INDEFINITE GREEN'S FUNCTIONS AND ELEMENTARY SOLUTIONS

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Linear differential equations both ordinary and partial are often studied by means of Green's functions. One reason for this is that linearity permits superposition of solutions. A Green's function describes the "effect" of a point source, and the description of line, surface, or volume sources is achieved by superposing, that is to say, integrating, this function over the source distribution.

For equations with constant coefficients the use of integral transforms permits the calculation of such source functions in the form of integrals. Only in the simplest cases is explicit evaluation by elementary functions possible, and this has perforce led to the use of asymptotic estimates, which so thoroughly pervade the domain of applied mathematics.

Our purpose in this note is to point out a basic general property of point source solutions, which has apparently been little noticed. This property is, that the exponents of decay with time of the elementary solution are related by Hamiltonian duality to the differential polynomial. To illustrate the asymptotic behaviour so determined, in some simple cases, we give examples of equations of wave motion, diffusion, and dispersion, with one space variable.

To prepare for a separate treatment of the time variable, we begin with a self-contained section on Green's functions for ordinary differential equations with constant coefficients. It is shown how any solution of such an equation can be expressed in terms of one function of a single variable.

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1. Green's functions for ordinary differential equations.

Let t be the independent variable, and let $\delta(t)$ denote the Dirac delta-function, more strictly the measure which represents a unit mass concentrated at the origin. Suppose that an ordinary differential operator L, with constant coefficients, is given, of the form

$$Ly(t) = y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y . \tag{1.1}$$

and consider the nonhomogeneous equation

$$Ly(t) = \delta(t) , \tag{1.2}$$

with auxiliary initial condition

$$y(t) \equiv 0 \qquad t < 0 .$$

We note first that our--unique--solution $y(t)$ satisfies $Ly = 0$ for $t \neq 0$. For t negative it is identically zero, while for t positive it is some linear combination of the n exponential solutions $e^{\lambda_i t}$, where λ_i ranges through the roots of the characteristic equation

$$P(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \dots + a_n = 0 . \tag{1.3}$$

Let us integrate the differential equation over an interval containing the origin. On the right hand side we obtain the Heaviside step function

$$H(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

and so find

$$y^{(n-1)} + a_1 y^{(n-2)} + \dots + a_n \int y dt = H(t) .$$

Now let us take the discontinuity of both sides around $t = 0$.

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We find

$$(y^{(n-1)}) = 1 :$$

here the parentheses indicate the difference of the two one-sided limits.

Thus our solution satisfies the conditions

$$y(0) = y^{(1)}(0) = \ldots = y^{(n-2)}(0) = 0 ,$$

and

$$y^{(n-1)}(0+) = 1 .$$

The unique solution with these initial values, which has continuous derivatives up to the order $n - 2$ and vanishes for $t < 0$, is known as the indefinite Green's function of the operator L, and is denoted by $G(t)$.

A discontinuity of the $(n - 1)^{\underline{st}}$ order derivative is typical of Green's functions not only for initial value problems but also for two-point boundary value problems. It is noteworthy that the indefinite Green's function for an operator with constant coefficients is a function of a single argument rather than a kernel with two variables as is usual in boundary value problems.

A most useful property of $G(t)$ is that any solution whatever of $Ly = f(t)$ can be expressed by means of it. Thus it contains, in itself, the entire range of possibilities described by the differential equation.

We first search for a particular integral, that is, for some solution of the non-homogeneous equation. The essential property of the delta-function on the right side of (1.2) is that for any function $f(t)$ whatever

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau = \int_{0}^{t+\varepsilon} f(\tau)\,\delta(t-\tau)d\tau, \qquad \varepsilon > 0 , \quad t > 0 .$$

Combining this with the property

$$LG(t-\tau) = \delta(t-\tau) ,$$

and with the linearity of the equation, which permits the superposition of solutions, we see that the finite convolution

$$y_o(t) = \int_0^{t+\varepsilon} G(t-\tau) f(\tau) d\tau = \int_0^{t} G(t-\tau) f(\tau) d\tau$$

is, formally, a particular integral. It is easy to verify that its initial data are

$$y_o(0) = y_o^{(1)}(0) = \ldots = y_o^{(n-1)}(0) = 0 .$$

We remark that this convolution integral is just the solution to which the method of variation of parameters would lead for this equation. To complement this solution with solutions of the homogeneous equation, we first note that the particular integral has initial value zero together with its first $n - 1$ derivatives. The Green's function itself satisfies

$$G^{(h)}(0+) = \delta_h^{n-1}$$

where δ_h^{n-1} denotes the Kronecker delta symbol, unity if $h = n - 1$, zero otherwise. To find solutions of the homogeneous equation, which span its n-dimensional vector space, we rewrite the equation (1.2) as follows:

$$\begin{aligned} &(G^{(k)} + a_1 G^{(k-1)} + \ldots + a_k G)^{(n-k)} \\ (1.2') \qquad &= -(a_{k+1} G^{(n-k-1)} + \ldots + a_n G) + \delta(t) . \end{aligned}$$

Thus the jump of

$$(G^{(k)} + a_1 G^{(k-1)} + \ldots + a_k G)^{(n-k-1)}$$

across the origin, is unity. We note that derivatives of the

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combination within the parentheses, of lower order than $n - k - 1$, are continuous. Since $G(t) \equiv 0$ for negative t, we see that the relation

$$\left[\left(G^{(k)} + a_1 G^{(k-1)} + \ldots + a_k G\right)^{(h)}\right]_{0+} = \delta_h^{n-1-k}$$

certainly holds for $h < n - k - 1$. For $n - k \leq h \leq - 1$, the remaining values of interest, the same formula is a consequence of (1.2'). Since the delta-function and its derivatives are zero for all non-zero values of t, the limit as t tends to zero from above is zero. Differentiating $h - n - k$ times on both sides of (1.2'), we verify the result.

The explicit solution of the initial value problem

$$Ly = f(t)$$

$$y^{(h)}(0+) = c_h \qquad h = 0, 1, \ldots n - 1$$

now is given by

$$y(t) = \int_0^t G(t-\tau)f(\tau)d\tau$$

$$+ \sum_{h=0}^{n-1} C_h \sum_{i=0}^{n-h-1} a_i G^{(n-i-1)}(t) \qquad , \quad a_o \equiv 1 .$$

For example, the differential equation of harmonic vibrations of frequency ω, $\ddot{y} + \omega^2 y = 0$, has the Green's function

$$G(t) = \frac{1}{\omega} \sin \omega t .$$

The solution of the initial value problem

$$\ddot{y} + \omega^2 y = f(t), \qquad t > 0$$

with $y(0) = c_o$, $y^{(1)}(0) = c_1$ is

$$y(t) = \int_0^t \frac{\sin\omega(t-\tau)}{\omega} f(\tau)d\tau$$

$$+ c_o \cos\omega t + c_1 \frac{\sin\omega t}{\omega} .$$

An explicit formula for $G(t)$ is available, if the roots $\lambda_1, \ldots, \lambda_n$ of the characteristic equation (1.3) are known. Assuming for simplicity that these roots are distinct, we choose a representation

$$G(t) = \sum_k g_k e^{\lambda_k t}$$

The initial conditions for $G(t)$ as $t \to 0+$ yield

$$\sum_k g_k \lambda_k^h = \delta_{n-1}^h , \qquad h = 0, 1, \ldots, n-1 .$$

The solution of these n equations for the g_k leads to Vandermonde determinants, and to the formula

$$g_k = \frac{1}{\prod_{j \neq k} (\lambda_k - \lambda_j)} = \frac{1}{\left.\frac{\partial P}{\partial \lambda}\right|_{\lambda=\lambda_k}}$$

