operatorname{Aut} H^2(BT, \mathbf{Z}_p)$ as the group generated by the pseudo-reflections of W_H and W_{K^p}. Then

$$(\phi_1\psi)^* H^*(BK, \mathbf{Z}_p) = H^*(BT, \mathbf{Z}_p)^W$$

and clearly W_H and W_{K^p} are subgroups of W. But, by **[8]**, we have

$$|W| = r_1 r_2 \ldots r_n = |W_{K^p}|.$$

Therefore, $W = W_{K^p}$ and $W_H \subseteq W_{K^p}$. If $W_H = W_{K^p}$ then ϕ_1^* would be an isomorphism and

$$H^*(K/H, \mathbf{Z}_p) = H^0(K/H, \mathbf{Z}_p) \approx \mathbf{Z}_p.$$

So $W_H \neq W_{K^p}$ and, similarly, $W_{K^p} \neq W_G$. Thus, we have proven the following:

THEOREM 1.1. *Let H, G, and K satisfy the conditions of the conjecture. Then the rank of K equals the rank of G. Furthermore, if K is not homotopy equivalent to H or G then, for p large enough, the formal Weyl group of K is a proper subgroup of the Weyl group of G and contains the Weyl group of H as a proper subgroup.*

2. Applications of the main theorem. All of the pairs of H and G satisfying the hypotheses of the conjecture are listed in [**6**]. We can use the results of the previous chapter to prove the conjecture for most of those pairs. Specifically, we will show that the conjecture is true for the following sublist of cases.

Weyl groups of Lie groups and their subgroups
for which the conjecture is true

	W_G	W_H
a)	A_n	$A_i \times A_{n-i-1}$
b)	B_n	D_n
c)	C_n	$C_i \times C_{n-i}$
d)	D_n	$D_i \times D_{n-i}$
e)	D_n	D_{n-1}
f)	D_n	A_{n-1}
g)	E_6	$A_1 \times A_5$
h)	E_6	D_5
i)	E_6	$A_2 \times A_2 \times A_2$
j)	E_7	$A_1 \times D_6$
k)	E_7	A_7
l)	E_7	$A_2 \times A_5$
m)	E_8	D_8
n)	E_8	$A_1 \times E_7$
o)	E_8	A_8
p)	E_8	$A_2 \times E_6$
q)	E_8	$A_4 \times A_4$
r)	F_4	$A_2 \times A_2$
s)	F_4	B_4
t)	G_2	$A_1 \times A_1$
u)	G_2	A_2

Note. Cases (s) and (u) were covered in [**12**] with stronger results; therefore, the arguments for these will be omitted here.

The representations of the Weyl groups used here are taken from [**2**]. For the rest of the chapter the superscript "p" in W_{K^p} is unnecessary and will be suppressed.

Classical cases.

a) $W_G = A_n$, $W_H = A_i \times A_{n-i-1}$.

Let $\mathbf{R}^{n+1}$ have the orthonormal basis $\{e_0, e_1, \ldots, e_n\}$. Then A_n has a representation such that the reflections correspond to transpositions of pairs of the e_i's. Therefore, the representation of W_K is also generated by such transpositions; hence, W_K is isomorphic to a product of A_j's. Consequently, if $A_i \times A_{n-i-1} = W_K$ we get $W_H = W_K$ or $W_K = W_G$.

b) $W_G = B_n$, $W_H = D_n$.

With $\mathbf{R}^n$ having the orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ the roots of a representation of B_n are $\{\pm e_j, \pm e_j \pm e_k\}$; those of D_n are $\{\pm e_j \pm e_k\}$, $j \neq k$. If $D_n \neq W_K$ then another root of B_n is a root of W_K. Without loss of generality let e_1 be such a root. Since reflections in Coxeter groups send roots to roots then

$$e_1 - 2[\langle e_1, e_1 - e_j\rangle/\|e_1 - e_j\|^2](e_1 - e_j) = e_j$$

is also a root of W_K. So the roots of W_K are $\{\pm e_i, e_i \pm e_j\}$ and $W_K = B_n$.

c) $W_G = C_n$, $W_H = C_i \times C_{n-i}$.

Since C_n is isomorphic to B_n, the representation in (b) above will be used. $C_i \times C_{n-i}$ then has roots $\{\pm e_j, \pm e_j \pm e_k\}$, $j \neq k$, with $j \leqq i$ if and only if $k \leqq i$. So any root r_0 in W_K not in W_H must be of the form $\pm(e_{j_0} + e_{k_0})$ or $\pm(e_{j_0} - e_{k_0})$ with $j_0 \leqq i$ and $k_0 > i$.

Case 1). $r_0 = \pm(e_{j_0} + e_{k_0})$. Let S_r denote the reflection along the root r. Then other roots of W_K are, for all $j \leqq i$, $k > i$

$$\begin{aligned}
&S_{(e_j - e_{j_0})} S_{(e_k - e_{k_0})}(e_{j_0} + e_{k_0}) \\
&= S_{(e_j - e_{j_0})}[\,(e_{j_0} + e_{k_0}) - \langle e_k - e_{k_0}, e_{j_0} + e_{k_0}\rangle(e_k - e_{k_0})\,] \\
&= S_{(e_j - e_{j_0})}[\,(e_{j_0} + e_{k_0}) + (e_k - e_{k_0})\,] \\
&= S_{(e_j - e_{j_0})}(e_{j_0} + e_k) \\
&= (e_{j_0} + e_k) - \langle e_j - e_{j_0}, e_{j_0} + e_k\rangle(e_j - e_{j_0}) \\
&= (e_{j_0} + e_k) + (e_j - e_{j_0}) \\
&= e_j + e_k
\end{aligned}$$

and

$$\begin{aligned}
&S_{(e_j - e_{j_0})} S_{(e_k + e_{k_0})}(e_{j_0} + e_{k_0}) \\
&= S_{(e_j - e_{j_0})}[\,(e_{j_0} + e_{k_0}) - \langle e_k + e_{k_0}, e_{j_0} + e_{k_0}\rangle(e_k + e_{k_0})\,] \\
&= S_{(e_j - e_{j_0})}[\,(e_{j_0} + e_{k_0}) - (e_k - e_{k_0})\,] \\
&= S_{(e_j - e_{j_0})}(e_{j_0} - e_k) \\
&= (e_{j_0} - e_k) - \langle e_j - e_{j_0}, e_{j_0} - e_k\rangle(e_j - e_{j_0})
\end{aligned}$$

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$$= (e_{j_0} - e_k) + (e_j - e_{j_0})$$
$$= e_j - e_k.$$

Hence $W_K = W_G$.

