T. Sugatani
Nagoya Math. J.
Vol. 81 (1981), 73–78

RINGS OF CONVERGENT POWER SERIES AND WEIERSTRASS PREPARATION THEOREM

TAKASI SUGATANI

§0.

Let B be a B-ring with a nonarchimedean valuation $|\ |$, i.e., B is an integral domain satisfying the following conditions: (i) B is bounded ($|a| \leqq 1$ for every $a \in B$), (ii) the boundary $\partial(B) = \{a \in B; |a| = 1\}$ forms a multiplicative group. Let Z_+ denote the set of all nonnegative integers. Let $n \in Z_+$. Let $x_1, \cdots, x_n$ be n variables over B. We denote by $A_n = B\langle x_1, \cdots, x_n\rangle$ the set of all elements which can be written in the form

$$\sum_\nu a_\nu x^\nu ,$$

where $a_\nu \in B$ for all $\nu \in Z_+^n$ and $|a_\nu| \to 0$ as $\nu_1 + \cdots + \nu_n \to \infty$. We define a norm $\|\ \|$ on A_n: For $g = \sum a_\nu x^\nu \in A_n$, let $\|g\| = \max\{|a_\nu|\}$. Let m be the maximal ideal of B and $k = B/m$ be the residue field. Let τ be the canonical mapping of B onto k. Then τ can be extended to an epimorphism from A_n to a polynomial ring $k[x_1, \cdots, x_n]$ in the usual manner. We assume, throughout this paper, the B-ring B is complete. We shall identify $A_{n-1}\langle x_n\rangle$ with A_n so that each element g of A_n has an expression $\sum g_i x_n^i$, where $g_i \in A_{n-1}$ for all $i \in Z_+$ and $\|g_i\| \to 0$ as $i \to \infty$. For any $s \in Z_+$, let P_s denote the set of all polynomials of $A_{n-1}[X_n]$ of degree $< s$. One can see several properties on a B-ring in [2], [4].

In this paper, we shall prove Weierstrass Preparation Theorem for A_n. We shall obtain Weierstrass Form Theorem and Scherung Theorem for A_n also.

Before going into discussions, the author wishes to express his thanks to Professor Hisasi Morikawa for his valuable suggestions and constant encouragement in the presentation of this paper.

Received June 1, 1979.

§1.

To consider Weierstrass Preparation Theorem we need some information on unit elements of A_n. We prove

PROPOSITION 1.1. *Let* $g = \sum a_\nu x^\nu \in A_n$. *Then* g *is a unit element of* A_n *if and only if*

$$\begin{cases} |a_{0,\dots,0}| = 1 \,, \\ |a_\nu| < 1 \text{ for each } \nu \neq (0, \cdots, 0) \,. \end{cases} \tag{1.1}$$

Proof. Let g be a unit element of A_n then there exists an element u of A_n such that $gu = 1$. It follows

$$1 = \tau(gu) = \tau(g)\tau(u) \in k[x_1, \cdots, x_n] \,.$$

Hence (1.1) holds. Conversely, suppose g satisfies (1.1). Then it can be seen that the inverse element of g is given by

$$g^{-1} = (a_{0,\dots,0})^{-1}[1 + \sum_1^\infty (g'')^i] \,, \tag{1.2}$$

where

$$-g'' = (a_{0,\dots,0})^{-1} \sum_{\nu \neq (0,\cdots,0)} a_\nu x^\nu \,.$$

With this the proof is complete.

From this proposition, we have the followings:

Remark 1.2. If $n \geqq 1$, then A_n is not a quasi-local ring.

Proof. For a contradiction, we assume A_n is a quasi-local ring. Then it follows the set M of all nonunit elements of A_n forms a maximal ideal. By Proposition 1.1, for instance, x_1 and $1 + x_1$ belong to M. Then 1 belongs to M, a contradiction.

Let $g = \sum g_i x_n^i \in A_{n-1}\langle x_n \rangle$. Let $s \in Z_+$. We say that g is general (allgemein) in x_n of order s if g_s is a unit element of A_{n-1} and $\|g_i\| < 1$ for all $i > s$.

Remark 1.3. $g \in A_n$ is general in x_n of order $s \geqq 0$ if and only if

$$\tau(g) = \tau(g_0) + \tau(g_1)x_n + \cdots + \tau(g_s)x_n^s \tag{1.3}$$

for which $\tau(g_i) \in k[x_1, \cdots, x_{n-1}]$ for each $i = 0, \cdots, s-1$, and $\tau(g_s) \in k^* = k - \{0\}$.

Proof. By Proposition 1.1, it is clear g_s is a unit element of A_{n-1} if and only if $\tau(g_s)$ is in k^*. It is easy to verify the other conditions on the coefficients of g.

