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THE PRIMITIVE IDEAL SPACE OF A C^*-ALGEBRA

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Introduction. The commutative Gelfand-Naimark Theorem says that any commutative C^*-algebra A is isomorphic to the ring $C_0(M, C)$ of all continuous complex-valued functions tending to zero outside of compact sets of a locally compact Hausdorff space M. A very important part of this theorem is an intrinsic and also a complete characterization of M as exactly the primitive ideal space of A in the hull-kernel (or weak-star) topology. In the non-commutative case, $A \cong \Gamma_0(M, E)$—the ring of sections tending to zero outside of compact subsets of a locally compact Hausdorff space M with values in the stalks or fibers E. Furthermore, this representation reduces to $E = M \times C$ and $\Gamma_0(M, M \times C) \cong C_0(M, C)$ if A is commutative (see [**1**]). However, in the non-commutative case it has not been possible to determine the topology of M precisely. Here, a partial answer is given to this very troublesome flaw in the general Gelfand-Naimark Theorem.

Consider a C^*-algebra A and its primitive ideal space B in the hull-kernel topology. Any topological space B has a complete regularization $\phi: B \rightarrow M$, where M is a completely regular topological space with the universal property that any continuous map of B into a completely regular topological space factors uniquely through ϕ. The points $m \in M$ can be identified with ideals of A by

$$m = \bigcap \{b \in B | \phi(b) = m\}.$$

Here the main interest will be in the case when A does not contain an identity, in which case in general neither B nor M need be compact. If $\mathscr{K}$ is any family of compact subsets of M that is closed under finite unions and with $\bigcup \mathscr{K} = M$, then a Hausdorff one point compactification of M is obtained by taking $\{M\backslash K | K \in \mathscr{K}\}$ as a neighborhood basis of the point of infinity. The finest one point compactification of M is obtained by taking $\mathscr{K}$ as all compact subsets of M. However, let

$$\mathscr{K} = \{\{m \in M| \,||a + m|| \geqq \lambda\} | a \in A; \lambda > 0\}.$$

Then by the non-commutative Gelfand-Naimark Theorem (see [**4**, p. 119, 8.13]), there is a fiber-bundle or sheaf-like structure

$$\pi : E \equiv \bigcup \{\frac{A}{m} | m \in M\} \rightarrow M$$

$$\hat{a}: M \rightarrow E, a(m) = a + m \in \pi^{-1}(m) \quad a \in A, M \in M.$$

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Let $\Gamma_0(M, E)$ denote all continuous cross sections $\sigma: M \to E$ (with $\pi \circ \sigma =$ identity) vanishing at infinity on this particular one point compactification of M given by the above $\mathscr{K}$. The non-commutative Gelfand-Naimark theorem (see [**1**, p. 119, 8.13]) asserts that $\hat{A} = \Gamma_0(M, E)$. The objective of this note is to show that the $\mathscr{K}$-compactification is the finest one, provided each primitive ideal of A does not contain the center of A. The general case still remains an open question.

1. Spaces of ideals. Some of the facts from [**5**] and [**6**] are used without proof. It will be convenient to take equivalent, but slightly different definitions than were used previously in [**5**] and [**6**] for some of the basic invariants associated with a C^*-algebra.

1.1 *Notation.* For any C^*-algebra A whatever with or without an identity, consider its primitive ideal space Prim $A \equiv B$, its centroid R, and the maximal ideal space Y of the centroid (see [**5**] and [**6**]). There is a map $F: B \to Y$, $F(b) = \{r \in R | rA \subseteq b\}$. For each $p \in Y$, define an ideal $m(p)$ of A by

$$m(p) = \bigcap \{b \in B | F(b) = p\} = \bigcap F^{-1}(\{p\}).$$

Let M be the set of ideals $M = \{m(p) | p \in F(B)\}$. Now define a map $\phi: B \to M$ by $\phi(b) = m(F(b))$. (This is the *complete regularization* map used in the introduction.) (For various properties of R, Y, F, and M, see [**6**].) Let $Z \equiv$ center A. If $1 \in A$, then $R = Z$ and the map ϕ simply becomes $\phi(b) = \bigcap\{q \in B | q \cap Z = b \cap Z\}$. There is a one-to-one correspondence $F(B) \to M$ given by $p \to m(p)$. Consequently, let us simply transfer the topology from $F(B)$ to M by means of this identification.