Hence

$$G(t) = \sum_k \frac{e^{\lambda_k t}}{\prod_{j\neq k} (\lambda_k - \lambda_j)} , \qquad t > 0 .$$

For large values of t, only the root or roots λ_k with

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maximal real part will be significant. If there is only one such root λ_{max} then

$$G(t) \sim \frac{e^{\lambda_{max} t}}{\left.\frac{\partial P}{\partial \lambda}\right|_{\lambda = \lambda_{max}}} .$$

2. Elementary solutions of partial differential equations. For simplicity let us consider equations in one dimension of space, that is, in two independent variables x and t. An elementary solution of such an equation will satisfy, by definition,

$$(2.1) \qquad L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u = \delta(x)\delta(t) ,$$

and so represents an initial disturbance concentrated at a single point, the origin. If n is the order of the highest time-derivative which occurs, the initial values zero for the solution and its first $n - 2$ time derivatives are prescribed. From the differential equation we may again deduce that the $n - 1^{st}$ order initial time derivative is the distribution $\delta(x)$.

We shall assume, as in § 1, that the coefficient of the highest time derivative is 1; this now excludes space derivation from the leading term.

Let us take the bilateral Laplace transform of (2.1): if we write

$$(2.2) \qquad \tilde{u}(t, y) = \int_{-\infty}^{\infty} e^{-xy} u(t, x) dx$$

then we obtain

$$(2.3) \qquad L\left(\frac{\partial}{\partial t}, y\right)\tilde{u} = \delta(t) ,$$

an ordinary differential equation for $u(t, y)$.

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We denote by $G(t, y)$ the Green's function for this equation. If we denote by $p_k(y)$ the n roots of the auxiliary equation in p

$$(2.4) \qquad L(p, y) = 0 ,$$

and if we suppose they are all distinct, then

$$(2.5) \qquad G(t, y) = \sum_{k=1}^{n} \frac{e^{p_k(y)t}}{\left.\frac{\partial L}{\partial p}\right|_{p=p_k}}$$

The unique elementary solution of the partial differential equation is now found by the inverse Laplace transform:

$$k(t, x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G(t, y) e^{xy} dy$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sum_{k} \frac{e^{xy+p_k(y)t}}{\frac{\partial L}{\partial p_k}} dy ,$$

Here c may be any sufficiently large positive number.

The convergence of this integral is dependent upon the behaviour of the roots $p_k(y)$ as $y \to \pm i\infty$ and we therefore restrict attention to those equations which satisfy a suitable regularity condition. The simplest such condition is that

$$(2.7) \qquad R\, p_k(y) < - \delta , \qquad \delta > 0 ,$$

for $|y| \to \infty$ in sectors

$$|\arg y \pm \tfrac{\pi}{2}| < \varepsilon ,$$

and we shall refer to such equations as regular. Diffusion, and dispersion are physical processes which ordinarily lead to regular equations.

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We now consider the general initial value problem

$$(2.8) \qquad Lu = f(t,x) = \int\int f(\tau,\xi)\delta(t-\tau)\delta(x-\xi)d\tau\, d\xi ,$$

with initial conditions

$$(2.9) \qquad u_{(t)}^{(h)}(0,x) = c_h(x) , \qquad h = 0, 1, \ldots, n-1 .$$

Let us apply the bilateral Laplace transform in x to the unknown function $u(t,x)$:

$$(2.10) \qquad \tilde{u}(t,y) = \int_{-\infty}^{\infty} e^{-xy} u(t,x)dx$$

and to the data of the problem:

$$\tilde{f}(t,y) = \int_{-\infty}^{\infty} e^{-xy} f(t,x)dx ,$$

$$\tilde{c}_h(y) = \int_{-\infty}^{\infty} e^{-xy} c_h(x)dx .$$

We find the ordinary differential equation

$$L(\frac{\partial}{\partial t}, y)\tilde{u} = \tilde{f}(t,y)$$

with initial conditions

$$\tilde{u}_{(t)}^{(h)}(0,y) = \tilde{c}_h(y) .$$

From § 1 we see that the solution is

$$\tilde{u}(t,y) = \int_{0}^{t} G(t-\tau,y)\tilde{f}(\tau,y)d\tau$$

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$$+ \sum_{n=0}^{n-1} c_h(y) \sum_{i=0}^{n-h-1} a_i(y)\, G_{(t)}^{(n-i-1)}(t, y) .$$

Here the Green's function $G(t, y)$ depends on the transformed variable y, and the coefficients $a_i(y)$ of the transformed differential equation are polynomials in y.

The inverse transform

$$u(t, x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{xy} \tilde{u}(t, y) dy$$

leads back to the solution of the general initial problem. To carry out this inversion we must evaluate the inverse transform of the product of $G(t-\tau, y)$ with certain other functions of y. But the transform of a product is the convolution of the transforms. We therefore find bilateral convolutions of the elementary solution with the data functions, in the form

$$(2.11) \qquad u(t, x) = \int_{-\infty}^{\infty} \int_{0}^{t} k(t-\tau, x-\xi) f(\tau, \xi) d\xi d\tau$$

$$+ \sum_{h=0}^{n-1} \int_{-\infty}^{\infty} c_h(\xi) \sum_{i=0}^{n-h-1} a_i\left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial}{\partial t}\right)^{n-i-1} k(t, x-\xi) d\xi .$$

The differentiations with respect to ξ indicated in the sum on the right-hand side can equally well be transferred to the initial data $c_h(\xi)$. This formula shows how the elementary solution of the partial differential equation leads by superposition and differentiation to the most general solution of an initial value problem.

3. <u>The normal curve.</u> The algebraic curve

$$(3.1) \qquad L(p, y) = 0$$

associated with the differential operator is known as the <u>normal</u>

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curve. It is the graph of an algebraic function p implicitly defined by (3.1). This function [or its graph] has n branches

$$p = p_k(y) \qquad k = 1, \ldots, n .$$

We shall consider only the case in which the branches are all totally real; that is, for each real y there are n real (distinct) values of p. This class of operators includes many hyperbolic and parabolic types of physical significance. A number of examples of these are discussed in later sections.

4. The method of steepest descent. Explicit evaluation of the elementary solutions is possible only in a few especially simple cases such as the equation of the string and of heat flow. In more complicated examples, a clear description of the behaviour of the elementary solution can be achieved by asymptotic estimates. We shall be particularly concerned with estimates wherein x and t become large with the ratio

$$(4.1) \qquad \alpha = \frac{x}{t}$$

remaining fixed.

With the substitution $x = \alpha t$ the elementary solution becomes a sum of integrals

$$(4.2) \qquad k(t, x) = \frac{1}{2\pi i} \sum_k \int_{c-i\infty}^{c+i\infty} \frac{e^{t(\alpha y + p_k(y))}}{\frac{\partial L}{\partial p_k}} dy ,$$

and these are of the type adapted to the steepest descent method as $t \to \infty$.

Let us set

$$g_k(y) = \alpha y + p_k(y) .$$

Then the steepest descent estimate is found by first locating a saddle-point, or col, of the exponential factor. This is a stationary point of the real part of $g_k(y)$. By the Cauchy-

Riemann equations, such a point y_o is stationary for the imaginary part as well and thus

$$g_k'(y_o) = 0 .$$

The lines of steepest decrease of exp $g_k(y)$ are the level lines of Im $g_k(y)$, and we are required to deform the path of integration into a steepest line through a saddle-point. If this is done, the asymptotic estimate is

$$(4.3) \qquad \sqrt{\frac{2\pi}{t\, g_k''(y_o)}} \; \frac{e^{t\, g_k(y_o)}}{\left.\frac{\partial L}{\partial p_k}\right|_{y=y_o}} \left[1 + 0\left(\frac{1}{t}\right)\right] ,$$

assuming that g_k'' does not vanish at the saddle-point y_o.

