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BERNSTEIN-TYPE INEQUALITIES WITH BOMBIERI NORM

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ABSTRACT. If $P(z) = \sum_{k=0}^{n} a_k z^k$ is an univariate polynomial with degree n then Bombieri norm of P is defined by

$$[P]_n = \left(\sum_{k=0}^{n} \frac{|a_k|^2}{\binom{n}{k}} \right)^{1/2}$$

where $\binom{n}{k}$ denotes the binomial coefficient.

In the present paper we give, under assumptions on the roots of P, optimal Bernstein-type inequalities for the ratio between Bombieri norm of P and that of its derivative P'.

We also give such inequalities for the polar derivatives of P defined by

$$P_1(\alpha, z) = nP(z) - (z - \alpha)P'(z), \alpha \in \mathbb{C}.$$

If P is an univariate polynomial with degree n, a well-known theorem due to Bernstein asserts that

$$\frac{\|P'\|_\infty}{\|P\|_\infty} \leq n$$

where $\|\cdot\|_\infty$ denotes the L_∞-norm, defined by

$$\|P\|_\infty = \max_{|z|=1} |P(z)|.$$

Of course this estimate is optimal since equality holds for $P(z) = z^n$.

This theorem inspired numerous works on polynomials and trigonometric polynomials. Many of them proposed to sharpen Bernstein inequality under assumptions on the roots of P.

Malik (see [10]) proved that if P has no root inside the disc $\{|z| < R\}$ with $R \geq 1$ then

$$\frac{\|P'\|_\infty}{\|P\|_\infty} \leq \frac{n}{1+R}$$

with equality for $P(z) = (z+R)^n$; a result due to Lax (see [9]) in the case $R = 1$. Let us mention that the optimal upper-bound for this ratio when P has all roots in smaller discs (that is, $R < 1$) is not known.

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It can be deduced (see [5]) from Malik's result applied to the reciprocal polynomial that if P has all roots in the disc $\{|z| \leq R\}$ with $R \leq 1$ then

$$\frac{\|P'\|_\infty}{\|P\|_\infty} \geq \frac{n}{1+R}$$

with equality for $P(z) = (z+R)^n$, but the problem remains open when $R > 1$.

Such inequalities are known for the L_p-norms, $1 \leq p < +\infty$, defined by

$$\|P\|_p = \left(\int_0^{2\pi} |P(e^{it})|^p \frac{dt}{2\pi} \right)^{1/p}.$$

The general one is due to Zygmund (see [13]): if P has degree n, then

$$\frac{\|P'\|_p}{\|P\|_p} \leq n.$$

Here again, $P(z) = z^n$ gives equality.

De Bruijn (see [4]) proved the following optimal refinement: if P has degree n and no root inside the unit disc $\{|z| < 1\}$, then

$$\frac{\|P'\|_p}{\|P\|_p} \leq nC_p,$$

with

$$C_p = \frac{1}{\|1+z\|_p} = \frac{1}{2}\left(\frac{\sqrt{\pi}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+\frac{1}{2})} \right)^{1/p}.$$

It is easily seen that equality holds for $P(z) = z^n + 1$.

Further refinements do not seem to be known for the L_p-norms, except for $p = 2$. In this case the L_2-norm coincides with the l_2-norm on the coefficients of the polynomial; that is,

$$\|P\|_2 = \left(\int_0^{2\pi} |P(e^{it})|^2 \frac{dt}{2\pi} \right)^{1/2} = \left(\sum_{k=0}^{n} |a_k|^2 \right)^{1/2} \text{ if } P(z) = \sum_{k=0}^{n} a_k z^k.$$

Rahman (see [11]) got with this norm an optimal estimate for polynomials with degree n having no root inside the disc $\{|z| < R\}$ with $R \leq 1$, namely

$$\frac{\|P'\|_2}{\|P\|_2} \leq \frac{n}{\sqrt{1+R^{2n}}}$$

with equality for $P(z) = z^n + R^n$. For $R > 1$ Govil and Rahman (see [7]) proved the estimate

$$\frac{\|P'\|_2}{\|P\|_2} \leq \frac{n}{\sqrt{1+R^2}}$$

and conjectured the following one:

$$\frac{\|P'\|_2}{\|P\|_2} \leq n \max\left\{ \frac{1}{\|z^n + R^n\|_2}, \frac{\|(z+R)^{n-1}\|_2}{\|(z+R)^n\|_2} \right\},$$

a conjecture which is still open.