Case 2). $r_0 = \pm(e_{j_0} - e_{k_0})$. As in Case 1) above, other roots for W_K are, for all $j \leqq i$, $k > i$

$$S_{(e_j - e_{j_0})}S_{(e_k - e_{k_0})}(e_{j_0} - e_{k_0})$$
$$= S_{(e_j - e_{j_0})}[\,(e_{j_0} - e_{k_0}) - \langle e_k - e_{k_0}, e_{j_0} - e_{k_0}\rangle(e_k - e_{k_0})\,]$$
$$= S_{(e_j - e_{j_0})}[\,(e_{j_0} - e_{k_0}) - (e_k - e_{k_0})\,]$$
$$= S_{(e_j - e_{j_0})}(e_{j_0} - e_k)$$
$$= (e_{j_0} - e_k) - \langle e_j - e_{j_0}, e_{j_0} - e_k\rangle(e_j - e_{j_0})$$
$$= (e_{j_0} - e_k) + (e_j - e_{j_0})$$
$$= e_j - e_k$$

and

$$S_{(e_j - e_{j_0})}S_{(e_k + e_{k_0})}(e_{j_0} - e_{k_0})$$
$$= S_{(e_j - e_{j_0})}[\,(e_{j_0} - e_{k_0}) - \langle e_k + e_{k_0}, e_{j_0} - e_{k_0}\rangle(e_k + e_{k_0})\,]$$
$$= S_{(e_j - e_{j_0})}(e_{j_0} + e_k)$$
$$= e_j - e_k.$$

Hence $W_K = W_G$ once again.

d) $W_G = D_n$, $W_H = D_i \times D_{n-i}$.

Recall that D_n has roots $\{\pm e_j \pm e_k\}$, $1 \leqq j < k \leqq n$ and $D_i \times D_{n-i}$ has roots $\{\pm e_j \pm e_k\}$ with $1 \leqq j < k \leqq i$ or $i + 1 \leqq j < k \leqq n$. From the calculations in (c) above we see that there is no intermediate Coxeter subgroup between W_H and W_G.

e) $W_G = D_n$, $W_H = D_{n-1}$.

D_{n-1} has all of the roots of D_n except those of the form $\pm e_i \pm e_n$. Again the calculations in (c) verify the conjecture for this case.

f) $W_G = D_n$, $W_H = A_{n-1}$.

A_{n-1} has $\{\pm(e_j - e_k)\}$ for its roots. The first calculation in part (c), Case 2, above shows that if W_K has any root in W_G not in W_H then $W_K = W_G$.

Exceptional cases. Unfortunately, the Weyl groups of the exceptional Lie groups do not have such "nice" representations. So for the following cases, I find it easier to use Poincaré polynomials.

Note that if $H^*(BG, \mathbf{Q})$ has type $[2i_1, 2i_2, \dots, 2i_n]$ and $H^*(BK, \mathbf{Q})$ has type $[2j_1, 2j_2, \dots, 2j_n]$ then

$$P_0(G/K, t) = \frac{(1 - t^{2i_1}) \dots (1 - t^{2i_n})}{(1 - t^{2j_1}) \dots (1 - t^{2j_n})}.$$

The strategy now will be to determine possible candidates for W_K and to show that for each candidate $P_0(K/H, t)$ or $P_0(G/K, t)$ is not a finite polynomial.

For example, consider $W_G = E_j (j = 6, 7, \text{ or } 8)$. If r_1 and r_2 are two roots of the representation of E_i as a Coxeter group then the value of $\langle r_1, r_2 \rangle / \|r_1\| \, \|r_2\|$ is either 0, ± 1, or $\pm 1/2$. Therefore, W_K must be a product of A_i's, D_j's, E_K's, and the trivial group.

g) $W_G = E_6$, $W_H = A_1 \times A_5$.

h) $W_G = E_6$, $W_H = D_5$.

$H^*(BE_6, \mathbf{Q})$ has type [4, 10, 12, 16, 18, 24]. The candidates for W_K are listed below along with corresponding types for $H^*(BK, \mathbf{Q})$, and the appropriate polynomials.

W_K	$H^*(BK, \mathbf{Q})$ type	Poincaré polynomial
$A_1 \times D_5$	[4, 4, 8, 12, 16, 10]	$P_0(G/K, t) = \dfrac{(1 - t^{18})(1 - t^{24})}{(1 - t^4)(1 - t^8)}$
A_6	[4, 6, 8, 10, 12, 14]	$P_0(G/K, t) = \dfrac{(1 - t^{16})(1 - t^{18})(1 - t^{24})}{(1 - t^6)(1 - t^8)(1 - t^{14})}$
D_6	[4, 8, 12, 16, 20, 12]	$P_0(G/K, t) = \dfrac{(1 - t^{10})(1 - t^{18})(1 - t^{24})}{(1 - t^8)(1 - t^{20})(1 - t^{12})}$

By substituting $t = e^{\pi i/2}$ in the first and third polynomials and $t = e^{\pi i/7}$ in the second we see that none of them are finite polynomials.

i) $W_G = E_6$, $W_H = A_2 \times A_2 \times A_2$.