We shall be interested chiefly in the exponents $g_k(y_o)$ which depend upon α since $y_o = y_o(\alpha)$. At great distances or time intervals, this exponent will provide the most significant information on the magnitude of the solution. In particular, that integral of the sum for which this exponent is largest will dominate all the others. The two assumptions of regularity and total reality, which we have made, will ensure that there are real saddle-points, leading to real exponential decrease towards infinity.

The saddle-points are located by solving

$$g_k'(y) = \alpha + p_k'(y) = 0 ,$$

that is to say,

$$(4.4) \qquad -p_k'(y) = \alpha$$

for $y = y_o(\alpha)$. The exponent g_k is then the compounded function $g_k(y_o(\alpha))$. Now this process is just the Legendre

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transformation in which $g_k(y_o(\alpha))$ is the dual of $-p_k(y)$. In geometrical terms $g_k(y_o(\alpha))$ is the p-intercept of the tangent with slope $-\alpha$ to the k^{th} branch of the normal curve, the point of contact being $y_o(\alpha)$. In the language of mechanics, $g_k(y_o(\alpha))$ is the Hamiltonian corresponding to the Lagrangian $-p_k(y)$. We commented in the introduction on the basic nature of this duality, which is so well known in mechanics, continuum mechanics, and wave motion, and which is equally significant for our elementary solutions.

In the following example explicit formulae are possible. The heat equation for a one-dimensional medium travelling with velocity c is

$$u_t = - cu_x + u_{xx} .$$

To the Lagrangian

$$p = - cy + y^2$$

there corresponds the Hamiltonian

$$g = - \frac{1}{4}(\alpha - c)^2$$

and the exact elementary solution

$$k(t, x) = \frac{1}{2\sqrt{\pi t}} \exp - \frac{(x-ct)^2}{4t} .$$

To justify the deformation of contours necessary for the steepest descent calculation two types of singularities in the complex plane must be considered. These are the poles and branch points of the integrands of (4.2). The denominators $\partial L/\partial p$ are algebraic functions of the form

$$\prod_{j \neq k} (p_k(y) - p_j(y))$$

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and they will vanish only when two or more roots $p_k(y)$ coincide. Thus poles are automatically branch points.

The only possible real branch point, under our assumptions, is the origin.

At a branch point certain of the roots $p_k(y)$ are permuted among themselves by any small circuit enclosing the branch point. Thus if the contours of all of the integrals referring to these roots are carried over the branch point, the integrals will be permuted among themselves without change in their sum. Thus if all of the steepest descent contours lie either to the right or all lie to the left of the branch point, there will be no contribution from that branch point. We shall say, in this case, that the contours are consistent relative to the branch point. As yet no example of an inconsistent set of contours has been found. It would be interesting to determine explicitly the class of equations for which this consistency holds.

5. Subcharacteristics. Equations of the type we are studying represent some combination of wave motion and diffusion or dispersion. Typical of such processes is the splitting of a point source effect into a number of travelling waves which diffuse or disperse as time goes on. The paths in space-time of these waves are indicated by local maximum values of the dual exponents $g_k(y_o(\alpha))$ as functions of space time directions α. Such directions are known as subcharacteristics, and we now show how they can be found from the series expansions of the roots $p_k(y)$ about the origin.

Let us suppose

$$p_k(y) = a_o + a_1 y + a_2 y + \dots, \quad y \to 0 ,$$

and that $a_2 \neq 0$. (In all physical examples known to us a_2 is positive.) Then

$$g = a_o + (\alpha + a_1)y + a_2 y^2 + \dots$$

while the steepest descent leads asymptotically to

$$0 = g' = \alpha + a_1 + 2a_2 y + \dots$$

so that

$$y \sim -\frac{\alpha + a_1}{2a_2}, \qquad \alpha \sim -a_1,$$

and in turn

$$g \sim a_o - \frac{(\alpha + a_1)^2}{2a_2} \dots, \qquad \alpha \sim -a_1 .$$

Evidently g has a maximum at $\alpha = -a_1$ and the contribution to the elementary solution is a Gaussian wave

$$(5.1) \qquad \sqrt{\frac{\pi}{ta_2}} \frac{1}{\frac{\partial L}{\partial p_k}} e^{a_o t - \frac{(x + a_1 t)^2}{4a_2 t}} [1 + 0(\frac{1}{t})]$$

The subcharacteristic line is $x + a_1 t = 0$. We note that $\partial L / \partial p_k$ is a polynomial of degree $n - 1$ in $\alpha = x/t$, which does not vanish in a neighbourhood of $\alpha = -a_1$.

We have seen how the initial terms of the power series about the origin for the characteristic roots $p_k(y)$ determine the position and nature of the subcharacteristics. Now the expansions for large y lead in a similar fashion to asymptotic estimates for large values of $|\alpha|$, that is, when t is small compared with $|x|$. Such estimates indicate physically the onset at a field point of disturbances generated elsewhere. Though we shall not include details there is one circumstance of particular interest to which we shall refer.

The operator L is hyperbolic provided that the highest x-derivative is also of order n. It is regularly hyperbolic provided that no multiple roots appear for $y \neq 0$; this implies that for large y

$$p_k(y) = a_1^k y + a_o^k + 0(\frac{1}{y}) .$$

The corresponding characteristic lines $\alpha = - a_1^k$ determine the region of influence of the disturbance—that is, the region outside of which it is zero identically. This can be shown for our integrals by completing the contour with a semicircle to the right and establishing that the contribution is zero for $t < |x| / |a_1^k|$. Thus the outermost characteristic on each side is the boundary of the region of disturbance.

6. Examples of elementary solutions and their asymptotic estimates.

Example 1. The Klein-Gordon equation

$$u_{tt} = u_{xx} + c^2 u \tag{6.1}$$

has the normal curve

$$p^2 = y^2 + c^2 , \tag{6.2}$$

a rectangular hyperbola, with two branches both totally real. (See figure 1.) Let us take as branch 1

$$p_1(y) = +\sqrt{y^2 + c^2} , \tag{6.3}$$

and as branch 2

$$p_2(y) = -\sqrt{y^2 + c^2} . \tag{6.4}$$

In the $y = u + iv$ plane we set up a two-sheeted Riemann surface, with branch points at $y = \pm ic$, and assign branch 1 to the upper sheet, branch 2 to the lower sheet.