For polynomials with all roots in such discs, estimates are known (see [5]) but they seem to be far from optimality.

In the present paper, we give such inequalities with Bombieri norm. Introduced in [2], this norm depends on the degree of the polynomial on which it applies. If $P(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial with degree n then Bombieri norm of P is defined by

$$[P]_n = \left(\sum_{k=0}^{n} \frac{|a_k|^2}{\binom{n}{k}} \right)^{1/2}$$

where $\binom{n}{k}$ denotes the binomial coefficient.

We give estimates for the ratio $\frac{[P']_{n-1}}{[P]_n}$ under each of the above assumptions on the roots of P. All of these estimates are optimal and we give the extremal polynomials. We also derive optimal bounds for the ratio between Bombieri norm of P and that of its polar derivative with respect to any point $\alpha \in \mathbb{C}$. This polar derivative is defined by

$$P_1(z) = nP(z) - (z - \alpha)P'(z).$$

In this case, the roots of P are assumed to stay inside or outside discs or half-planes related to α.

In Section 1, we study elementary properties of Bombieri norm with respect to polynomial derivatives and we state our main theorem. This theorem is proved in Section 2 and estimates are derived in Section 3 for polar derivatives.

1. **Bombieri norm and polynomial derivatives.** If $P(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial with degree n then Bombieri norm of its derivative is given by

$$[P']_{n-1} = \left(\sum_{k=0}^{n} \frac{k^2 |a_k|^2}{\binom{n-1}{k-1}} \right)^{1/2}$$
$$= \left(n \sum_{k=0}^{n} k \frac{|a_k|^2}{\binom{n}{k}} \right)^{1/2}.$$

It is easy to see that Bombieri norm of P and P' are connected by the following identity.

PROPOSITION 1. *If $P(z) = \sum_{k=0}^{n} a_k z^k$ has degree n, then*

$$n^2 [P]_n^2 = [P']_{n-1}^2 + [(P^*)']_{n-1}^2$$

where P^* denotes the reciprocal polynomial of P, defined by

$$P^*(z) = z^n \overline{P(1/\bar{z})} = \sum_{k=0}^{n} \bar{a}_{n-k} z^k.$$

Of course the reciprocal polynomial P^* has degree at most n, less than n if $P(0) = 0$. So $(P^*)'$ may have degree less than $n-1$, but in Proposition 1 we still consider its Bombieri norm at degree $n-1$.

From Proposition 1 follows that if P is self-reciprocal (that is, if $P^* = \lambda P$ with $\lambda \in \mathbb{C}$, $|\lambda| = 1$) and thus if P has all roots of modulus 1, then

$$\frac{[P']_{n-1}}{[P]_n} = \frac{n}{\sqrt{2}}.$$

Note that the L_∞-norm has this property as well: the ratio $\frac{\|P'\|_\infty}{\|P\|_\infty}$ is constant (equal to $\frac{n}{2}$) for self-reciprocal polynomials with degree n (see [5]). However, it is easy to see that it is not the case for the L_2-norm.

From Proposition 1 we deduce that if P has degree n, then

$$\frac{[P']_{n-1}}{[P]_n} \leq n.$$

Of course, this bound is optimal since equality holds for $P(z) = z^n$. Hence the ratio is close to n if the roots of P are small, and it follows from Proposition 1 that it is small (as small as wanted) if the roots of P are big. The quantitative formulation of this behaviour is part of our main theorem below.

