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AN EPI-REFLECTOR FOR UNIVERSAL THEORIES

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A construction of an epi-reflector by injective hull techniques is given which applies to the class of models of any universal theory with the Amalgamation Property and there yields a weak but functorial type of algebraic closure. Various completions such as the boolean envelope and quotient field constructions are identified as such injective hulls over epimorphic injections. Forms of the Amalgamation Property are also shown to eliminate various pathologies of epimorphisms and equalizers.

1. **Persistent monomorphisms.** We shall adopt the terminology of Kelly [6] except for four changes:

(1) it will apply to the dual situation (of monomorphisms);

(2) we define a monomorphism f to be *persistent* if for every g with domain that of f there is an h and a monic k with $hf=kg$;

(3) if $f=rn$ is a regular factorization we shall also call r the *dominion* of f (following Isbell [4]) and n the *antidominion* of f;

(4) if $f=gh$ we call h a *restriction* of f by g.

Our first task is to prove Kelly's results without assuming the existence of pushouts. The key to this is:

LEMMA 1.1. *If $f:A\to B$ is a persistent monomorphism and $a, b:A\to C$ are morphisms then there are $c, d:B\to D$ and a monic $g:C\to D$ with $cf=ga$ and $df=gb$.*

Proof. There is $c_1:B\to D_1$ and monic $g_1:C\to D_1$ with $g_1a=c_1f$. Then there is $d:B\to D$ and monic $g_2:D_1\to D$ with $g_2(g_1b)=df$. Let $c=g_2c_1$, $g=g_2g_1$. Thus g is monic.

By replacing Kelly's uses of (the dual of) his Lemma 5.9 by applications of our Lemma 1.1 in [6] we easily obtain:

THEOREM 1.2. *Let $\mathscr{C}$ be a category admitting regular factorizations and with regular monomorphisms persistent. Then regular monomorphisms are closed under composition and restriction, antidominions are epic, and every strong monomorphism is regular.*

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It is easy to verify that persistent monomorphisms are closed under composition, restriction, and transference by pushouts. A more interesting result is:

LEMMA 1.3. *Let gf be epic where g is persistent monic. Then f is epic.*

Proof. Let $af=bf$. Now by 1.1 there is a monic h and morphisms c, d with $cg=ha$ and $dg=hb$. Thus $cgf=dgf$ and so $c=d$. Hence $ha=hb$ and so $a=b$.

2. **Transference in bicategories.** Let X be an abstract class of monomorphisms. An object D is called *X-injective* if $(u, D):(B, D)\to(A, D)$ is surjective for all $u:A\to B$ in X. A morphism $f \in X$ is called *X-essential* if $gf \in X$ implies $g \in X$. An *X-injective hull* of A is an X-essential X-morphism $A\to B$ with B X-injective. Any two X-injective hulls of A are isomorphic (see [1]).

We say that X is *transferable* if for any $f \in X$ and g with domain that of f there is $k \in X$ and a morphism h with $hf=kg$. A useful result is:

LEMMA 2.1. *Let X be transferable and closed under restriction. Then any epic X-morphism is X-essential.*

Proof. Suppose that $gf \in X$ where $f \in X$ is epic. Now by transference there is $h \in X$ and a morphism k with $hf=kgf$. Thus $h=kg$ and so $g \in X$.

From now on let $\mathscr{C}$ be a *bicategory* in the sense of [7] with class I of *injections* and S of *surjections*. Let E be the class of epimorphisms of $\mathscr{C}$. If $X=I$ we omit the prefix "X-" in the above definitions, and if $X=E \cap I$ we replace it by "epi-". General material on bicategories can be found in [7] and [8].

As usual we say that $\mathscr{C}$ has *enough injectives* if for any $A \in \mathscr{C}$ there is $A\to B$ in I with B injective. It is easy to see that if $\mathscr{C}$ has enough injectives then injections are transferable. We note also that in $\mathscr{C}$ injections are transferable if and only if I satisfies ($E4$) in the terminology of Banaschewski [1].