It will be convenient to view A as an ideal and R as a subalgebra of $M(A)$, the multiplier (or double centralizer) of A (see [**6**]). Then $A \subseteq R + A \subseteq M(A)$. (In a C^*-algebra such as $M(A)$, the sum of a closed ideal and a closed subalgebra can be shown to be closed [**10**, p. 18, 1.8.3]). Let $A_1 = R + A$. The above objects B, M, ϕ, and F are defined for any C^*-algebra whatever with or without identity. For the C^*-algebra A_1 they will be denoted by B_1, M_1, ϕ_1, and F_1. Since center $A_1 = R$, the centroid of A_1 is also R (so that $R_1 = R$, $Y_1 = Y$). Let $F_1: B_1 \to Y$. Since $1 \in A_1$, $F_1(B_1) = Y$.

1.2. Since M is indexed by the subset $F(B) \subset Y$ and M_1 by all of Y, there is a natural injection defined by

$$i: M \to M_1, \; i(m(p)) = m_1(p) \quad p \in F(B)$$

where $m_1(p) = \bigcap \{b_1 \in B_1 | b_1 \cap R = p\}$. Since the topologies on M and M_1 were transferred from $F(B) \subset F_1(B_1)$, the map i is a homeomorphism onto its image.

1.3. For $b \in B$, define $\bar{b} \equiv \{\alpha \in A_1 | \alpha A \subseteq b\}$ and set $\bar{B} = \{\bar{b} | b \in B\}$. Then it can be shown directly or it follows from [**6**] that $\bar{b} = \{\alpha \in A_1 | \alpha A + A\alpha \subseteq b\}$ and that $\bar{B} = \{J \in B_1 | A \not\subseteq J\}$. Now define a function

$$j: B \to B_1, j(b) = \bar{b} \quad b \in B.$$

The symbol "$\triangleleft$" will denote ideals. Since $A \triangleleft A_1$, j is a homeomorphism of B onto the open subset $\bar{B}$ of B_1. Furthermore, $\bar{b} \cap A = b$.

In the next paragraph some of the relations between the objects ϕ, F, M, B defined for A and ϕ_1, F_1, M_1, B_1 and A_1 are explained.

1.4. Since center $A_1 = R$, for any $J \in B_1$, $F_1(J) = J \cap R$. In particular, for $b \in B$ and $\bar{b} = j(b)$, $\bar{b} \cap R = F_1(b)$. But by definition of $\bar{b}$ and $F(b)$, $\bar{b} \cap R = \{r \in R | rA \subseteq b\} = F(b)$. Thus $F_1 j = F$. It follows from the definition of $m_1(p)$ that $m_1(p) \supseteq p$ and that $m_1(p) \cap R = p$.

1.5. Some facts about the hull-kernel topology, the primitive ideals, and the norm of an arbitrary C^*-algebra A (with or without an identity) have to be recalled.

(*a*) For $b \in B$, and $a + b \in A/b$, $||a + b||$ is the quotient norm $||a + b|| = \inf \{||a + c|| \, | c \in b\}$. The sets

$$\{b \in B | \, ||a + b|| > \lambda\} \quad 0 < \lambda, a \in A$$

form a basis for the hull-kernel open sets [**11**, p. 257, 4.9.15].

(b) Each subset of the form $\{b \in B | \, ||a + b|| \geqq \lambda\}$ is compact [**11**, p. 258, 4.9.19]. (All these latter sets are also closed if and only if B is Hausdorff [**11**, p. 258, 4.9.19].)

(c) For any closed subset $\Delta \subset B$ and any $a \in A$, $\sup \{||a + b|| \, | b \in \Delta\}$ exists [**11**, p. 256, Theorem 4.9.14]. If $m \triangleleft A$ is any closed ideal, then Prim $A/m = \{b/m | b \in \Delta\}$, where $\Delta \subset B$ is the closed set $\Delta = \{b \in B | b \supseteq m\}$, i.e., the hull of m. For any C^*-algebra $\bar{A}$, and any $\bar{a} \in \bar{A}$,

$$||\bar{a}|| = \sup \{||\bar{a} + J|| \, | J \in \text{Prim } \bar{A}\}.$$

Take $\bar{A} = A/m$ and $\bar{a} = a + m$. The last two facts imply that $||a + m|| = \sup \{||a + b|| \, | m \subseteq b \in B\}$.