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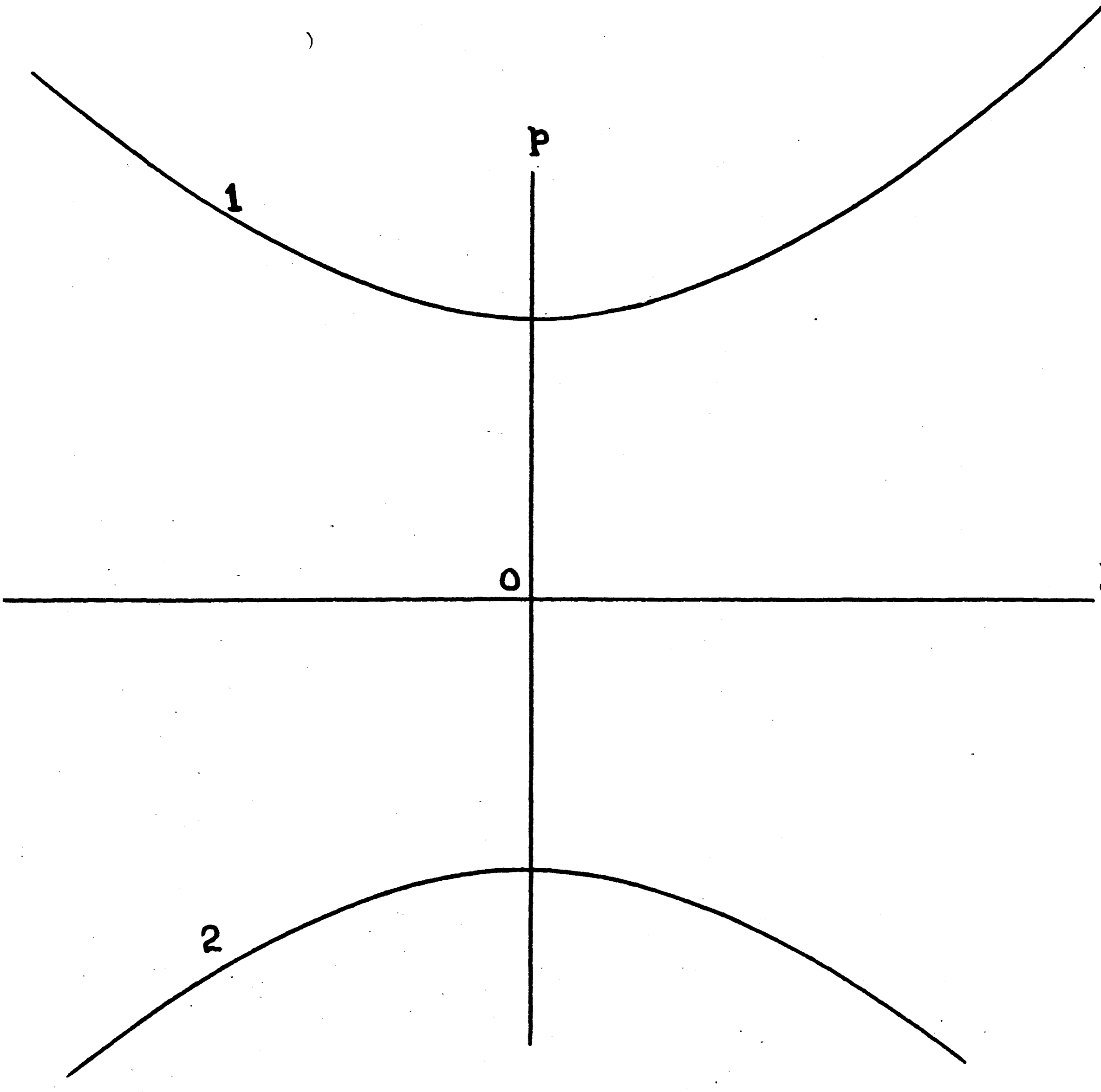

FIGURE 1

The Normal Curve of the Klein-Gordon Equation

The elementary solution is

$$(6.5) \qquad k(t,x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{t(\alpha y + \sqrt{y^2 + c^2})}}{2\sqrt{y^2+c^2}} \, dy$$

$$- \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{t(\alpha y - \sqrt{y^2 + c^2})}}{2\sqrt{y^2+c^2}} \, dy$$

where the first integral is on the upper sheet of the Riemann surface, the second integral on the lower sheet. In this simple example the elementary solution can be expressed in terms of a Bessel function. There are two cases to discuss.

Case I. $\alpha > 1$ or $|x| > t$

In this case we can prove $u = 0$. Since, for $|y|$ large

$$(6.6) \qquad \begin{aligned} g_1(y) &\sim y(\alpha + 1) && \text{in right half plane} \\ g_1(y) &\sim y(\alpha - 1) && \text{in left half plane} \\ g_2(y) &\sim y(\alpha - 1) && \text{in right half plane} \\ g_2(y) &\sim y(\alpha + 1) && \text{in left half plane} \end{aligned}$$

we can deform both integrals into left semicircles at infinity, each of which gives zero. Since, for both integrals the branch points are passed over in the deformation, these cancel, and the elementary solution is zero. This exemplifies the fact that the region of influence for a hyperbolic equation is bounded by the characteristics.

Case II. $\alpha < 1$ or $|x| < t$

In this case the second integral gives zero, since it can be deformed into a right semicircle at infinity, and the branch points are not passed over in the deformation.

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The first integral gives [3, p. 248]

$$\frac{1}{2} I_o(c\sqrt{t^2 - x^2})$$

Thus the elementary solution is

$$(6.7) \qquad k(x, t) = \frac{1}{2} I_o(ct\sqrt{1 - \alpha^2})\, H(t - x)$$

$$= \frac{1}{2} I_o(c\sqrt{t^2 - x^2})\, H(t - x)$$

Now using the result in [2] p. 86, #5

$$I_o(x) \sim \frac{e^x}{\sqrt{2\pi x}}$$

we obtain

$$(6.8) \qquad k(t, x) = \frac{e^{ct\sqrt{1 - \alpha^2}}}{2\sqrt{2\pi ct}\,(1 - \alpha^2)^{\frac{1}{4}}}\, H(t - x) .$$

This same result may be obtained using the asymptotic procedure outlined in § 4, the curve dual to the rectangular hyperbola being a circle.

Example 2. The heat equation

$$(6.9) \qquad u_t = u_{xx} - hu$$

has the normal curve

$$(6.10) \qquad p = y^2 - h ,$$

a parabola, with one branch, and totally real.

The elementary solution is given by

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(6.11) $$k(t,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{t(\alpha y + y^2 - h)}\,dy .$$

The asymptotic formula for this integral turns out to be exact. From [4], #710.0, we find

(6.12) $$k(t,x) = \frac{e^{-th}\, e^{-\frac{x^2}{4t}}}{2\sqrt{\pi t}}$$

Example 3. Stokes' equation. This equation, which governs the propagation of sound waves in a viscous medium, is

(6.13) $$u_{tt} = u_{xx} + u_{xxt} ,$$

and it has the normal curve

(6.14) $$y^2 - p^2 + y^2 p = 0 .$$

The normal curve has two branches both totally real. We define branch 1 by

(6.15) $$p_1(y) = \frac{y}{2}[y - \sqrt{y^2 + 4}] ,$$

and branch 2 by

(6.16) $$p_2(y) = \frac{y}{2}[y + \sqrt{y^2 + 4}] .$$

(See figure 2.)

In the $y = u + iv$ plane there are branch points at $y = \pm 2i$. We shall set up a two-sheeted Riemann surface, one sheet for each of the two branches.