THEOREM 2. *Let P be a polynomial with degree n.*
If P has no root inside the disc $\{|z| < R\}$ then

$$\text{(1)} \qquad \frac{[P']_{n-1}}{[P]_n} \leq \frac{n}{\sqrt{1+R^{2n}}} \quad \text{if } R \leq 1$$

$$\text{(2)} \qquad \leq \frac{n}{\sqrt{1+R^2}} \quad \text{if } R \geq 1.$$

If P has all roots in the disc $\{|z| \leq R\}$ then

$$\text{(3)} \qquad \frac{[P']_{n-1}}{[P]_n} \geq \frac{n}{\sqrt{1+R^2}} \quad \text{if } R \leq 1$$

$$\text{(4)} \qquad \geq \frac{n}{\sqrt{1+R^{2n}}} \quad \text{if } R \geq 1.$$

All of these estimates are optimal since equality holds in (1) and (4) for $P(z) = z^n + R^n$, in (2) and (3) for $P(z) = (z+R)^n$.

2. **Proof of the main theorem.** We first observe that estimates (1) and (2) are derived from the last two ones by applying Proposition 1 to the reciprocal polynomial. Hence we only have to prove estimates (3) and (4). For this, we need the following theorem of comparison between Bombieri norms of polynomials under assumptions on the values.

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THEOREM 3. *Let P and Q be two polynomials with degree n. Assume that P has all roots in the closed unit disc $\{|z| \leq 1\}$ and that $|P(z)| \geq |Q(z)|$ for every z, $|z| = 1$. Then*

$$[P]_n \geq [Q]_n.$$

PROOF. Arguing from continuity, we may assume that P has all roots in the open unit disc $\{|z| < 1\}$. Then, for $\alpha \in \mathbb{C}$, $|\alpha| > 1$, we consider the two polynomials $R_1 = \alpha P - Q$ and $R_2 = \bar{\alpha} P - Q$. The fonction $f = \frac{Q}{P}$ being holomorphic on $\{|z| \geq 1\}$, the maximum modulus principle ensures

$$|P(z)| \geq |Q(z)| \text{ for every } z, \ |z| \geq 1.$$

Hence the leading coefficient in P is bigger than or equal to that in Q so R_1 and R_2 have degree n, and they clearly have all roots in the open unit disc.

Then we apply the following lemma, immediate consequence of Grace's apolarity theorem (see [1] for details), in which $[\cdot,\cdot]_n$ denotes Bombieri scalar product defined by

$$\left[\sum_{k=0}^{n} c_k z^k, \sum_{k=0}^{n} d_k z^k\right]_n = \sum_{k=0}^{n} \frac{c_k \overline{d_k}}{\binom{n}{k}}.$$

LEMMA 4. *Let P and Q be two polynomials with degree n having all roots in the open unit disc $\{|z| < 1\}$, then*

$$[P, Q]_n \neq 0.$$

Here Lemma 4 gives

$$[R_1, R_2]_n \neq 0$$

that is,

$$\alpha^2 [P]_n^2 - 2\alpha \Re([P, Q]_n) + [Q]_n^2 \neq 0.$$

This being valid for every α, $|\alpha| > 1$, this trinomial in α has its roots of modulus at most 1. Therefore its trailing coefficient is not bigger than its leading one; that is,

$$[P]_n \geq [Q]_n.$$

This completes the proof of Theorem 3. ■

Before using Theorem 3 to prove estimates (3) and (4), let us make three remarks.

REMARK 1. We have seen that assumptions of Theorem 3 imply that $|P(z)| \geq |Q(z)|$ for every z, $|z| \geq 1$. It may be asked whether this condition yields by itself the conclusion $[P]_n \geq [Q]_n$. The answer is negative, as shown by the following counter-example.