(d) In (c) above, actually there exists an ideal $q \in B$ with $||a|| = ||a + q||$. Similarly, for an ideal $m \subset A$, $||a + m|| = ||a + q||$ for some $m \subseteq q \in B$. Both of these assertions follow from [**11**, p. 256, 4.9.14].

The next proposition is stated in slightly greater generality than actually later used in order to emphasize that it involves no topological considerations. It also should be noted that any closed ideal in a C^*-algebra is the intersection of all the primitive ideals containing it; thus in later applications $m = \bigcap \phi^{-1}(m)$ below.

1.6. PROPOSITION. *Suppose that A is any C^*-algebra* (with or without identity), *that B = Prim A, that M is any set of closed ideals of A, and that $\phi: B \to M$ is* *any surjective function whatever subject only to the restriction that*

$$\phi(b) = m \Leftrightarrow m \subseteq b \quad b \in B, m \in M.$$

Then for each real $\lambda > 0$ *and* $a \in A$,

(i) $\phi(\{b \in B|\ ||a + b|| \geqq \lambda\}) = \{m \in M|\ ||a + m|| \geqq \lambda\}$;

(ii) $\phi(\{b \in B|\ ||a + b|| > \lambda\}) = \{m \in M|\ ||a + m|| > \lambda\}$.

Proof. (i) For $b \in B$ with $||a + b|| \geqq \lambda$, it follows from $\phi(b) \subseteq b$ that $||a + \phi(b)|| \geqq ||a + b|| \geqq \lambda$. Hence

$$\phi(\{b|\ ||a + b|| \geqq \lambda\}) \subseteq \{m|\ ||a + m|| \geqq \lambda\}.$$

Conversely, if $||a + m|| \geqq \lambda$ for some $m \in M$, then 1.5(d) shows that there exists a $q \in B$ with $m \subseteq q$ and $||a + q|| = ||a + m||$. Consequently, also

$$\{m|\ ||a + m|| \geqq \lambda\} \subseteq \phi(\{b|\ ||a + b|| \geqq \lambda\}).$$

(ii) If $||a + b|| > \lambda$, then $\phi(b) \subseteq b$ and $||a + \phi(b)|| \geqq ||a + b|| > \lambda$. Whereas if $m \in M$ with $||a + m|| > \lambda$, then 1.5 (c) shows that $||a + m|| \geqq ||a + b|| > \lambda$ for some $m \subseteq b \in B$. Thus

$$\phi(\{b|\ ||a + b|| > \lambda\}) = \{m \in M|\ ||a + m|| > \lambda\}.$$

Although not needed for later purposes, the previous proof actually proves the next corollary. It is stated with possible later generalizations to more general than C^*-algebras in mind.

1.7. Corollary. *Assume that A is a Banach algebra with $\phi:B \rightarrow M$ as in* 1.6 *and assume that* 1.5 (c) *holds. Then*

(i) $\{m \in M|\ ||a + m|| > \lambda\} \subseteq \phi(\{b \in B|\ ||a + b|| \geqq \lambda\}) \subseteq \{m \in M|\ ||a + m|| \geqq \lambda\}$;

(ii) $\phi(\{b \in B|\ ||a + b|| > \lambda\}) = \{m \in M|\ ||a + m|| > \lambda\}$.

1.8. *Remark.* If in the previous corollary $||a + m|| = \lambda$, it may be impossible to find a $b \in B$ with $b \supseteq \phi(b) = m$ and $||a + b|| = \lambda$.

1.9. Corollary. *In addition assume that $\phi:B \rightarrow M$ is the complete regularization map of the primitive ideal space B of a C^*-algebra and that B has the hull-kernel topology while M has the topology defined in* 1.2. *Then* $\{m|\ ||a + m|| \geqq \lambda\}$ *is a compact subset of M. If ϕ is an open map then* $\{m \in M|a + m|| > \lambda\}$ *is open in M.*

Proof. By 1.5 (b) and the fact that ϕ is continuous, the set

$$\{m \in M|\ ||a + m|| \geqq \lambda\}$$

is compact. (Note that it is closed because M is Hausdorff since $F(B)$ is.) By 1.5 (a), $\{b|\ ||a + b|| > \lambda\}$ is open in B. But then 1.6 (ii) shows that so is $\{m|\ ||a + m|| > \lambda\}$ provided ϕ is open.

2. The main theorem. For the remainder assume that A does not contain an identity. Then B is locally compact while B_1 is compact.