Here $H^*(BH, \mathbf{Q})$ has type [4, 4, 4, 6, 6, 6]. Below we list possible candidates for W_K, the corresponding types of $H^*(BK, \mathbf{Q})$, and the appropriate Poincaré polynomials.

W_K	$H^*(BK, \mathbf{Q})$ type	Poincaré polynomial
$A_2 \times A_4$	[4, 4, 6, 6, 8, 10]	$P_0(K/H, t) = \dfrac{(1 - t^8)(1 - t^{10})}{(1 - t^4)(1 - t^6)}$
$A_2 \times D_4$	[4, 4, 6, 8, 8, 12]	$P_0(K/H, t) = \dfrac{(1 - t^8)(1 - t^8)(1 - t^{10})}{(1 - t^4)(1 - t^6)(1 - t^6)}$

By substituting $t = e^{\pi i/3}$ we see that neither is the Poincaré polynomial of a finite complex.

j) $W_G = E_7$, $W_H = A_1 \times D_6$.

It is known that $H^*(BG, \mathbf{Q})$ and $H^*(BH, \mathbf{Q})$ have respective types [4, 12, 16, 20, 24, 28, 36] and [4, 4, 8, 12, 16, 20, 12].

W_K	$H^*(BK, \mathbf{Q})$ type	Poincaré polynomial
$A_1 \times E_6$	[4, 4, 10, 12, 16, 18, 24]	$P_0(K/H, t) = \dfrac{(1 - t^{10})(1 - t^{18})(1 - t^{24})}{(1 - t^8)(1 - t^{20})(1 - t^{12})}$
A_7	[4, 6, 8, 10, 12, 14, 16]	$P_0(K/H, t) = \dfrac{(1 - t^6)(1 - t^{10})(1 - t^{14})}{(1 - t^4)(1 - t^{12})(1 - t^{20})}$
D_7	[4, 8, 12, 16, 20, 24, 14]	$P_0(G/K, t) = \dfrac{(1 - t^{28})(1 - t^{36})}{(1 - t^8)(1 - t^{14})}$

Note that in the first two cases $P_0(K/H, e^{\pi i/2})$ is not defined while in the third case $P_0(G/K, e^{\pi i/4})$ doesn't exist.

k) $W_G = E_7$, $W_H = A_7$.
For this the only candidate might be $W_K = D_7$, but this was eliminated in part (j) above.

l) $W_G = E_7$, $W_H = A_2 \times A_5$.
As we have seen, D_7 cannot be a candidate for W_K. A_7 can't be used either since this would give

$$P_0(K/H, t) = \frac{(1 - t^{14})(1 + t^{16})}{(1 - t^4)(1 - t^6)}$$

which is undefined for $t = e^{\pi i/3}$.

m) $W_G = E_8$, $W_H = D_8$.
If $W_K = A_8$ then

$$P_0(K/H, t) = \frac{(1 - t^6)(1 - t^{10})(1 - t^{14})(1 - t^{18})}{(1 - t^{20})(1 - t^{24})(1 - t^{28})(1 - t^{16})}$$

which is obviously not finite.

For the next four cases the only candidate for W_K is D_8, which would give $H^*(BK, \mathbf{Q})$ a type [4, 8, 12, 16, 20, 24, 28, 16]. But in each instance $P_0(K/H, t)$ is undefined for some value of t, hence is incompatible with a finite complex.

n) $W_G = E_8$, $W_H = A_1 \times E_7$.
For this

$$P_0(K/H, t) = \frac{(1 - t^8)(1 - t^{16})}{(1 - t^4)(1 - t^{36})}$$

which is undefined for $t = e^{\pi i/3}$.

o) $W_G = E_8$, $W_H = A_8$.
Here $t = e^{\pi i/3}$ shows that

$$P_0(K/H, t) = \frac{(1 - t^{20})(1 - t^{24})(1 - t^{28})(1 - t^{32})}{(1 - t^6)(1 - t^{10})(1 - t^{14})(1 - t^{18})}$$

isn't a finite polynomial.

p) $W_G = E_8$, $W_H = A_2 \times E_6$.
Here

$$P_0(K/H, t) = \frac{(1 - t^8)(1 - t^{16})(1 - t^{20})(1 - t^{28})}{(1 - t^4)(1 - t^6)(1 - t^{10})(1 - t^{18})}$$

which is also undefined for $t = e^{\pi i/3}$.

q) $W_G = E_8$, $W_H = A_4 \times A_4$.
For this case $t = e^{\pi i/5}$ doesn't return a value for

$$P_0(K/H, t)$$
$$= \frac{(1 - t^{12})(1 - t^{16})(1 - t^{20})(1 - t^{24})(1 - t^{28})(1 - t^{16})}{(1 - t^6)(1 - t^{10})(1 - t^4)(1 - t^6)(1 - t^8)(1 - t^{10})}$$

r) $W_G = F_4$, $W_H = A_2 \times A_2$.
The only possible candidate for W_K here might be D_4 but a quick scan of the root system shows that this doesn't contain W_H as a subgroup.

t) $W_G = G_2$, $W_H = A_1 \times A_1$.
This case yields stronger results. Since $H^*(BG_2, \mathbf{Q})$ has type [4, 12] and $H^*(BH, \mathbf{Q})$ has type [4, 4] then

$$P_0(G/H, t) = \frac{(1 - t^{12})}{(1 - t^4)}.$$

Let $F \to G/H \to B$ be a compact fibering of G/H with F connected. Then since

$$P_0(G/H, t) = P_0(F, t)P_0(B, t)$$

we get

$$P_0(B, t) = \frac{(1 - t^{12})}{(1 - t^a)}$$

and

$$P_0(F, t) = \frac{(1 - t^a)}{(1 - t^4)}$$

for some a. This implies $4|a$ and $a|12$; hence, a is equal to 4 or 12. Therefore, G/H is connectedwise prime.