Given R, $0 < R < 1$ and $\alpha \in \mathbb{C}$, $|\alpha| > 1$, we consider the polynomials

$$P(z) = (z + \alpha)(z + R)$$

and

$$Q(z) = (z + \alpha)(Rz + 1).$$

It is easily seen that $|P(z)| \geq |Q(z)|$ for every z, $|z| \geq 1$, but one checks that if $\alpha = \frac{1}{R}$ then $[P]_2 < [Q]_2$.

This proves that there is no monotone integral representation for Bombieri norm; that is,

$$[P] = f\Big(\int\int g(z)h\big(|P(z)|\big)\,dz\Big)$$

with f and h monotone and g non-negative, taking into account the values of the polynomial only on $\{|z| \geq 1\}$ and neither is there with the values only on the unit disc $\{|z| \leq 1\}$ as seen with the reciprocal polynomial.

Let us mention that the only known such representation for Bombieri norm is due to Boyd (see [3]) and takes into account the values of the polynomial on the whole complex plane. This integral representation is the following one:

$$[P]_n = \left(\frac{\pi}{n+1}\int_0^{+\infty}\int_0^{2\pi}\frac{|P(re^{it})|^2}{(1+r^2)^{n+2}}\,r\,dr\,dt\right)^{1/2}.$$

REMARK 2. As shown in [1], Lemma 4 is also valid for half-planes whose boundary contains the origin. Moreover, for such half-planes, it is valid with the Euclidean scalar product as well (the Euclidean scalar product is the one associated with the L_2-norm). Hence an analogue of Theorem 3 may be stated as follows:

THEOREM 5. *Let P and Q be two polynomials with degree n and let H be a closed half-plane whose boundary ∂H contains the origin. Assume that P has all roots in H and that $|P(z)| \geq |Q(z)|$ for every z on ∂H, then*

$$[P]_n \geq [Q]_n$$

and

$$\|P\|_2 \geq \|Q\|_2.$$

Once again it is easily seen that the previous assumptions cannot be replaced by the weaker condition "$|P(z)| \geq |Q(z)|$ for every z in the closed complement of H".

REMARK 3. Following the proof of Theorem 3, one can check that this theorem remains valid if Q has degree less than n (and so does Theorem 5), provided that we still consider its Bombieri norm at degree n.

We can now establish estimates (3) and (4). We begin with estimate (3). From Proposition 1 follows that estimate (3) is equivalent to the following one:

$$R[P']_{n-1} \geq [(P^*)']_{n-1}.$$

Since P has all roots in the closed unit disc, so does P' by Lucas theorem. Thus, in order to apply Theorem 3 with polynomials RP' and $(P^*)'$ we only have to use the following lemma, due to Malik (see [10]) and of which we give here a more direct proof.

LEMMA 6. *Let P be a polynomial with all roots in the disc $\{|z| \leq R\}$ with $R \leq 1$, then*

$$R|P'(z)| \geq |(P^*)'(z)|$$

for every z, $|z| = 1$.

PROOF. Let $z_0 \in \mathbb{C}$, $|z_0| = 1$. It is easily seen that

$$|(P^*)'(z_0)| = |nP(z_0) - z_0P'(z_0)|$$

where n is the degree of P.

To prove Lemma 6 we may assume that $R < 1$ (the result with $R = 1$ will follow by continuity), which ensures by Lucas theorem that $P'(z_0) \neq 0$; so we can write

$$\frac{|(P^*)'(z_0)|}{|P'(z_0)|} = \left|n\frac{P(z_0)}{P'(z_0)} - z_0\right| = \left|\frac{1-\xi}{\xi}\right|$$

with $\xi = \frac{1}{n}z_0\frac{P'(z_0)}{P(z_0)}$.

Writing $\alpha_1, \ldots, \alpha_n$ for the roots of P we have

$$\xi = \frac{1}{n}\sum_{j=1}^{n}\frac{z_0}{z_0 - \alpha_j} = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{1-\beta_j}$$

with $\beta_j = \frac{\alpha_j}{z_0}$, $1 \leq j \leq n$. Note that $|\beta_j| = |\alpha_j| \leq R$, $1 \leq j \leq n$.