2.1. LEMMA. *In the notation of the previous section there is a commutative diagram*

$$\begin{tikzcd} B \arrow[r, tail, "j"] \arrow[d, "\phi"'] & B_1 \arrow[d, "\phi_1"] \\ M \arrow[r, tail, "i"'] & M_1 \end{tikzcd}$$

Proof. For $b \in B$, and the corresponding $\bar{b} = \{\alpha \in A_1 | \alpha A \subseteq b\}$, we have $\bar{b} \cap R = \{r \in R | rA \subseteq b\} \equiv p$, where $F(b) = p$. But by the definition of ϕ and m (or ϕ_1 and m_1) we have

$$\phi_1(\bar{b}) = m_1(p) = \cap \{J | J \in B_1, J \cap R = \bar{b} \cap R = p\}.$$

Since, $\phi(b) = m(p)$, $i(m(p)) = m_1(p)$, it follows that $\phi_1 j(b) = i\phi(b)$. Thus $\phi_1 j = i\phi$ and the diagram commutes.

2.2. LEMMA. *Let center* $A \equiv Z$. *Then* $B_1 \backslash \bar{B} = \{p + A | p \in Y, Z \subseteq p\}$.

Proof. If $Z \not\subseteq p$, then $p + Z = R$, and $p + A = R + A$. If $Z \subseteq p$, then

$$\frac{R + A}{p + A} \cong \frac{R}{R \cap (p + A)} = \frac{R}{p} = \mathbf{C}.$$

Thus $p + A \in B_1 \backslash \bar{B}$ if $Z \subseteq p$. Conversely, since $\bar{B} = \{J \in B_1 | A \not\subseteq J\}$, if $I \in B_1 \backslash \bar{B}$, then $A \not\subseteq I \subset R + A$. Thus $I = R \cap I + A$ with $(R + A)/I = R/R \cap I$. Thus the latter is $\mathbf{C}$ and $p = R \cap I \in Y$. Hence $I = p + A$.

The first three conclusions of the next proposition would immediately follow from the commutativity of the diagram in 2.1 in case ϕ was one-to-one.

2.3. PROPOSITION. *For an arbitrary C*-algebra, the following hold:*

(i) $\phi_1^{-1}(i(M)) \subseteq \bar{B}$.

(ii) *For any* $I \in \phi_1^{-1}(i(M)) \backslash \bar{B}$, *there exists a* $b \in B$ *with* $\phi(I) = \phi(\bar{b})$.

(iii) *Furthermore,* $I \cap R = \bar{b} \cap R \in Y$, *and*

(iv) $Z \subseteq b$.

Proof. (i) The commutativity of the diagram in 2.1, i.e., $i\phi = \phi_1 j$, implies that $\phi_1^{-1}(i(M)) \supseteq \bar{B}$.

(ii) By 2.2, $I = p + A$ for some $p \in Y$ with $Z \subseteq p$. Since $I \in \phi_1^{-1}(i(M))$, $\phi_1(I) = i(m)$ for some $m \in M$. Since $M = \phi(B)$, there exists a $b \in B$ with $m = \phi(b)$. Thus $\phi_1(I) = i\phi(b) = \phi_1 j(b) = \phi_1(\bar{b})$.

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(iii) For any $I \in B_1$, $\phi_1(I) \cap R = I \cap R \in Y$. Set $p = \phi_1(I) \cap R$. Thus $p = \phi_1(\bar{b}) \cap R = \bar{b} \cap R = F(b)$.

(iv) Suppose $Z \not\subseteq b$. Then $z \in Z \backslash b$ implies that $zA \not\subseteq b$ and hence $z \notin \bar{b}$. Thus $Z \not\subseteq b \cap R = p$, a contradiction. Thus $Z \subseteq b$.

Some additional information about p, $\bar{b}$, and I is contained in the next corollary.

2.4. COROLLARY. *Under the assumptions and with the notation of the previous proposition for* $p = I \cap R = \bar{b} \cap R$ *as in* (iii), *the following hold*

(v) b *is non-modular* $\Rightarrow \bar{b} = b + p$;

(vi) $1 = e + b \in A/b$ *for some* $e \in A \Rightarrow \bar{b} = R(1 - e) + b$;

(vii) $I = A + p; \bar{b} \subset I$.