3. Other cases. Unfortunately, it is possible for H to be a maximal subgroup of G of maximal rank without W_H being a proper reflection subgroup of W_G. These cases include, among others, $W_G = C_n$ with

$W_H = A_{n-1}$ and $W_G = B_n$ with $W_H = B_{n-1}$ or $B_1 \times D_{n-1}$. However, the conjecture still remains valid here for some specific cases of G or H.

a) $G = Sp(n)$, $H = U(n)$.

Here $W_G = C_n$ and $W_H = A_{n-1}$. If K is not homotopy equivalent to H or G then $W_K = D_n$. So, in cohomology with rational coefficients, we have $H^*(BT)$ generated by $x_1, \ldots, x_n$ with dim $x_1 = 2$ and

$H^*(BSp(n))$ maps isomorphically onto $S(x_1^2, \ldots, x_n^2)$;

$H^*(BU(n))$ maps isomorphically onto $S(x_1, \ldots, x_n)$; and

$H^*(BK)$ maps isomorphically onto the ring generated by $S(x_1^2, \ldots, x_n^2)$ and $x_1x_2 \ldots x_n$.

(Here $S(a_1, \ldots, a_n)$ denotes the ring of symmetric functions on $(a_1, \ldots, a_n)$.)

But $Sp(n)$ and $U(n)$ are torsion free which means that the statements above concerning $H^*(BT)$, $H^*(BSp(n))$, and $H^*(BU(n))$ are true even with integer coefficients. Therefore, there must be classes $s_1, \ldots, s_n, s'$ in $H^*(BK, \mathbf{Z})$ which map onto the symmetric functions $\sigma_1(x_1^2, \ldots, x_n^2)$, $\sigma_2(x_1^2, \ldots, x_n^2), \ldots, \sigma_n(x_x^2, \ldots, x_n^2)$ and $kx_1x_2 \ldots x_n$ respectively in $H^*(BT, \mathbf{Z})$.

LEMMA 3.1. *If* $k \equiv 1 \pmod 2$ *then the conjecture holds for* $G = Sp(n)$, $H = U(n)$.

Proof. If $k \equiv 1 \pmod 2$ then in cohomology with mod 2 coefficients the image of $(\phi_1\psi)^*$ contains the ring generated by $S(x_1^2, \ldots, x_n^2)$ and $x_1x_2 \ldots x_n$ (reduced modulo 2) in $H^*(BT)$. Let t' be the mod 2 reduction of s' and let $c_1, \ldots, c_n$ be the mod 2 Chern classes of $H^*(BU(n))$. Then $\phi_1^*t' = c_n$ and ϕ^* maps $H^*(BSp(n))$ isomorphically onto the ring generated by $c_1^2, c_2^2, \ldots, c_n^2$.

Since $W_G = C_n$ and $W_K = D_n$ then

$$\begin{aligned} P_0(G/K, t) &= \frac{(1 - t^4)(1 - t^8) \ldots (1 - t^{4(n-1)})(1 - t^{4n})}{(1 - t^4)(1 - t^8) \ldots (1 - t^{4(n-1)})(1 - t^{2n})} \\ &= 1 + t^{2n}. \end{aligned}$$

So dim $G/K = 2n$.

Now consider the diagram

$$\begin{tikzcd} K/U(n) \arrow[d] & K/U(n) \arrow[d] & \\ Sp(n)/U(n) \arrow[r, "\rho"] \arrow[d, "\pi"] & BU(n) \arrow[r, "\phi"] \arrow[d, "\phi_1"] & BSp(n) \arrow[d] \\ Sp(n)/K \arrow[r, "\rho'"] & BK \arrow[r, "\phi_2"] & BSp(n) \end{tikzcd}$$

Since $U(n)$ and $Sp(n)$ have no 2 torsion we have $H^*(SP(n)/U(n), Z_2)$ generated by $\rho^*c_1, \rho^*c_2, \ldots, \rho^*c_n$, none of which is zero, with the relation $(\rho^*c_i)^2 = 0$. (See for example **[5]**.) So

$$\rho^*\phi_1^*t' = \rho^*c_n \neq 0.$$

Since c_n maps to $x_1x_2 \ldots x_n$ in $H^*(BT)$, Sq^2c_n maps to $(x_1 \ldots x_n) \times (x_1 + x_2 + \ldots + x_n)$, i.e., $Sq^2c_n = c_nc_1$. Therefore

$$0 \neq (\rho^*c_n)(\rho^*c_1) = \rho^*Sq^2c_n = \rho^*\phi^*Sq^2t' = \pi^*(\rho'^*Sq^2t').$$

But this gives a contradiction since

$$\dim (\rho'^*Sq^2t') = 2n + 2$$

which is greater than the dimension of $Sp(n)/K$.

Now we can use this lemma to prove the conjecture for a few cases.

THEOREM 3.2. *If $G = Sp(n)$ and $H = U(n)$ then the conjecture is true for $n = 1, 2,$ or* 3.

Proof. The proof for $n = 1$ follows from the fact that there are no intermediate subgroups between the trivial group and C_1.