One checks that the Möbius transform $t_1 : z \longrightarrow \frac{1}{1-z}$ maps the disc $D_R = \{|z| \leq R\}$ onto the disc D with center $\frac{1}{1-R^2}$ and radius $\frac{R}{1-R^2}$. Hence each complex number $\frac{1}{1-\beta_j}$, $1 \leq j \leq n$ belongs to D and so does ξ by convexity.

Therefore it remains to show that

$$\left|\frac{1-z}{z}\right| \leq R$$

for every z in D.

This follows from the fact that the Möbius transform $t_2 = -t_1^{-1} : z \longrightarrow \frac{1-z}{z}$ maps D onto D_R, which completes the proof of Lemma 6 and thus that of estimate (3). ■

To complete the proof of Theorem 2 we must now establish estimate (4). From Proposition 1 follows that estimate (4) is equivalent to the following one:

$$R^n[P']_{n-1} \geq [(P^*)']_{n-1}.$$

Since P has all roots in $\{|z| \leq R\}$, the polynomial $Q(z) = P(Rz)$ has all roots in the unit disc. So estimate (3) gives

$$\frac{[Q']_{n-1}}{[Q]_n} \geq \frac{n}{\sqrt{2}}$$

that is, with Proposition 1,

$$[Q']_{n-1} \geq [(Q^*)']_{n-1}.$$

But $Q'(z) = RP'(Rz)$ and $R \geq 1$, so that

$$[Q']_{n-1} \leq R^n [P']_{n-1}$$

by definition of Bombieri norm.

On the other hand a straightforward calculation gives

$$Q^*(z) = R^n P^*\Big(\frac{z}{R}\Big)$$

and

$$(Q^*)'(z) = R^{n-1}(P^*)'\Big(\frac{z}{R}\Big).$$

Therefore, since $R \geq 1$,

$$[(Q^*)']_{n-1} \geq [(P^*)']_{n-1}.$$

This completes the proof of Theorem 2. ■

3. **Polar derivatives.** Let us first give the following identity, analogue of Proposition 1 for polar derivatives.

PROPOSITION 7. *Let P be a polynomial with degree n and $\alpha \in \mathbb{C}$, $\alpha \neq 0$. Then*

$$n^2(1+|\alpha|^2)[P]_n^2 = [P_1(\alpha, z)]_{n-1}^2 + |\alpha|^2 \Big[P_1\Big(-\frac{1}{\bar{\alpha}}, z\Big)\Big]_{n-1}^2.$$

PROOF. We only need to deal with polynomials P such that $P(0) \neq 0$. The general result will follow by continuity. We write

$$P_1(\alpha, z) = \big((P^*)'\big)^*(z) + \alpha P'(z)$$

and

$$P_1\Big(-\frac{1}{\bar{\alpha}}, z\Big) = \big((P^*)'\big)^*(z) - \frac{1}{\bar{\alpha}} P'(z).$$

Then

$$P_1(\alpha, z) - P_1\Big(-\frac{1}{\bar{\alpha}}, z\Big) = \frac{1+|\alpha|^2}{\bar{\alpha}} P'(z)$$

and

$$P_1(\alpha, z) + |\alpha|^2 P_1\Big(-\frac{1}{\bar{\alpha}}, z\Big) = (1+|\alpha|^2)\big((P^*)'\big)^*(z).$$

Hence, since $[(P^*)']_{n-1} = \Big[\big((P^*)'\big)^*\Big]_{n-1}$, Proposition 1 gives

$$|\alpha|^2 \Big[P_1(\alpha, z) - P_1\Big(-\frac{1}{\bar{\alpha}}, z\Big)\Big]_{n-1}^2 + \Big[P_1(\alpha, z) + |\alpha|^2 P_1\Big(-\frac{1}{\bar{\alpha}}, z\Big)\Big]_{n-1}^2 = n^2(1+|\alpha|^2)^2 [P]_n^2.$$

Expanding Bombieri norms in the left-hand side of this equality, the result follows. ∎

REMARK. Viewing, with suitable conventions, the classical derivative as the polar derivative with respect to the point at infinity; that is,

$$P'(z) = \lim_{\alpha \to \infty} \frac{P_1(\alpha, z)}{\alpha} = \frac{P_1(\infty, z)}{\infty},$$

Proposition 1 appears as the limit case $\alpha = 0$ or $\alpha = \infty$ in Proposition 7.