The expression for the elementary solution is

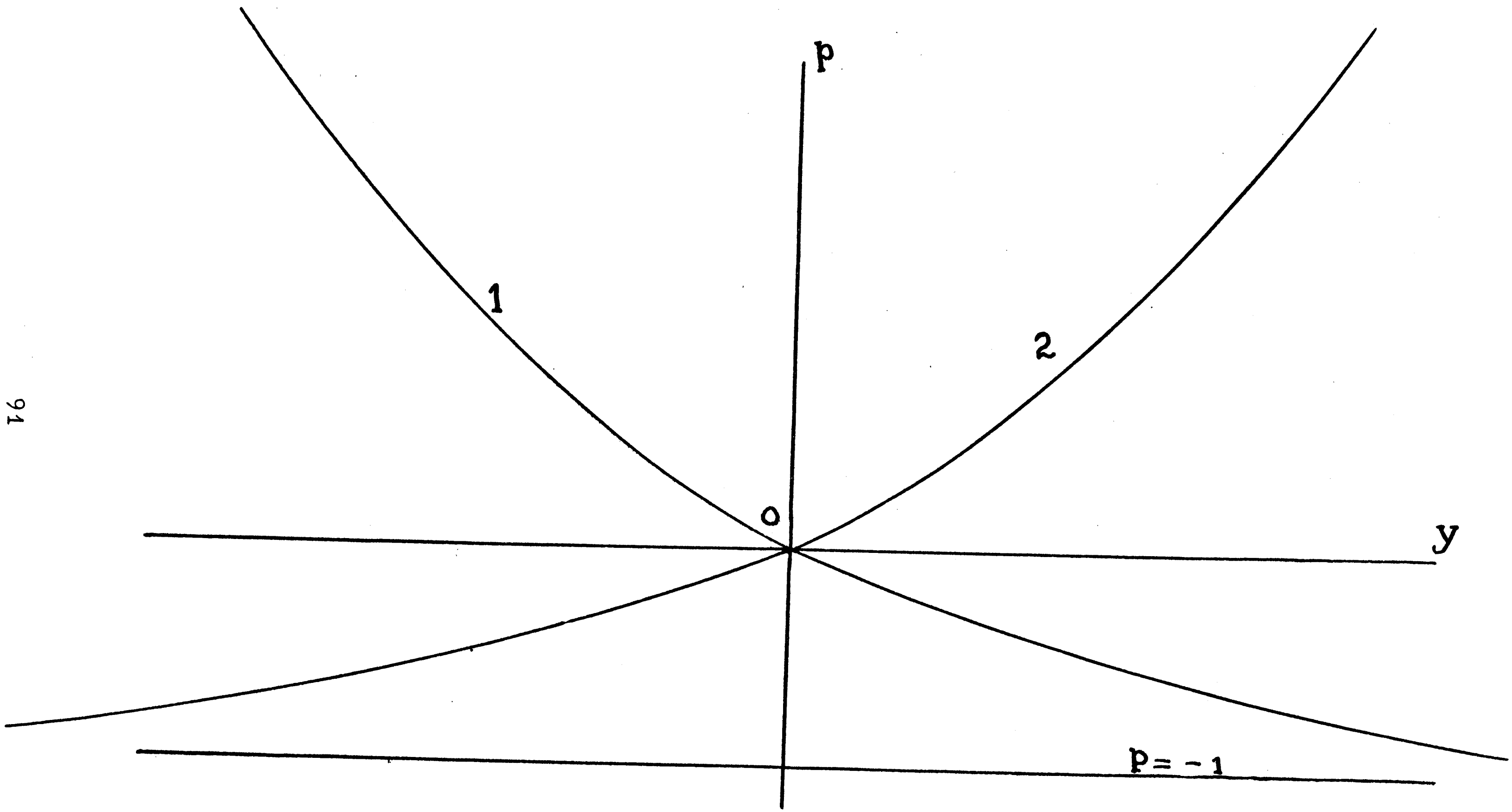

FIGURE 2

The Normal Curve of Stokes' Equation

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$$(6.17) \qquad k(t,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{tg_1(y)}}{Q_1(y)}\,dy + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{tg_2(y)}}{Q_2(y)}\,dy,$$

where

$$g_1(y) = \alpha y + p_1(y)$$

$$g_2(y) = \alpha y + p_2(y)$$

$$Q_1(y) = y\sqrt{y^2+4}$$

$$Q_2(y) = -y\sqrt{y^2+4}$$

The first integral lies on the upper sheet and the second integral on the lower.

The saddle point y_o is the value of y for which

$$g'(y) = \alpha + p'(y) = 0$$

and we see that branch 1 has a real saddle point for all α. As the slope of branch 1 at the origin is 1, the saddle point of branch 1 lies to the right of the origin if $\alpha < 1$, and to the left of the origin if $\alpha > 1$. We deform the first integral into a steepest line through the saddle point, that is, into the curve

$$\ell m\, g_1(y) = \ell m\, g_1(y_o) = 0,$$

since $g_1(y_o)$ is real.

For $|y|$ large, we have

$$(6.18) \qquad \begin{aligned} g_1(y) &\sim \alpha y - 1 && \text{in right half plane} \\ g_1(y) &\sim \alpha y + y^2 + 1 && \text{in left half plane} \\ g_2(y) &\sim \alpha y - 1 && \text{in left half plane} \\ g_2(y) &\sim \alpha y + y^2 + 1 && \text{in right half plane.} \end{aligned}$$

Thus the family of steepest lines $\ell m\ g_1(y) = k,\ -\infty < k < \infty$, far out from the origin on the right are the straight lines

$$v = \frac{k}{\alpha}$$

and far out from the origin on the left are the rectangular hyperbolas

$$v = \frac{k}{\alpha + 2u}.$$

The steepest line through the saddle point, corresponding to $k = 0$, is shown in figure 3, for a typical value of α. It is seen that for all values of α, when we deform the original path of integration into the path $\ell m\ g_1(y) = 0$, we pass over the branch points. We can now obtain an asymptotic estimate of the first integral using 4. 3.

The second integral is more easily disposed of. From (6. 18) we see that the second integral can be deformed into the left semicircle at infinity which will give zero. Since the branch points are passed over, these contribute, but cancel with the branch point contribution of the first integral. In other words the deformed contours are consistent relative to the branch points. The poles at the origin for each branch make no contribution. This is easily shown by combining the two integrals into a single integral, and it is found that the resulting integrand has no pole at the origin.

Using 4. 3 we find

$$(6.19) \qquad k(t, x) \sim \frac{1}{2\pi i}\sqrt{\frac{2\pi}{t g_1''(y_0)}}\ \frac{e^{t g_1(y_0)}}{Q_1(y_0)}.$$

To find the saddle point and the exponent we have to solve the cubic

$$(6.20) \qquad y^3 + \frac{\alpha}{2} y^2 + 4y + 2(\alpha - \frac{1}{\alpha}) = 0.$$