Proof. (v) It will be shown that $\bar{b} = b + p$. Suppose $r - c \in \bar{b} = \{\alpha \in A_1 | \alpha A \subseteq b\}$ with $r \in R$, $c \in A$. Then for $x, y \in A$

$$rxy = xry = cxy = xcy \text{ modulo } (b).$$

Thus $(cx - xc)A \subseteq b$. But $b = \{a \in A | aA \subseteq b\}$. Hence $cx - xc \in b$ for any $x \in A$. Thus $c + b \in$ center A/b. At this point our additional hypothesis that b is non-modular has to be invoked. Thus center $A/b = 0$ and $c \in b$. But then $(r - c)A \subseteq b$ if and only if $rA \subseteq b$, or $r \in F(b) = p$. Thus $\bar{b} = b + p \subset A + p = I$, and conclusion (v) follows.

(vi) Clearly, $[R(1 - e) + b]A \subseteq b$, hence

$$R(1 - e) + b \subseteq \{\alpha \in A_1 | \alpha A \subseteq b\} = \bar{b}.$$

Conversely, if $r - c \in \bar{b}$, then $r - c = r(1 - e) + (re - ce) + (ce - c)$, where the last two terms are in b. Thus $\bar{b} = R(1 - e) + b$.

(vii) By 2.2, $I = A + p$. By (v) and (vi), $\bar{b} \subset I$. Alternatively, since $b \subset I = \{\alpha \in A_1 | \alpha A_1 \subseteq I\}$, also $\bar{b} \subset I$.

2.5. THEOREM. *Consider a C^*-algebra A with center Z, primitive ideal space B in the hull-kernel topology, and $\phi : B \rightarrow M$ its complete regularization. Assume that $Z \not\subseteq b$ for every primitive ideal b of A. Then an arbitrary compact subset $K \subset M$ is contained in a compact subset of M of the form*

$$K \subset \{m \in M | \ ||a + m|| \geqq \lambda\}$$

for some positive real $\lambda > 0$.

Proof. Since i is continuous (in fact, a homeomorphism) and K compact, also $i(K)$ is compact in M_1. Since M_1 is Hausdorff, $i(K)$ is closed. Since ϕ_1 is continuous, $\phi_1^{-1}(i(K))$ is also closed. But B_1 is compact and hence $\phi_1^{-1}(i(K))$ is compact. As a consequence of 2.3 (iv) and the assumption that $Z \not\subseteq b$ for all $b \in B$, it follows that $\phi_1^{-1}(i(K)) \subseteq \bar{B}$. However, for any $\bar{b} \in B$, $A \not\subseteq \bar{b}$ because $\bar{b} \cap A = b$. Let j^* be the corestriction $j^* : B \rightarrow \bar{B}$ of j to its image $\bar{B}$. Since j^* is a homeomorphism, $j^{*-1}(\bar{B} \cap \phi_1^{-1}(i(K))) \subset B$ is compact. If $\phi_1{}^*$ is

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the restriction of ϕ_1 to $\bar{B}$, then $\phi^{*-1}(i(K)) = \bar{B} \cap \phi^{*-1}(i(K))$ and the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, tail, two heads, "j^*"] \arrow[d, two heads, "\phi"'] & \bar{B} \arrow[d, "\phi_1^*"] \\ M \arrow[r, tail, "i"'] & M_1 \end{tikzcd}$$

Consequently, $\phi^{-1}(K) = j^{*-1}\phi_1^{*-1}(i(K))$ is a compact subset of B. By 1.5 (a), there is a $\rho > 0$ and a finite subset $\mathscr{F} \subset A$ such that

$$\phi^{-1}(K) \subseteq \cup \{b \in B | \, ||c + b|| > \rho, c \in \mathscr{F}\}.$$

Set $a = \sum\{c^*c | c \in \mathscr{F}\}$. In any C^*-algebra A, if $0 < x < y \in A$, then also $0 < ||x|| < ||y||$. Thus $||(c^* + b)(c + b)|| = ||c^*c + b|| \leqq ||a + b||$. Thus if $||c + b|| < \rho$, then

$$\rho^2 \leqq ||c + b||^2 = ||c^*c + b|| \leqq ||a + b||$$

and $\phi^{-1}(K) \subseteq \{b \in B| \, ||a + b|| \geqq \lambda\}$ with $\lambda = \rho^2$. For $||a + b|| \geqq \lambda$, set $m = \phi(b)$. Then $b \supseteq m$ and $||a + m|| \geqq ||a + b||$. Thus

$$K = \phi(\phi^{-1}(K)) \subseteq \{\phi(b)| \, ||a + b|| \geqq \lambda\}$$
$$\subseteq \{m| \, ||a + m|| \geqq \lambda\}.$$

The next result is stated only for the sake of completeness.