Of course Proposition 7 yields the following optimal general estimate for Bombieri norm of the polar derivative of any polynomial P with degree n with respect to any point α in $\mathbb{C}$:

$$\frac{[P_1(\alpha, z)]_{n-1}}{[P]_n} \leq n\sqrt{1 + |\alpha|^2},$$

with equality for $P(z) = (z + \frac{1}{\alpha})^n$.

It can be seen in [8] that polar derivation appears in a natural way by subjecting the complex plane to Möbius transformations

$$t(z) = \frac{az + b}{cz + d},$$

with a, b, c, d in $\mathbb{C}$, $ad - bc \neq 0$. Indeed, if P is a polynomial with degree n and α a point in $\mathbb{C}$, consider the Möbius transformation

$$t_\alpha(z) = \frac{\alpha z + 1}{z - \bar{\alpha}}.$$

Writing $A = \begin{pmatrix} \alpha & 1 \\ 1 & -\bar{\alpha} \end{pmatrix}$ for the matrix associated with t_α, consider the polynomial $C_{A,n}(P)$ with degree at most n defined by

$$C_{A,n}(P) = (z - \bar{\alpha})^n P\Big(\frac{\alpha z + 1}{z - \bar{\alpha}}\Big).$$

Taking the derivative of this polynomial and subjecting the complex plane to to the reverse Möbius transformation, it is easy to check that one finds precisely the polar derivative of P with respect to α; that is,

$$C_{A,n}\big(P_1(\alpha, z)\big) = \big(C_{A,n-1}(P)\big)'(z).$$

Looking at polar derivatives through Möbius transformations makes very attractive the use of Bombieri norm. Indeed, as proved simultaneously Frot (see [6]) and Reznick (see [12]), Bombieri norm and the associated scalar product have the following remarkable property for Möbius transformations.

THEOREM 8. *Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible matrix of order 2 with complex coefficients and let P and Q be two polynomials with degree n. Writing*

$$C_{M,n}(P)(z) = (cz + d)^n P\Big(\frac{az + b}{cz + d}\Big),$$

we have

$$[C_{M,n}(P), Q]_n = [P, C_{\bar{M}^t,n}(Q)]_n.$$

Theorem 8 implies that Bombieri norm is stable by polynomial composition with unitary Möbius transformations; that is,

$$[C_{M,n}(P)]_n = [P]_n$$

if M is unitary. The Möbius transformation t_α involved in the polar derivative of P with respect to α is "almost unitary"; that is, its matrix A satisfies

$$A\bar{A}^t = (1 + |\alpha|^2)I_2$$

where I_2 denotes the identity matrix of order 2. Thus Theorem 8 yields

$$[C_{A,n}(P)]_n = (1 + |\alpha|^2)^{\frac{n}{2}}[P]_n$$

and

$$\left[C_{A,n-1}\big(P_1(\alpha, z)\big)\right]_{n-1} = (1 + |\alpha|^2)^{\frac{n-1}{2}}[P_1(\alpha, z)]_{n-1}$$

that is,

$$\left[\big(C_{A,n}(P)\big)'\right]_{n-1} = (1 + |\alpha|^2)^{\frac{n-1}{2}}[P_1(\alpha, z)]_{n-1}.$$

In order to use these identities to deduce from Theorem 2 analogous estimates for polar derivatives, we have to study how the zeros of $C_{A,n}(P)$ depend on those of P. Precisely, we have to find, for every $R \geq 0$, the region S_R in which (respectively outside of which) the zeros of P have to be assigned to stay in order to make sure that those of $C_{A,n}(P)$ are in $D_R = \{|z| \leq R\}$ (respectively outside of D_R).