Suppose $n = 2$. Then we see, say by using Poincaré polynomials, that $K/U(2)$ and $Sp(2)/K$ are simply connected Poincaré-Wall complexes of dimension 2 and 4 respectively. Furthermore, the polynomials show us that $H^*(Sp(2)/K)$ contains only torsion submodules for $* = 1, 2,$ or 3. We can use duality and the universal coefficient theorem to show that

$$H^*(Sp(2)/K) = 0$$

in these dimensions. We can sum this up, using mod 2 coefficients as

$$H^i(K/U(2), \mathbf{Z}_2) = \begin{cases} \mathbf{Z}_2 & i = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

$$H^i(Sp(2)/K, \mathbf{Z}_2) = \begin{cases} \mathbf{Z}_2 & i = 0, 4 \\ 0 & \text{otherwise.} \end{cases}$$

In the spectral sequence of

$$Sp(2)/K \rightarrow BK \rightarrow BSp(2)$$

we see E_2 is evenly graded. Thus $E_2 = E_\infty$ and $H^*(BK, \mathbf{Z}_2)$ is evenly graded. Consequently, from the spectral sequence of

$$K/U(2) \rightarrow BU(2) \rightarrow BK,$$

since everything is evenly graded, we find that ϕ_1^* is a monomorphism. Hence $k \equiv 1 \pmod 2$ and the lemma applies.

Suppose now $n = 3$. Then we have

$$\begin{aligned} P_0(Sp(3)/K, t) &= \frac{(1 - t^4)(1 - t^8)(1 - t^{12})}{(1 - t^4)(1 - t^8)(1 - t^6)} \\ &= 1 + t^6 \\ P_0(K/U(3), t) &= \frac{(1 - t^4)(1 - t^8)(1 - t^6)}{(1 - t^2)(1 - t^4)(1 - t^6)} \\ &= 1 + t^2 + t^4 + t^6 \\ P_0(Sp(3)/U(3), t) &= \frac{(1 - t^4)(1 - t^8)(1 - t^{12})}{(1 - t^2)(1 - t^4)(1 - t^6)} \\ &= 1 + t^2 + t^4 + 2t^6 + t^8 + t^{10} + t^{12}. \end{aligned}$$

Consider the mod 2 Serre spectral sequence of

$$K/U(3) \rightarrow Sp(3)/U(3) \rightarrow Sp(3)/K.$$

Suppose $Sp(3)/K$ has no 2 torsion. Then the differential d_r on E_r is the zero homomorphism for $2 \leqq r \leqq 6$. Therefore, we see that $E_2 = E_\infty$ and $K/U(3)$ has no 2 torsion. Suppose $Sp(3)/K$ has 2 torsion. By duality and the universal coefficient theorem, we see that

$$H_1(Sp(3)/K) \approx H_4(Sp(3)/K) \approx H_5(Sp(3)/K) = 0$$

and

$$H_2(Sp(3)/K) \approx H_3(Sp(3)/K).$$

Therefore, $Sp(3)/K$ has 2 torsion if and only if

$$H^2(Sp(3)/K, \mathbf{Z}_2) \neq 0.$$

Since the image of $H^2(Sp(3)/K, \mathbf{Z}_2)$ does not vanish in E_∞,

$$H^2(Sp(3)/K, \mathbf{Z}_2) = \mathbf{Z}_2.$$

We can then deduce that

$$H^3(Sp(3)/K, \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$$

and

$$H^2(K/U(3), \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2.$$

So $K/U(3)$ has 2 torsion and, as for $Sp(3)/K$,

$$H^3(K/U(3), \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2.$$

Consequently, $E_2^{3,3} = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2.$

But since the "homogeneous spaces" are 1-connected and satisfy Poincaré duality, the classes in $E_2^{3,3}$ remain through E_∞. This contradicts

$$H^6(Sp(3)/U(3), \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2.$$

So $K/U(3)$ and $Sp(3)/K$ must have no 2 torsion and hence, as in the $n = 2$ case, $k \equiv 1 \pmod 2$.

b) $G = SO(2n + 1)$, $H = SO(2n) \times SO(2(n - m) + 1)$.

Here $W_G = B_n$ and $W_H = B_{n-1}$ (for $m = 1$) or $B_m \times D_{n-m}$ (for $m > 1$). Recall that a representation of B_n has roots $\{\pm e_i, \pm e_i, \pm e_j\}$ with $1 \leqq i, j \leqq n$. $B_m \times D_{n-m}$ has roots $\{\pm e_i, \pm e_j, \pm e_k\}$ for $1 \leqq i \leqq m$ and $1 \leqq j < k \leqq m$ or $m < j < k \leqq n$. If W_K has a root of the form $\pm e_i \pm e_j$ with $i \leqq m$ and $j > m$ then, by applying reflections, we see that W_K and W_G have the same roots, hence are isomorphic. However, if we add to the roots of W_H the vector e_p with $p > m$, we get the root system of $B_m \times B_{n-m}$. So if there is a K, not homotopy equivalent to G or H, which satisfies the assumptions of the conjecture then $W_K = B_m \times B_{n-m}$. Nevertheless, there are still instances where the conjecture is valid.

THEOREM 3.3. *If*

$$G = SO(2n + 1) \quad \textit{and} \quad H = SO(2m) \times SO(2(n - m) + 1)$$

the conjecture holds for $n \geqq 2$; $m = 1$ *or* 2.

Proof. The proof for the $m = 1$ case will be given here. The proof for $m = 2$ is very similar. See [**9**] for details. Suppose the theorem were not true for $m = 1$. From their Weyl groups we see that

$H^*(BG, Q)$ has type $[4, 8, 12, \ldots, 4n]$;

$H^*(BH, Q)$ has type $[4, 8, 12, \ldots, 4(n-1), 2]$; and

$H^*(BK, Q)$ has type $[4, 8, \ldots, 4(n-1), 4]$.

Therefore

$$P_0(K/H, t) = \frac{1 - t^4}{1 - t^2}.$$

From the 1-connectedness of K/H and the Poincaré polynomials of BH and BK we have

$$H^j(K/H, \mathbf{Z}) = \begin{cases} \mathbf{Z} & j = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

See also [**11**].