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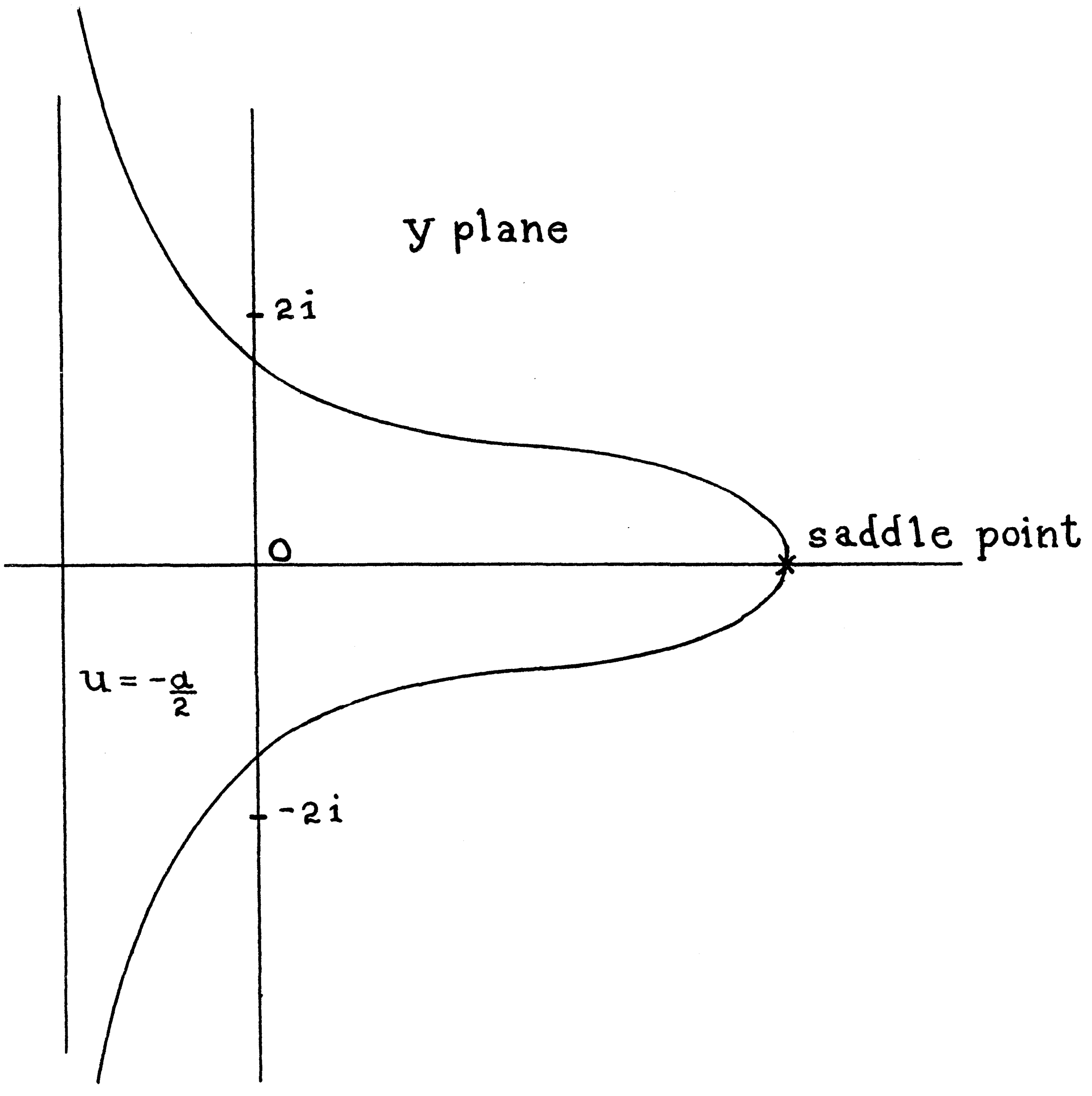

FIGURE 3

Stokes' Equation. The Steepest Line Through the Saddle Point for a Typical Value of α.

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By Cardan's method, we obtain

$$y_o = -\frac{\alpha}{6} + \left(-\frac{q}{2} + \frac{\sqrt{\Delta}}{2}\right)^{\frac{1}{3}} - \left(\frac{q}{2} + \frac{\sqrt{\Delta}}{2}\right)^{\frac{1}{3}}$$

where

$$\Delta = \frac{\alpha^4}{27} + \frac{31\alpha^2}{27} + \frac{112}{27} + \frac{4}{\alpha^2},$$

$$q = \frac{\alpha^3}{108} + \frac{4\alpha}{3} - \frac{2}{\alpha},$$

and

$$g(y_o(\alpha)) = \frac{3\alpha y_o^2 + 2y_o(\alpha^2-1)}{2(y_o + \alpha)}$$

In addition

$$g_1''(y_o) = 1 - \frac{y_o^3 + 6y_o}{(y_o^2 + 4)^{3/2}}$$

Even this asymptotic formula is so complicated as to require further approximation for special values of α.

Case I: $\alpha \sim 0$. In this case

$$\sqrt{\Delta} \sim \frac{2}{\alpha}$$

$$q \sim \frac{-2}{\alpha}$$

$$y_o \sim \left(\frac{2}{\alpha}\right)^{1/3}$$

$$g_1(y_o(\alpha)) \sim -1 + 3\left(\frac{\alpha}{2}\right)^{2/3}$$

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$$g_1''(y_0) \sim \left(\frac{\alpha}{2}\right)^{4/3}$$

$$Q_1(y_0) \sim \left(\frac{2}{\alpha}\right)^{2/3}$$

and we find

$$k(t,x) = \frac{1}{2\sqrt{3\pi t}} \exp\left[- t + 3t \left(\frac{x}{2t}\right)^{2/3}\right] . \tag{6.21}$$

Case II: $\alpha \sim \infty$. Here

$$\sqrt{\Delta} \sim \frac{\alpha^2}{3\sqrt{3}}$$

$$q \sim \frac{\alpha^3}{108}$$

$$y_0 + \alpha \sim \frac{\alpha}{2}$$

$$g_1(y_0(\alpha)) \sim -\frac{\alpha^2}{4} + 1$$

$$g_1''(y_0) \sim \frac{96}{\alpha^4}$$

$$Q_1(y_0) \sim \frac{\alpha^2}{4}$$

and we find

$$k(t,x) = \frac{1}{2\sqrt{3\pi t}} \exp\left[t - \frac{x^2}{4t}\right] . \tag{6.22}$$

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Case III: $\alpha \sim 1$. We find

$$\Delta \sim \frac{28}{3} - \frac{50}{9}(\alpha - 1)$$

$$q \sim \frac{-71}{108} + \frac{121}{36}(\alpha - 1)$$

$$y_o \sim -k(\alpha - 1)$$

(6.23) $$g(y_o(\alpha)) \sim \frac{3k - 4k}{2}(\alpha - 1)^2 \,\bar{\sim}\, -.579(\alpha - 1)^2$$

where $k = \dfrac{13704 - 363\sqrt{21}}{13254} \doteq .909$

Using the information from the three cases we can sketch $g(y_o(\alpha))$ (figure 4).

Finally, we exhibit the solution of the initial value problem in its full generality, as an example of the general formula 2.11 of section 2. Let

$$u_{tt} - u_{xx} - u_{xxt} = f(x, t) \qquad t > 0$$

and let

$$u(O, x) = c_o(x), \quad u_t(0, x) = c_1(x).$$

Then

$$u(t, x) = \int_0^t \int_{-\infty}^{\infty} k(t - \tau, x - \xi) f(\tau, \xi) d\tau\, d\xi$$

$$+ \int_{-\infty}^{\infty} c_o(\xi) \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial \xi^2}\right) k(t, x-\xi) d\xi$$