Writing $z_1, \ldots, z_n$ for the zeros of P, those of $C_{A,n}(P)$ are clearly $t_\alpha^{-1}(z_1), \ldots, t_\alpha^{-1}(z_n)$. So $S_R = t_\alpha(D_R)$; that is,

- the closed disc with center $\Omega = \frac{1+R^2}{R^2-|\alpha|^2}\alpha$ and radius $\frac{R(1+|\alpha|^2)}{|\alpha|^2-R^2}$ if $R < |\alpha|$,
- the closed half-plane limited by the mediatrix Δ of $(\alpha, -\frac{1}{\bar{\alpha}})$ and containing $-\frac{1}{\bar{\alpha}}$ if $R = |\alpha|$,
- the closed complement of the disc with center $\Omega = \frac{1+R^2}{R^2-|\alpha|^2}\alpha$ and radius $\frac{R(1+|\alpha|^2)}{R^2-|\alpha|^2}$ if $R > |\alpha|$.

We can now state the following theorem, analogue of Theorem 2 for polar derivatives.

THEOREM 9. *Let P be a polynomial with degree n and* $\alpha \in \mathbb{C}$. *Writing* $t_\alpha(z) = \frac{\alpha z+1}{z-\bar{\alpha}}$ *and* $S_R = t_\alpha(\{|z| \leq R\})$ *for every* $R \geq 0$, *we have:*

- *if P has no root in the interior of* S_R, *then*

$$\frac{[P_1(\alpha, z)]_{n-1}}{[P]_n} \leq n\sqrt{\frac{1 + |\alpha|^2}{1 + R^{2n}}} \text{ if } R \leq 1$$
$$\leq n\sqrt{\frac{1 + |\alpha|^2}{1 + R^2}} \text{ if } R \geq 1.$$

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- *if P has all roots in* S_R, *then*

$$\frac{[P_1(\alpha, z)]_{n-1}}{[P]_n} \geq n\sqrt{\frac{1+|\alpha|^2}{1+R^2}} \text{ if } R \leq 1$$
$$\geq n\sqrt{\frac{1+|\alpha|^2}{1+R^{2n}}} \text{ if } R \geq 1.$$

All of these etimates are optimal.

PROOF. We just apply Theorem 2 to $C_{A,n}(P)$ and use the above identities for Bombieri norm of the involved polynomials. These estimates are optimal since those of Theorem 2 are and extremal polynomials are found by composition with the reverse Möbius transformation t_α^{-1} from polynomials extremal for the estimates of Theorem 2. ■

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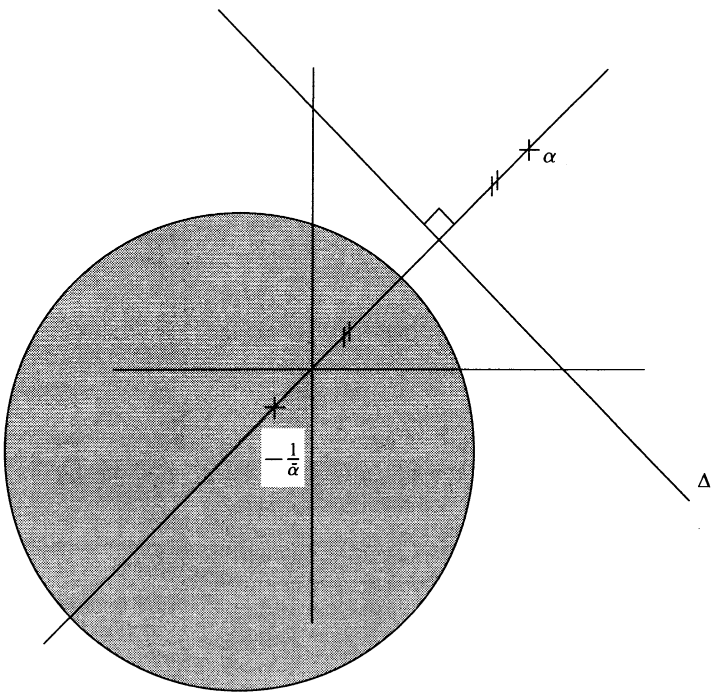