For the remainder of this proof, we will be using mod 2 coefficients.

From [**5**] we have

$$H^*(BG) = \mathbf{Z}_2[W_2, W_3, \ldots, W_{2n+1}] \quad \text{with } \dim W_j = j;$$

$$H^*(BH) = \mathbf{Z}_2[w_2', w_2, w_3, \ldots, w_{2n-1}] \quad \text{with } \dim w_2' = 2 \text{ and}$$

$$\dim w_j = j; \text{ and}$$

$$(1) \quad \phi^* W_j = w_j + w_{j-2} w_2'$$

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where $w_0 = 1$; $w_j = 0$ for $j \notin \{0, 2, 3, \ldots, 2n - 1\}$.

We will now show that i^* is not surjective. Consider now the Serre spectral sequence of

$$K/H \xrightarrow{i} BH \to BK.$$

Suppose i^* is surjective. Then $E_2 = E_\infty$ and we get

$$\begin{aligned} P_2(BK, t) &= P_2(BH, t)/P_2(K/H, t) \\ &= 1/[\,(1 - t^2)(1 - t^2)(1 - t^3)(1 - t^4) \\ &\qquad \times (1 - t^5) \ldots (1 - t^{2n-1})(1 + t^2)\,] \\ &= 1/[\,(1 - t^2)(1 - t^3)(1 - t^4)(1 - t^4) \\ &\qquad \times (1 - t^5) \ldots (1 - t^{2n-1})\,]. \end{aligned}$$

From (1) we know $H^*(BK)$ has classes $\underline{v}_2, \underline{v}_3, \ldots, \underline{v}_{2n-1}$ where

$$\phi_1^* \underline{v}_j = w_j + w_{j-2} w_2'.$$

From $P_2(BK, t)$ above we see that there exists a nontrivial class $\underline{v}_4' \in H^4(BK)$ not generated by $\underline{v}_4$ and $\underline{v}_2^2$. Since $\underline{v}_4'$ is not in the span of $\underline{v}_4$ and $\underline{v}_2^2$ we can assume without loss of generality that

$$\phi_1^* \underline{v}_4' = a w_2 w_2' + b (w_2')^2$$

with a or b nonzero. It is now straightforward to show that $\underline{v}_4'$, $\underline{v}_2, \ldots, \underline{v}_{2n-1}$ are algebraically independent since their images under ϕ_1^* are algebraically independent. Hence,

$$H^*(BK) = \mathbf{Z}_2[\underline{v}_4', \underline{v}_2, \underline{v}_3, \ldots, \underline{v}_{2n-1}].$$

But if i^* is surjective then we see that $w_{2n-1} w_2'$ is not in the range of ϕ_1^* since

$$\phi_1^*(\underline{v}_{2n-1} \underline{v}_2) = (w_{2n-1} + w_{2n-3} w_2')(w_2 + w_2')$$

and ϕ_1^* maps no other product of $\underline{v}_j$'s to an expression with terms of the form $w_{2n-1} w_2$ or $w_{2n-1} w_2'$. This is a contradiction of

$$\phi^* W_{2n+1} = \phi_1^* \phi_2^* W_{2n+1} = w_{2n-1} w_2'.$$

Therefore i^* is not surjective.

Since i^* is not onto then, from the spectral sequence of

$$K/H \to BH \to BK,$$

we see that there are classes v_2 and v_2' in $H^*(BK)$ such that $\phi_1^* v_2 = w_2$ and $\phi_1^* v_2' = w_2'$. Using this fact, equation (1), and a little induction we find that there are classes $v_3, v_4, \ldots, v_{2n-1}$ in $H^*(BK)$ with $\phi_1^* v_i = w_i$. Hence ϕ_1^* is onto.

Now consider the following diagram.

$$\begin{tikzcd}
H \approx \Omega BH \arrow[r, "\Omega\phi_1"] \arrow[d] & \Omega BK \approx K \arrow[d] \\
EH \approx PBH \arrow[r, "P\phi_1"] \arrow[d] & PBK \arrow[d] \\
BH \arrow[r, "\phi_1"] & BK
\end{tikzcd}$$

This induces a transformation τ^* from the cohomology spectral sequence of $PBK \to BK$ to that of $EH \to BH$. We know (for example, see [**5**]) that there are classes x_1', x_1, $x_2, \dots, x_{2n-2}$ in $H^*(H)$ such that w_2', $w_2, \dots, w_{2n-1}$ are their respective images by transgression and such that

$$H^*(H) = \Delta[x_1', x_1, \dots, x_{2n-2}]$$

(that is, $H^*(H)$ has a simple system of generators $\{x_1', x_1, \dots, x_{2n-2}\}$). Since PBK is contractible we know that there is an element y_{i-1} in E_r, for some r, such that $d_r y_{i-1}$ is the nontrivial image of v_i from $H^*(BK) \to E_2 \to E_r$. Then $\tau^* y_{i-1}$ "kills" $\phi_1^* v_i = w_i$ in the spectral sequence of $EH \to BH$. Therefore, $\tau^* y_{i-1}$ must be the image of x_{i-1} in $H^*(H) \to E_2$. We may think of y_{i-1} as being in $H^*(K)$. So now we have classes y_1', y_1, $y_2, \dots, y_{2n-2}$ in $H^*(K)$ such that

$$(\Omega\phi_1)^* y_1' = x_1', \quad (\Omega\phi_1)^* y_i = x_i,$$

and their images by transgression are v_2', v_2, $v_3, \dots, v_{2n-1}$, respectively.