$$+ \int_{-\infty}^{\infty} c_1(\xi)\, k(t, x-\xi) d\xi .$$

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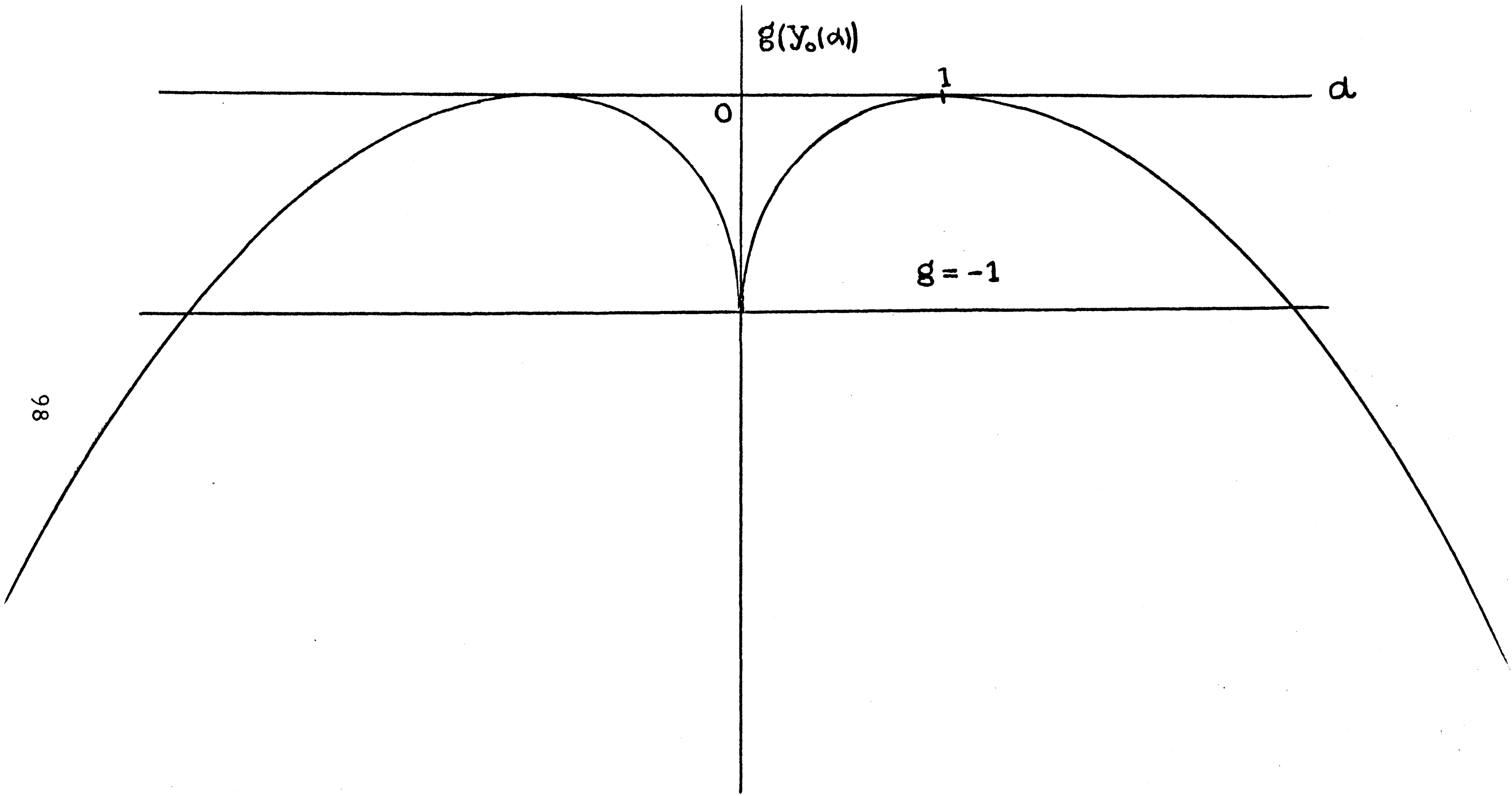

FIGURE 4

The Dual Curve for Stokes' Equation

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Example 4. The hydromagnetic boundary layer equation. This equation is

$$(6.24) \qquad u_{tt} - 2a^2 u_{txx} + (a^4 - \varepsilon^2) u_{xxxx} - c^2 u_{xx} = 0$$

and it has the normal curve

$$(6.25) \qquad p^2 - 2a^2 py^2 + (a^4 - \varepsilon^2) y^4 - c^2 y^2 = 0 \ .$$

This consists of two approximately parabolic branches, both totally real, crossing at the origin (figure 5).

We define branch 1 by

$$(6.26) \qquad p_1(y) = y \, [a^2 y + (\varepsilon^2 y^2 + c^2)^{\frac{1}{2}}]$$

and branch 2 by

$$(6.27) \qquad p_2(y) = y \, [a^2 y - (\varepsilon^2 y^2 + c^2)^{\frac{1}{2}}] \ .$$

In the y-plane there are branch points at $y = \pm i \frac{c}{\varepsilon}$, and we set up a two-sheeted Riemann surface, one sheet for each of the two branches. The elementary solution is given by (6.17), where, in this case

$$g_1(y) = \alpha y + p_1(y)$$

$$g_2(y) = \alpha y + p_2(y)$$

$$Q_1(y) = 2(p_1 - a^2 y^2)$$

$$Q_2(y) = 2(p_2 - a^2 y^2) \ .$$

Since the slope of branch 2 at the origin is $-c$, the saddle point for branch 2 lies to the right of the origin for $0 < \alpha < c$, and to the left of the origin for $c < \alpha < \infty$. The saddle point for branch 1 lies to the left of the origin for all α.

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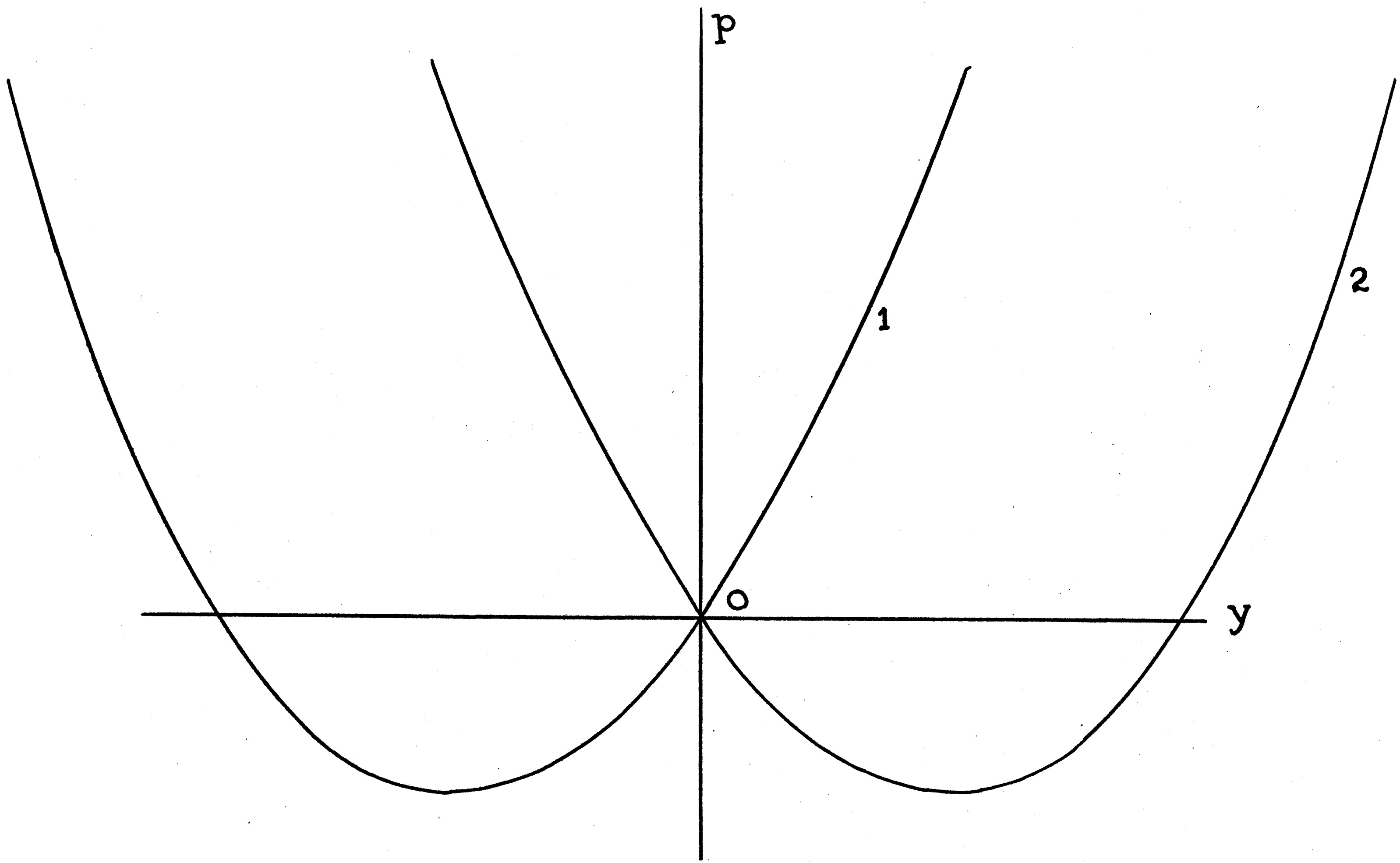

