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CONSTRUCTION PRINCIPLE AND TRANSFINITE INDUCTION UP TO ε_0

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Abstract

What we call here the "construction principle" is a principle on the ground of which some functionals can be defined; the domain and the range of such a functional consist of some "computable" functionals of various finite types. The principle above is considered here as the basis of the functional interpretation of transfinite induction up to ε_0. It is concretely repesented as the "term-forms", where every term-form is shown to be "computable" in some sense.

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It is well-known that the accessibility of the ordered structure which is a canonical representation of the ordinals below ε_0 (the first ε-number) cannot be proved in elementary number theory (see Gentzen (1943)), while it is provable if an analytic method is employed, namely, it is provable in first order arithmetic augmented by the Π^1_1-induction (see Gentzen (1943)). The full power of the Π^1_1-induction is not necessary, however, and attempts have been made to establish the accessibility along more concrete lines, for example in Gentzen (1936) and Takeuti (1975).

In this article we are to propose a theory of "construction principle", a principle on the ground of which some functionals can be defined; the domain and the range of such a functional consist of some "computable" functionals of various finite types. The principle above is considered here as the basis of the functional interpretation of transfinite induction up to ε_0.

The article begins with a glossary of the terminology and symbolism (Section 1). Section 2 consists in the interpretation of transfinite induction up to ε_0 in an arithmetic with infinite reasoning. Although the local technicalities used in this

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section are borrowed from Section 2 of Gentzen (1943), our scheme is a "uniform version" of the provability demonstration, so to speak, thus reaching up to ε_0. The reason why we devote one section to this endeavor is to facilitate our ultimate objective, namely, the functional interpretation of transfinite induction up to ε_0, an informal account of which is given in Section 3.

Our construction principle, the principle behind the functional interpretation of Section 3, is formulated and developed in Section 4. It is concretely represented as the "term-forms", where every term-form is shown to be "computable" in some sense. Section 5 concludes the functional interpretation; the functional construction given in Section 3 can be interpreted in the theory of the term-forms.

We have developed the theory of "construction principle" just to interpret ε_0. Mathematical relationships of this theory to other theories as well as further developments and applications in this line are left open as a problem to be worked on in the future.

Some theories of generalization over Gödel's computable functionals have appeared, a few of which are listed as references; they were developed for their respective purposes.

The idea of "construction principle" was first presented at the workshop on proof-theory which was taken place in May, 1979, at the Research Institute for Mathematical Sciences in Kyoto. The author is grateful to the participants for their valuable comments and discussions.

1. Preliminary definitions

Definition 1.1. The canonical well-ordering system $(\mathbf{E}, \prec)$ whose order-type is known to be ε_0 is defined as usual; 0 is the basic element and ω^a and $a_1 + \cdots + a_m$ are compound elements, where $a_1, \ldots, a_m$ are monomials. ω^a will also be written as $\exp(\omega, a)$. The equality relation $=$ and the order $\prec$ are defind as usual for the elements of $\mathbf{E}$, and we assume that the components (monomials) of an element of $\mathbf{E}$ are arranged in the non-increasing order.

The linearity of the order $\prec$ is easily established, hence will be assumed throughout.

ω_n abbreviates the following:

$$\omega_1 = \omega^0 = 1, \qquad \omega_2 = \omega^1 = \omega \qquad \text{and} \qquad \omega_{n+1} = \exp(\omega, \omega_n).$$

Definition 1.2. An $\mathbf{E}$-element a is said to be accessible (with regards to $\prec$) if, given any $\prec$-decreasing sequence from $\mathbf{E}$ led by a, there is a method (uniform for all such sequences) to show that it is finite. $\prec$ (or $(\mathbf{E}, \prec)$) is said to be accessible if there is a uniform method to establish that every $\mathbf{E}$-element is accessible.

Note. We do not specify the nature of the "method" here.

DEFINITION 1.3. $\mathrm{ht}(a)$, the height of an $\mathbf{E}$-element a, which is a natural number, is defined as follows.

$$\mathrm{ht}(0) = \mathrm{ht}(\omega_0) = 0, \quad \mathrm{ht}(1) = \mathrm{ht}(\omega_1) = 1, \quad \mathrm{ht}(\omega^a) = \mathrm{ht}(a) + 1,$$
$$\mathrm{ht}(a_1 + \cdots + a_m) = \max(\mathrm{ht}(a_1), \ldots, \mathrm{ht}(a_m))$$

$(= \mathrm{ht}(a_1)$; according to our assumption on the order of $a_1, \ldots, a_m)$.

Let f be a sequence from $\mathbf{E}$. $\mathrm{ht}(f)$ is defined to be $\mathrm{ht}(f(0))$.

In passing, the definition implies that for a decreasing sequence f

$$\mathrm{ht}(f(i)) \leqslant \mathrm{ht}(f), \quad \text{for } i = 0, 1, 2, \ldots.$$

DEFINITION 1.4. An $\mathbf{E}$-element consisting of equal monomials alone will be said to be homogeneous. $\omega^a + \cdots + \omega^a$ with l components will be abbreviated as $\omega^a l$. If $a = \exp(\omega, a_1)l_1 + \cdots + \exp(\omega, a_m)l_m$, $a_1, \ldots, a_m$ all distinct, then each $\exp(\omega, a_i)l_i$ is called a homogeneous term of a, a_1 the highest power of a and $\exp(\omega, a_1)l_1$ the homogeneous term of the highest power (of a).

$p(\omega^b l) = b$; $\mathrm{hp}(a)$ = the highest power of a; $\mathrm{hpt}(a)$ = the homogeneous term of the highest power of a.