§ 2.

In this section we shall show Weierstrass Form for A_n, which is a generalization of the result of Grauert-Remmert [1].

THEOREM 2.1 (Weierstrass Form for A_n). *Let $g \in A_n$ be general in x_n of order $s \geqq 0$. Then for each $f \in A_n$ there exists a unique pair $q \in A_n$, $r \in P_s$ satisfying*

(2.1) $$f = qg + r\,.$$

Further, we have

(2.2) $$\|f\| = \max\{\|q\|, \|r\|\}\,.$$

In order to prove this theorem we need the following lemmas. Lemma 2.3 is established for $K\langle x_1, \cdots, x_n\rangle$ (Satz 2.1 of [1]). But in our case it cannot be assumed that for a nonzero element f of $K\langle x_1, \cdots, x_n\rangle$ there exists a nonzero element a in K satisfying $\|af\| = 1$, because we take an arbitrary B-ring B as a coefficient ring. So, we prove at first Lemma 2.2 analogous to Theorem 3.20 in [3].

LEMMA 2.2. *Let $g \in A_n$ be general in x_n of order $s \geqq 0$. Then for $q \in A_n$ and $r \in P_s$ we have*

(2.3) $$\|qg + r\| \geqq \|q\|\,.$$

Proof. Let $q = \sum b_\nu x^\nu \in A_n$. Let $\mu = (\mu_1, \cdots, \mu_n) \in Z_+^n$ be the heighest indexterm of ν such that $\|q\| = |b_\nu|$. If $qg = \sum c_\nu x^\nu \in A_n$ then $\|q\| = \|qg\| = |c_{\mu'}|$, where $\mu' = (\mu_1, \cdots, \mu_{n-1}, \mu_n + s)$. If $r \in A_n$ such that the coefficient of $x^{\mu'}$ vanishes, then $\|qg + r\| \geqq \|q\|$. In particular this is true for all $r \in P_s$ and now (2.3) follows.

LEMMA 2.3. *Let $g \in A_{n-1}[x_n]$ be of degree s and the leading coefficient be a unit element of A_{n-1}. Then for each $f \in A_n$ there exists a pair $q \in A_n$, $r \in P_s$ satisfying*

(2.4) $$f = qg + r\,.$$

Proof. Let $f = \sum f_i x_n^i \in A_{n-1}\langle x_n\rangle$. It follows for each $i \geqq 0$ there exist

q_i and $r_i \in A_{n-1}[x_n]$ such that r_i is of degree $< s$ and $f_i x_n^i = q_i g + r_i$. Then (2.3) implies

$$\|f_i\| = \max\{\|q_i\|, \|r_i\|\} . \tag{2.5}$$

Let $r = \sum_0^\infty r_i$ and $q = \sum_0^\infty q_i$. Then we see that $q \in A_n$ and $r \in P_s$. With these q and r we obtain the equation (2.4).

Proof of Theorem 2.1. If $r \in gA_n \cap P_s$ then by Lemma 2.2, $0 = \|r - r\| \geqq \|r\|$. Therefore we have

$$gA_n \cap P_s = 0 ,$$

which shows the uniqueness of a pair q, r of (2.1).

We next prove

$$A_n = gA_n + P_s ,$$

by using Grauert-Remmert's method in [1]. In fact, let $g = \sum g_i x_n^i \in A_{n-1}\langle x_n \rangle$ and let $g = g^{(1)} + g^{(2)}$, where $g^{(1)} = \sum_0^s g_i x_n^i$. Then we have $\delta = \|g^{(2)}\| < 1$. We define a set of elements f_j, q_j and r_j of A_n in the following way: Let $f_0 = f = q_0 g^{(1)} + r_0$, where $r_0 \in P_s$. For $j \in Z_+$ we put $f_{j+1} = f_j - q_j g - r_j = q_{j+1} g^{(1)} + r_{j+1}$, where $r_{j+1} \in P_s$. This procedure is possible by Lemma 2.3. Then it follows

$$f_{j+1} = -q_j g^{(2)}$$

whence, by (2.5)

$$\|f_{j+1}\| = \delta\|q_j\| \leqq \delta\|f_j\| ,$$

therefore we have

$$\|f_{j+1}\| \leqq \delta\|f_j\| .$$

By induction on $j \geqq 0$, we have

$$\|f_j\| \leqq \delta^j\|f\|, \ \|q_j\| \leqq \delta^j\|f\|$$

and

$$\|r_j\| \leqq \delta^j\|f\| .$$

Putting $q = \sum_0^\infty q_j$ and $r = \sum_0^\infty r_j$, we have $q \in A_n$ and $r \in P_s$ satisfying (2.7) as required.