FIGURE 1

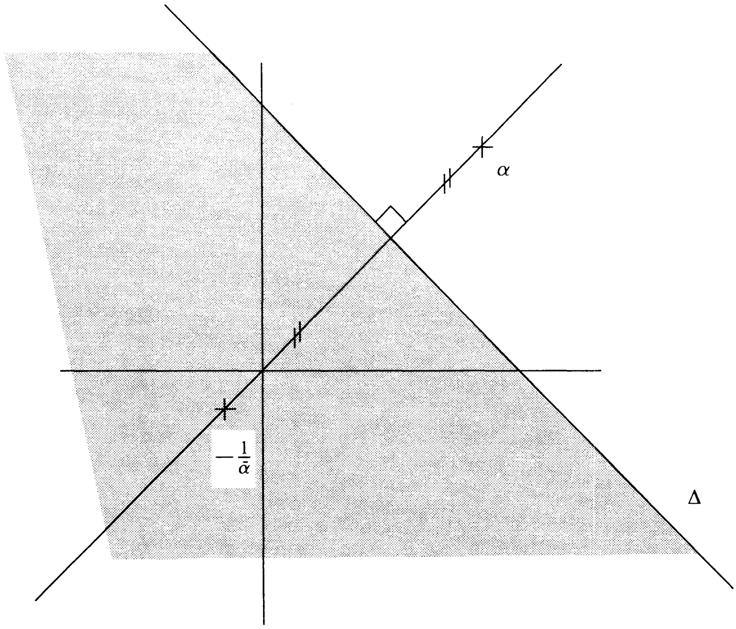

FIGURE 2

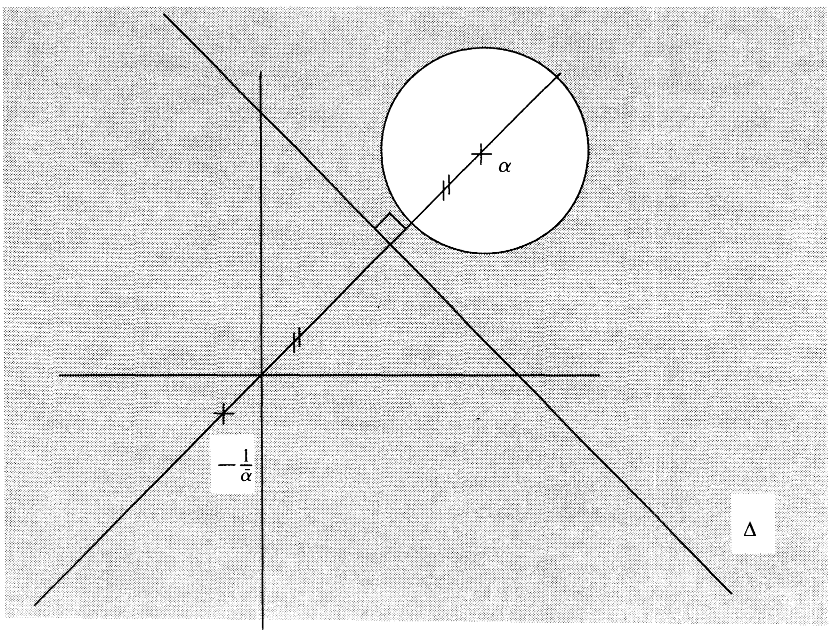

FIGURE 3

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