Every regular monomorphism is extremal and so is an injection: hence in future we shall talk of "regular injections". Now we call an object A *saturated* if every epic injection with domain A is an equivalence and *absolutely closed* if every injection with domain A is regular (the terminology is based on [4]). Clearly any epi-injective or absolutely closed object is saturated. For the converse we require the hypotheses of:

THEOREM 2.2. *Let $\mathscr{C}$ be a bicategory admitting regular factorizations and with injections transferable. Then* (*1*) *epic injections are transferable and* (*2*) *the notions of saturated, absolutely closed, and epi-injective agree.*

Proof. (1) Let $f:A\to B$ be an epic injection, $g:A\to C$ a morphism. Then there is $k \in I$ and a morphism h with $hf=kg$. Let $k=rs$ be a regular factorization. Then

$s \in I$, and s is epic by 1.2. Since r is strong and $hf=r(sg)$ with f epic, there is w with $wf=sg$. (2) Let A be saturated. Then A is absolutely closed as antidominions are epic. Now let $u:B\to C$ be an epic injection, $f \in (B, A)$. Then there is an epic injection $v:A\to D$ and a morphism g with $gu=vf$. Hence v is an equivalence and so $(v^{-1}g)u=f$.

Let $\mathscr{C}_0$ be the sub-bicategory of injections of $\mathscr{C}$. Then clearly $\mathscr{C}$ has the *Amalgamation Property* if and only if injections are transferable in $\mathscr{C}_0$. Also any $\mathscr{C}$-epic injection is $\mathscr{C}_0$-epic, and by the bicategory factorization any $\mathscr{C}_0$-regular injection is $\mathscr{C}$-regular. The converse is not in general true, but holds under the conditions of:

LEMMA 2.3. *Let $\mathscr{C}$ be a bicategory admitting products of pairs, $u:A\to B$ an injection of $\mathscr{C}$. Then:*

(1) *if u is $\mathscr{C}$-regular then u is $\mathscr{C}_0$-regular;*

(2) *if u is $\mathscr{C}_0$-epic then u is $\mathscr{C}$-epic.*

Proof. Let $f:B\to C$, and $B\times C, p, p'$ be a product of B and C. Then $(1,f)$ is the unique morphism $B\to B\times C$ with $p(1,f)=1_B$ and $p'(1,f)=f$: it is a coretraction and so an injection. (1) If u is the simultaneous equalizer in $\mathscr{C}$ of $f_i, g_i:B\to C_i$ for $i\in I$, then u is the simultaneous equalizer of the injections $(1,f_i)$, $(1,g_i):B\to B\times C_i$ for $i \in I$, and so is $\mathscr{C}_0$-regular (as injections are closed under restriction).

(2) If $fu=gu$ then $(1,f)u=(1,g)u$: hence $(1,f)=(1,g)$ and so $f=g$.

It follows from the proof of 2.1 that if $\mathscr{C}$ has the Amalgamation Property, then epic injections are essential. And we can combine 2.3, 1.2, and 1.3 to derive:

THEOREM 2.4. *Let $\mathscr{C}$ be a bicategory with the Amalgamation Property admitting regular factorizations and products of pairs. Then regular injections are closed under composition and restriction, antidominions of injections are epic, and epic injections are closed under restriction by injections.*

In particular this holds for any equational class with the Amalgamation Property.

3. **Construction of the epi-injective hull.** We shall call a class X of morphisms *normal* if X is closed under composition and whenever A_i, d_{ij}, $i\le j\le\alpha$ is a direct system where α is a limit ordinal such that (a) for every nonzero limit $\beta\le\alpha$ A_β, $(d_{i\beta}:i<\beta)$ is a direct limit of A_i, d_{ij}, $i\le j<\beta$ and (b) $d_{ij}\in X$ whenever $i<j<\alpha$, then $d_{i\alpha}\in X$ whenever $i<\alpha$.

It is easy to check that epimorphisms are normal (and even closed under fibred coproducts). Then we can prove:

THEOREM 3.1. *Let $\mathscr{C}$ be a co-well-powered bicategory admitting regular factorizations and direct limits for systems of injections, with injections transferable and normal. Then every object of $\mathscr{C}$ has an epi-injective hull.*

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Proof. Since epic injections are normal we can construct by transfinite induction an epic injection $u:A \to B$ such that there is no proper epic injection with domain B. Thus B is saturated and so epi-injective by 2.2(2). Finally u is essential by 2.1 and so epi-essential (as this is clearly just epic and essential).