Now look at $H \to K \to K/H$. We know from above that $(\Omega\phi_1)^*$ is surjective. So, since K/H is 1-connected, in the cohomology spectral sequence of this fibration we have $E_2 = E_\infty$. Hence, if y_2' is the image the generator of $H^2(K/H)$ is $H^2(K)$, the multiplicative properties of spectral sequences show us that

$$H^*(K) = [y_1', y_2', y_1, y_2, \dots, y_{2n-1}].$$

Suppose, in the spectral sequence of $PBK \to BK$, $d_r y_2' \neq 0$. (This must be true for some $r \geqq 2$.) Then

$$\tau^* d_r y_2' = d_r \tau^* y_2' = 0.$$

Since τ^* is injective for $E_2^{2,1}$ then r must be 3. So y_2' is also transgressive. Call its image in $H^3(BK)$ v_3'. Then (see [**4**]) we have

$$H^*(BK) = \mathbf{Z}_2[v_2', v_3', v_2, v_3, \dots, v_{2n-1}]$$

with the kernel of ϕ_1^* being $v_3'(H^*(BK))$.

Now let us derive some properties of $H^*(BK)$ with respect to the Steenrod squares. From [**3**] we have

$$Sq^i \underline{W}_j = \sum_{0 \leqq t \leqq i} \binom{j-i+t-1}{t} \underline{W}_{i-t} \underline{W}_{j+t} \text{ for } j > i$$

for $\underline{W}_j = W_j$ or w_j. Therefore

$$Sq^i \underline{v}_j = \sum_{0 \leqq t \leqq i} \binom{j-i+t-1}{t} \underline{v}_{i-t} \underline{v}_{j+t} + v_3' \text{ (other terms).}$$

(Here, and throughout the remainder of the proof, "other terms" will refer to a sum of products of the generators of $H^*(BK)$ not including those stated explicitly.)

Since

$$\phi_1^* Sq^1 v_2' = Sq^1 w_2' = 0$$

then $Sq^1 v_2' = \alpha v_3'$ for some α. Similarly, $Sq^1 v_3' = 0$.

$$\phi_1^* Sq^2 v_3' = Sq^2 \phi^* v_3' = 0;$$

consequently, there are β and ϵ such that

$$Sq^2 v_3' = v_3'(\beta v_2 + \epsilon v_2').$$

From the formula above $Sq^1 v_2 = v_3 + \delta v_3'$.

$$\begin{aligned}(v_3')^2 &= Sq^3 v_3' \\ &= Sq^1 Sq^2 v_3' \\ &= Sq^1(v_3'(\beta v_2 + \epsilon v_2') \\ &= v_3'(\beta(v_3 + \delta v_3') + \alpha\epsilon v_3').\end{aligned}$$

Therefore $\beta = 0$ and $\alpha = \epsilon = 1$ and we have $Sq' v_2' = v_3'$ and $Sq^2 v_3' = v_2' v_3'$.

From the Cartan formula and the results above we deduce a useful property.

PROPERTY 1). *If $v_{j_1} v_{j_2} \ldots v_{j_m} v_3'$ is a nonzero term of $Sq^k (v_{i_1} v_{i_2} \ldots v_{i_r} v_3')$ then $m \geqq r$ and, without loss of generality, the v_j's are ordered so that* $j_1 \geqq j_1 \geqq i_1, j_2 \geqq i_2, \ldots, j_r \geqq i_r$.

Now we are ready to finish the proof of the theorem. From equation (1) we have

$$\phi_2^* W_i = v_i + v_{i-2} v_2' + v_3' \text{ (other terms).}$$

In particular,

$$\phi_2^* W_{2n} = v_{2n-2} v_2' + v_3' \text{ (other terms).}$$

Also $Sq^2 W_{2n} = W_2 W_{2n}$. So now apply ϕ_2^* and get

(2) $\quad (v_2 + v_2')(v_{2n-2} v_2' + v_3' \text{ (other terms)})$

$$= Sq^2(v_{2n-2}v'_2 + v'_3 \text{ (other terms)}).$$

Using the Cartan formula we obtain

$$\begin{aligned} &Sq^2(v_{2n-2}v'_2) \\ &= Sq^2(v_{2n-2})v'_2 + Sq^1(v_{2n-2})(Sq^1v'_2) + v_{2n-2}(Sq^2v'_2) \\ &= Sq^2(v_{2n-2})v'_2 + v_{2n-1}v'_3 + v_{2n-2}(v'_2)^2 \\ &+ v'_3 \text{ (other terms)}. \end{aligned}$$

So, for equation (2) to hold, the $v_{2n-1}v'_3$ on the right-hand side must be "cancelled". Clearly, the only way this can be done is if the "cancelling" term comes from the $Sq^2(v'_3 \text{ (other terms)})$ part of (2). In particular, v_{2n-3} must be among the "other terms". But

$$Sq^2v_{2n-3} = v_2v_{2n-3} + \binom{2n-4}{2}v_{2n-1} + v'_3 \text{ (other terms)}.$$

So, in order to get the cancelling, we need $\binom{2n-4}{2}$ to be 1 (mod 2) and this can only happen if n is odd. This gives us our first partial conclusion: the conjecture is true if $n \equiv 0 \pmod 2$.

So now we continue, assuming $n \equiv 1 \pmod 2$. We must have

$$\phi_2^* W_{2n} = v_{2n-2}v'_2 + v_{2n-3}v'_3 + v'_3 \text{ (other terms)}.$$

Then

$$\begin{aligned} \phi_2^* W_{2n+1} &= \phi_2^* Sq^1 W_{2n} \\ &= v_{2n-1}v'_2 + v_{2n-2}v'_3 + v'_3 \text{ (other terms)}. \end{aligned}$$

Since

$$Sq^1 v_j = (j+1)v_{j+1} + v'_3 \text{ (other terms)}$$

and $W_{2n+1} = Sq^1 W_{2n}$ we get the following.