By the definition it is clear $\|q\| \leqq \|f\|$. Then we see $\|r\| = \|f - qg\| \leqq \max\{\|f\|, \|q\|\} = \|f\|$. Therefore we have $\|f\| \geqq \max\{\|q\|, \|r\|\}$. This proves

https://doi.org/10.1017/S0027763000019164 Published online by Cambridge University Press

half of (2.2) and the other half is obvious. Thus our theorem is completely proved.

§ 3.

THEOREM 3.1 (Weierstrass Preparation Theorem for A_n). *Let $g \in A_n$ be general in x_n of order $s \geqq 0$. Then there exist uniquely $u, a_0, \cdots, a_{s-2}$ and a_{s-1} satisfying the following conditions: u is a unit element of A_n, $a_0, \cdots, a_{s-1}$ are in A_{n-1} and*

$$g = u(x_n^s + a_{s-1}x_n^{s-1} + \cdots + a_1 x_n + a_0) . \tag{3.1}$$

Proof. By Theorem 2.1 there exists a unique pair $q \in A_n$, $r \in P_s$ satisfying

$$x_n^s = qg + r .$$

Applying Theorem 2.1 again, this time with $x_n^s - r$ instead of g, we obtain a unique pair $q' \in A_n$, $r' \in P_s$ satisfying

$$g = q'(x_n^s - r) + r' .$$

Then

$$g = q'qg + r' ,$$

therefore we must have $q'q = 1$ and $r' = 0$. In particular we have

$$g = q'(x_n^s - r) . \tag{3.2}$$

Put $-a_0, -a_1, \cdots, -a_{s-1}$ as the coefficients of r and $u = q'$. The uniqueness follows from the choice of r, which shows our assertion.

§ 4.

In this section we prove Scherung Theorem for A_n.

THEOREM 4.1. *Suppose the residue field k of B is infinite. Let f be in A_n and $\|f\| = 1$. Then there exists a B-automorphism σ of A_n such that $\sigma(f)$ is general in x_n.*

Proof. Let $f = \sum_0^\infty f_j$, where each f_j is the j-th homogeneous part of f. Then there exists f_s such that $\|f_s\| = 1$ and $\|f_j\| < 1$ for all $j > s$. Let $\tau(f_s) = \bar{f}_s$. Then $\bar{f}_s$ is a nonzero element of $k[x_1, \cdots, x_n]$. If $n = 1$ then the assertion is clear. Assume $n \geqq 2$. By our assumption that k is infinite, we can choose an element $(\bar{a}_1, \cdots, \bar{a}_{n-1}, \bar{a}_n) \in k^n$ satisfying

$$f_s(\bar{a}_1, \cdots, \bar{a}_n) \in k^* ,$$

where $\bar{a}_j = \tau(a_j)$ for $a_j \in B$, $j = 1, \cdots, n$. Here we may assume $|a_n| = 1$. Put $b = f_s(a_1, \cdots, a_n)$. Then $|b| = 1$. We define a B-algebra endomorphism σ such as

$$\sigma(x_j) = x_j + a_j x_n, \qquad j = 1, \cdots, n-1 ,$$
$$\sigma(x_n) = x_n .$$

Then σ^{-1} is given by

$$\sigma^{-1}(x_j) = x_j - a_j x_n, \qquad j = 1, \cdots, n-1 ,$$
$$\sigma^{-1}(x_n) = x_n .$$

It can be seen by easy calculations

$$\sigma(f) = \sum_0^\infty f_i^* x_n^i ,$$

where each $f_i^* \in A_{n-1}$ and $\|f_i^*\| < 1$ for all $i > s$. In particular f_s^* is a unit element of A_{n-1}, for the constant term is equal to b and the norm of the part of terms of degree $\geqq 1$ is less than 1. Therefore $\sigma(f)$ is general in x_n of order s. Thus σ is the B-automorphism to be desired.

References

[1] H. Grauert and R. Remmert, Nichtarchimedische Funktionentheorie, Weierstrass-Festschrift, Wissenschaftl. Abh. Arbeitsgem. f. Forsch. Nordrhein-Westfalen, **33** (1966), 393–476.

[2] H. Grauert and R. Remmert, Über die Methode der diskret bewerteten Ringe in der nichtarchimedischen Analysis, Inventiones Math., **2** (1966), 87–133.

[3] A. C. M. van Rooij, Non-Archimedean Functional Analysis, Marcel Dekker, Inc., New York, 1978.

[4] T. Sugatani, On quasi-noetherian rings which are not noetherian, Math. Rep. Toyama Univ., **1** (1978), 65–73.

Department of Mathematics
Faculty of Science
Toyama University