2.6. COROLLARY. *If A is any C^*-algebra with $\phi:B \rightarrow M$ an open map, then any compact subset $K \subset M$ is contained in one of the form*

$$K \subset \{m \in M| \, ||a + m|| \geqq \lambda\}$$

for some real $\lambda > 0$.

Proof. Since ϕ is open, by 1.5(a), sets of the form $\{m| \, ||c + m|| > \rho\} = \phi\{b| \, ||c + b|| > \rho\}$ with varying $c \in A$ and fixed real $\rho > 0$ provide an open cover of K in M. The rest of the proof is as before.

The next proposition isolates the obstruction to proving the main Theorem 2.5 in general.

2.7. PROPOSITION. *Consider an arbitrary C^*-algebra A with centroid R, and $\phi:B \rightarrow M$ as previously. Any compact subset $K \subset M$ is of the form*

$$K \subset \{m \in M| \, ||a + m|| \geqq \lambda\} \quad \lambda > 0$$

provided $B_1 = \text{Prim } (R + A) \supset \bar{B}$ (see 1.3) *satisfy the following condition:*

for all $I \in B_1 \backslash \bar{B}$ and all $r \in R$ with $||r + I|| \geqq \lambda$, there exists $a \in A$ such that for all $b \in B$, $||r + \bar{b}|| \leqq ||a + \bar{b}||$.

Proof. Throughout, let $\lambda > 0$ be any fixed real number. For $\alpha \in A_1$, let $\mathcal{O}(\alpha, \lambda) \subseteq B_1$ be the open set $\mathcal{O}(\alpha, \lambda) = \{I \in B_1 | \, ||\alpha + I|| > \lambda\}$. For $a \in A$ and $b \in B$ it follows from $b = A \cap \bar{b}$ that $A/b \cong (A + \bar{b})/\bar{b} \subseteq A_1/\bar{b}$ and hence that $||a + b|| = ||a + \bar{b}||$. Since $A \not\subseteq \bar{b}$ for all $b \in B$, $\phi_1^{-1}(i(K)) \cap \bar{B}$ is covered by open sets of the form $\mathcal{O}(a, \lambda)$ for varying $a \in A$. For $I \in \phi_1^{-1}(i(M))\backslash\bar{B}$, by 2.3, $I = \phi_1(\bar{b}) = A + p$ with $b \in B$, $I \supset \bar{b}$ and $p = I \cap R = \bar{b} \cap R \in Y$. Thus $\phi_1^{-1}(i(M))\backslash\bar{B}$ is covered by sets of the form $\mathcal{O}(r; \lambda)$ for various $r \in R$. Thus since it follows from the proof of Theorem 2.5 that $\phi_1^{-1}(i(K)) \subset B_1$ is compact, there are finite subsets $\mathcal{F}_1 \subset R$, $\mathcal{F}_2 \subset A$ such that

$$\phi_1^{-1}(i(K)) \subseteq \bigcup \{\mathcal{O}(r, \lambda) | r \in \mathcal{F}_1\} \cup \bigcup \{\mathcal{O}(a, \lambda) | a \in \mathcal{F}_2\}.$$

For each $r \in \mathcal{F}_1$, let $a = a(r)$ be the element given by the hypothesis. Then for any $I \in \mathcal{O}(r, \lambda)$, there exists a $b \in B$ with $\bar{b} \subset I$, with $\phi_1(\bar{b}) = \phi_1(I)$, and $\phi(b) = m \in K$. Furthermore,

$$\begin{aligned} ||a(r) + m|| &\geqq ||a(r) + b|| = ||a(r) + \bar{b}|| \\ &\geqq ||a(r) + I|| \geqq ||r + I|| > \lambda. \end{aligned}$$

On the other hand, any element $I \in \bar{B} \cap \phi_1^{-1}(i(K))$ is of the form $I = \bar{q}$ with $q \in B$. Hence if $\bar{q} \in \mathcal{O}(a, \lambda)$, then

$$||a + \phi(q)|| \geqq ||a + q|| = ||a + \bar{q}|| > \lambda.$$

Consequently

$$\begin{aligned} K \subset \{m \in M | \, ||a + m|| \geqq \lambda, a \in \mathcal{F}_2 \textit{ or} \\ a = a(r) \textit{ with } r \in \mathcal{F}_1\}, \end{aligned}$$

and the rest of the proof is as before.

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