PROPERTY 2). *For every nonzero term in $\phi_2^* W_{2n+1}$ of the form $v_{i_1}v_{i_2}\dots v_{i_m}v'_3$, $m > 1$, at least two of the i_j's are odd.* (*From the formula, we see that at least one i_j is odd. Two must be odd since the total dimension is odd.*)

Now consider $Sq^4 W_{2n+1} = W_4 W_{2n+1}$. Again apply ϕ_2^* to obtain

$$\begin{aligned} (3) \quad &Sq^4(v_{2n-1}v'_2 + v_{2n-2}v'_3 + v'_3 \text{ (other terms)}) \\ &= (v_4 + v_2v'_2)(v_{2n-1}v'_2 + v_{2n-2}v'_3 + v'_3 \text{ (other terms)}). \end{aligned}$$

Again use the Cartan formula.

$$\begin{aligned} &Sq^4(v_{2n-1}v'_2 + v_{2n-2}v'_3) \\ &= (Sq^4v_{2n-1})v'_2 + (Sq^3v_{2n-1})(Sq^1v'_2) + (Sq^2v_{2n-1})(Sq^2v'_2) \end{aligned}$$

$$+ (Sq^4 v_{2n-2})v_3' + (Sq^3 v_{2n-2})(Sq^1 v_3') + (Sq^2 v_{2n-2})(Sq^2 v_3')$$
$$= v_4 v_{2n-1} v_2' + v_3 v_{2n-1} v_3' + v_2 v_{2n-1}(v_2')^2$$
$$+ v_4 v_{2n-2} v_3' + v_2 v_{2n-2} v_2' v_3' + v_{2n-1}(v_3')^2 + v_3' \text{ (other terms)}.$$

Therefore, in order for equation (3) to be satisfied, we must "cancel" the $v_3 v_{2n-1} v_3'$ term above. This can only be done if one of the "other terms" in (3) is of the form $v_a v_b$ and $Sq^4(v_a v_b v_3')$ has $v_3 v_{2n-1} v_3'$ as a nonzero term.

From Properties 1 and 2 above we know that a and b are both odd and, without loss of generality, $a \leqq 3$ and $b \leqq 2n-1$. Therefore $a = 3$, $b = 2n - 5$, and $v_3 v_{2n-1}$ must come from $v_3(Sq^4 v_{2n-5})$. But

$$Sq^4 v_{2n-5} = v_4 v_{2n-5} + v_3 v_{2n-4} + \binom{2n-8}{2} v_2 v_{2n-3}$$
$$+ \binom{2n-6}{4} v_{2n-1} + v_3' \text{ (other terms)}.$$

Consequently, to "cancel" the $v_3 v_{2n-1} v_3'$ we must have

$$\binom{2n-6}{4} \equiv 1 \pmod 2.$$

Hence, $2n - 6 \equiv 4 \pmod 8$, i.e., $n \equiv 1 \pmod 4$ and we conclude that the conjecture is true for $n \equiv 3 \pmod 4$.

Now suppose $n \equiv 1 \pmod 4$ and continue.

$$\phi_2^* W_{2n+1} = v_{2n-1} v_2' + v_{2n-2} v_3' + v_3 v_{2n-5} v_3' + v_3' \text{ (other terms)}$$

where $v_3 v_{2n-5}$ is not among the "other terms".

$$\begin{aligned} Sq^4(v_3 v_{2n-5} v_3') &= (Sq^3 v_3)(Sq^1 v_{2n-5})v_3' + (Sq^2 v_3)(Sq^2 v_{2n-5})v_3' \\ &\quad + (Sq^1 v_3)(Sq^3 v_{2n-5})v_3' + v_3(Sq^4 v_{2n-5})v_3' \\ &= 0 + (v_2 v_3)(v_2 v_{2n-5})v_3' + 0 \\ &\quad + v_3(v_4 v_{2n-5} + v_3 v_{2n-4} + v_2 v_{2n-3} + v_{2n-1})v_3' \\ &\quad + \text{(other terms) } v_2' v_3' + \text{(other terms) } (v_3')^2. \end{aligned}$$

Note that $Sq^4(v_3 v_{2n-5} v_3')$ contains the nonzero term $v_2 v_3 v_{2n-3} v_3'$. To get equation (3) then, $\phi_2^* W_{2n+1}$ must contain a term other than $v_3 v_{2n-5} v_3'$ such that, when we apply Sq^4, we get a "cancelling" $v_2 v_3 v_{2n-3} v_3'$.

Case 1). Suppose that the term is of the form $v_a v_b v_3'$. Then Properties 1 and 2 show that a and b are odd, not greater than 3 and $2n - 3$ respectively, and $a + b = 2n - 2$. So $a = 3$ and $b = 2n - 5$, which is impossible, and we go to Case 2.

Case 2). The term is of the form $v_a v_b v_c v_3'$ with, say a even, b and c odd, and $b \leqq c$. Then by Property 1, since b and c are at least 3, $a \leqq 2$. So $a = 2$, $b = 3$, and consequently, $c = 2n - 7$. So the term must be $v_2 v_3 v_{2n-7} v_3'$.

The only way $v_2v_3v_{2n-3}v_3'$ can be produced from $Sq^4(v_2v_3v_{2n-7}v_3')$ is from the expression

$$v_2v_3(Sq^4v_{2n-7})v_3'.$$

But, in Sq^4v_{2n-7} we have the coefficient $\binom{2n-8}{4}$ for v_{2n-3} and, if $n \equiv 1 \pmod 4$ then $2n - 10 \equiv 0 \pmod 8$; hence,

$$\binom{2n-8}{4} \equiv 0 \pmod 2.$$

Therefore, the conjecture is valid for $n \equiv 1 \pmod 4$ and we are done.

Remark. For small values of n, classes such as v_{2n-7} and v_{2n-5} may not exist. In such cases, the proof is finished at the point where such classes are required.

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