The map $u:A \to B$ constructed above is actually a *reflection* of A in the full subcategory of epi-injectives. For if $g:A \to B'$ is such that B' is epi-injective then $g=hu$ for some h, which is unique as u is epic. Unlike the usual reflection theorems, 3.1 proves that the reflective subcategory is cogenerating in addition to producing a reflection.

4. **Metric spaces.** As an example let $\mathscr{C}$ be the category of metric spaces and contraction maps, with the bicategory structure defined by I=the isometric embeddings, S=the surjective maps. Clearly $\mathscr{C}$ admits regular factorizations, products of pairs, and direct limits of injections (with injections normal), but not for example infinite products. Also $\mathscr{C}$ has enough injectives by Garling [3] and so injections are transferable. It is easy to check that an injection $u:A \to B$ is regular just if uA is closed in B and epic just if uA is dense in B. Hence $\mathscr{C}$ is co-well-powered, and any complete space is saturated.

Now let $D=\{\frac{1}{2}^n : n<\omega\}$, $D^*=D \cup \{0\}$, $u:D \to D^*$ the inclusion, where D and D^* have the natural metrics (from the real line). It is easy to check that u is epic and that a space M is complete just if M is u-injective. Hence complete=epi-injective and the epi-injective hull is the usual completion.

5. **Varieties.** Other examples are provided by any variety with injections transferable. In particular let $\mathscr{C}$ be the variety of distributive lattices with 0, 1 and (0, 1)-homomorphisms. By [2, Lemma 2.1] and Stone [9] 2 is an injective separator and so $\mathscr{C}$ has enough injectives. As the full subcategory of boolean lattices is isomorphic to the variety of boolean algebras it follows that epimorphisms between boolean lattices are onto.

For every $\mathscr{C}$-lattice A there is a canonical injection $e:A \to 2^{(A,2)}$. The boolean sublattice of $2^{(A,2)}$ generated by eA is called the *boolean envelope* of A: clearly the injection into the envelope is epic. It now follows that a $\mathscr{C}$-lattice is saturated just if it boolean, and that the epi-injective hull is the boolean envelope.

The author does not know any characterizations of the epi-injective hull for other varieties with injections transferable and not all epimorphisms onto.

6. **Jónsson classes.** Let L be a language, T a universal L-theory with the Amalgamation Property, $\mathscr{M}_0(T)$ the category of models of T and embeddings between them. Clearly $\mathscr{M}_0(T)$ forms a bicategory with all morphisms injections. Using the last result of 2.4 it is not hard to show that the inclusion $A \to B$ of $\mathscr{M}_0(T)$ is epic precisely if every element of B is algebraic of reduced degree 1 over A in the sense of Jónsson [5]. Hence $\mathscr{M}_0(T)$ is co-well-powered by [5, Theorem 9.1] and so every

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element of $\mathscr{M}_0(T)$ has an epi-injective hull (as all hypotheses of 3.1 are satisfied).

For example let T be the theory of integral domains of characteristic 0. It can be shown that every epic embedding between fields in $\mathscr{M}_0(T)$ is surjective, and as every object has an epic extension to its field of quotients, it follows that the epi-injectives of $\mathscr{M}_0(T)$ are precisely the fields, and that the epi-injective hull is just the quotient field.

Other examples are given by T=the theory of torsion-free abelian groups, where the epi-injective hull is just the divisible closure (which is also the injective hull in $\mathscr{M}_0(T)$), and by the theory of cancellative abelian monoids, where the epi-injective hull is the enveloping group.

By purely category-theoretic techniques we can thus construct a rather weak (but functorial) algebraic closure for models of any theory of the above type. It will be shown in a subsequent paper that by a refinement of the ideas of [5] involving model-theoretic arguments one can in fact construct the full algebraically injective hull but show that the construction given in this paper is by no means superseded (even in model theory) as the two hulls agree for theories closed under direct product.

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