FIGURE 5

The Normal Curve for the Hydromagnetic Boundary Layer Equation

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We deform the first integral into the path $\operatorname{Im} g_1(y) = 0$, and the second integral into the path $\operatorname{Im} g_2(y) = 0$. These steepest paths both lie to the left of the branch points for all α somewhat as in figure 3. Thus there is cancellation of the branch point contributions. Just as in example 3, the pole at the origin does not contribute. An examination of figure 5 shows that the exponent for branch 1, for $\alpha > 0$, is greater in magnitude than the exponent for branch 2, and they are both negative. Thus the branch 2 integral in the elementary solution will dominate the branch 1 integral, which may therefore be neglected. The behaviour of both exponents as functions of α is illustrated in figure 6. Since the explicit calculation of this dual curve is complicated, we shall consider only approximations.

Case I: $\alpha \sim \infty$

If α is large then y_o is large and we find,

$$g_2(y) \sim \alpha y + (a^2 - \varepsilon)y^2 - \frac{c^2}{2\varepsilon} ,$$

and

$$y_o \sim - \frac{\alpha}{2(a^2 - \varepsilon)} .$$

Hence

$$g_2(y_o(\alpha)) \sim - \frac{\alpha^2}{4(a^2 - \varepsilon)} - \frac{c^2}{2\varepsilon}$$

$$g_2''(y_o) \sim 2(a^2 - \varepsilon)$$

$$Q_2(y_o) \sim - \frac{\varepsilon \alpha^2}{2(a^2 - \varepsilon)^2} - \frac{c^2}{\varepsilon}$$

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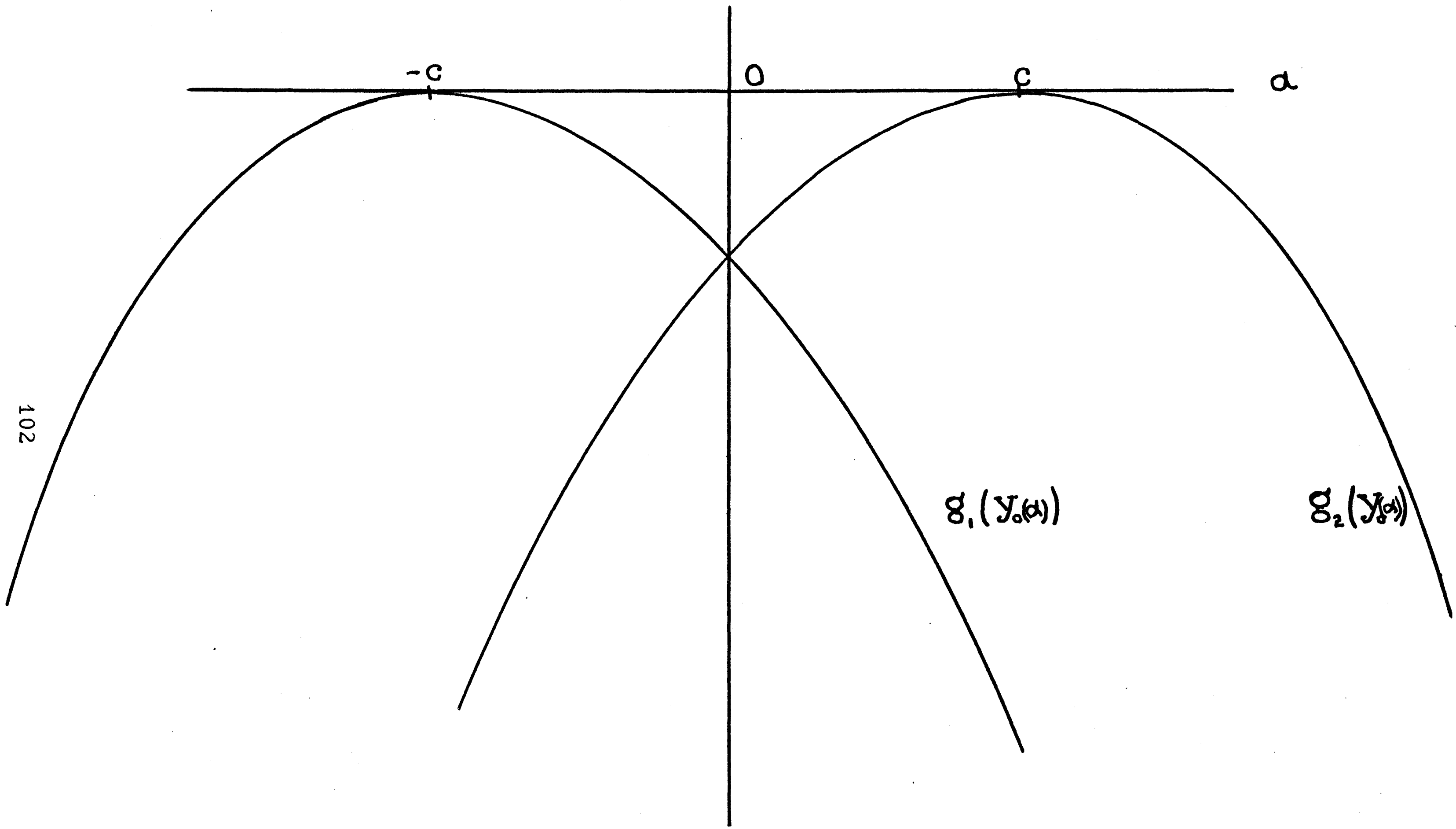

FIGURE 6

The Dual Curve for the Hydromagnetic Boundary Layer Equation

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and

$$(6.28)\quad k(x,t) = -\frac{t^{\frac{3}{2}}(a^2-\varepsilon)^{\frac{3}{2}}}{\sqrt{\pi}\,x^2}\exp\left(-\frac{c^2t}{2\varepsilon}-\frac{x^2}{4t(a^2-\varepsilon)}\right).$$

Case II: $\alpha \sim c$. It may be seen from figure 5 that if $\alpha \sim c$, then $y_o \sim 0$. Thus

$$p_2(y) \sim a^2y - cy$$

$$g_2(y) \sim (\alpha - c)y + a^2y$$

$$y_o \sim -\frac{\alpha - c}{2a^2}$$

$$(6.29)\qquad g_2(y_o(\alpha)) \sim -\frac{(\alpha-c)^2}{4a^2}$$

Figure 6 is obtained from the information in these two cases. By an easy calculation it is found that the double point of this curve is

$$-\frac{a^2c^2}{2\varepsilon^2} + \frac{c^2}{2\varepsilon^2}\sqrt{a^4-\varepsilon^2},$$

below the origin. This gives the time rate of attenuation at the source point.

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