2. An accessibility proof in an infinite system

Here we formulate an accessibility proof of $(\mathbf{E}, \prec)$ in a semiformal system with a restricted ω-rule. We take over Gentzen's demonstration of the accessibility proof for ω_n for each n (see Section 2 of Gentzen (1943)) in local technicalities. The difference lies in that in his case the demonstration consists of a finite (hence concrete) repetition of the derivations of a same sort, while here such a repetition be regarded as a principle, rather than a practice; this difference is what leads us to the accessibility proof for $(\mathbf{E}, \prec)$.

DEFINITION 2.1. Π will denote a formulation of arithmetic with function quantifications and the primitive recursive infinite rule, where the mathematical induction and the usual $\forall$-introduction is also admitted if the derivation up to it is finite. No sophisticated rules such as the choice rule are involved.

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We assume the arithmetization of $(\mathbf{E}, \prec)$ in Π. We shall use the following notational convention.

$a, b, c, \ldots, x, y, z, \ldots, r, \ldots$ range over the elements of $\mathbf{E}$; $m, n, \ldots, i, j, \ldots$ will range over natural numbers. $\forall x\ (E(x) \supset A)$ ($E(x)$ represents $x \in \mathbf{E}$) will often be abbreviated to $\forall x A$.

Definition 2.2. A formula in Π, $A_i(a)$, will be defined for each i, $i = 0, 1, 2, \ldots$.

$A_0(a)$: $\forall f$ (If f is a decreasing sequence from $\mathbf{E}$ led by a, then f is finite.) Suppose $A_i(a)$ has been defined. Let $A_i^*(a)$ abbreviate $\forall y\ (y \preccurlyeq a \supset A_i(y))$. Then,

$$A_{i+1}(a)\colon \forall x(A_i^*(x) \supset A_i^*(x + \omega^a)).$$

For the notational convenience, we shall write $A(i;\ a)$ for $A_i(a)$. The reader should be aware that $A(i;\ a)$ is not a single formula with a variable i, but a distinct formula for each i.

Theorem 1. $\forall n\ A(0;\ \omega_n)$ *is provable in* Π.

The theorem implies the

Conclusion. $\forall x\ (E(x) \supset A(0;\ x))$ *is provable in* Π. *In other words, the accessibility of* $(E, \prec)$ *is established in* Π.

Note. The "uniform" method in Definition 1.2 is here the reasoning admissible in Π.

Definition 2.3. $Pr(\{a\}B(a))$ stands for

$$\forall a(\forall x\ (x \prec a \supset B(x)) \supset B(a))$$

for any formula B.

Proposition 2.1. *The following* 1 *through* 6 *are provable in* Π *for each* k *without applications of the infinite rule; furthermore the proofs are primitive recursive in* k.

1. $A(0;\ 0)$,
2. $A(k + 1;\ 0) \to A(k + 1;\ \omega_0)$,
3. $Pr(\{x\}A(0;\ x))$,
4. $Pr(\{x\}A(k;\ x)) \to Pr(\{x\}A(k + 1;\ x))$,
5. $Pr(\{x\}A(k;\ x)) \to A(k + 1;\ 0)$,
6. *From* $Pr(\{x\}A(k;\ x))$ *and* $A(k + 1;\ 0) \to A(k + 1;\ a)$ *we can deduce* $A(k;\ 0) \to A(k;\ \omega^a)$.

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PROOF OF THEOREM 1 from Proposition 2.1. For each k,

7. $Pr(\{x\}A(k;\ x))$

by 3 and repeated applications of 4. For any fixed n, let k range over n, $n-1, \ldots, 1, 0$ consecutively in 5 and 6, where $a = \omega_{n-k}$ in 6. 2 with $k = n$ and consecutive applications of 5, 7 and 6 for $k = n, n-1, \ldots, 1, 0$ yield in succession:

$$A(n+1;\ 0) \to A(n+1;\ \omega_0),$$
$$A(n;\ 0) \to A(n;\ \omega_1),$$
$$\vdots$$
$$A(1;\ 0) \to A(1;\ \omega_n),$$
$$A(0;\ 0) \to A(0;\ \omega_{n+1}).$$

This last sequent and 1 above yield $A(0;\ \omega_n)$ in Π for every n; the derivations are primitive recursive in n. The infinite rule now yields $\forall n\, A(0;\ \omega_n)$ in Π.

PROOF OF THE CONCLUSION. For any a in $\mathbf{E}$, the least n such that $a \preccurlyeq \omega_n$ can be determined primitive recursively from a, say $g(a)$. So, for each a, $A(0;\ \omega_{g(a)})$ is provable in Π without infinite rule; furthermore the derivation is primitive recursive in a. (This fact can be established by modifying the last part of the proof of Theorem 1.) $A(0;\ \omega_{g(a)})$ implies $A(0;\ a)$. Now apply the infinite rule to obtain $\forall x(E(x) \supset A(0;\ x))$.

Notice that the theorem (as well as the conclusion) has been established by a single application of the infinite rule.

PROOF OF PROPOSITION 2.1. Since the proofs are essentially the same as those of Gentzen (1943), we shall briefly remark on one point.

4 of Proposition 2.1. According to Gentzen, we let d be $fu_1(b, c, a)$ and let n be $fu_2(b, c, a)$. By the definition of A, we have

(1) $A(k+1;\ d),\ A^*(k;\ c+\omega^d m) \to A^*(k;\ c+\omega^d(m+1))$,

hence by induction applied to m in $A^*(k;\ c+\omega^d m)$, we obtain

(2) $A(k+1;\ d),\ A^*(k;\ c) \to A^*(k;\ c+\omega^d n)$.

Notice that (1) and (2) can be established uniformly in (independently of) k. Deducing the required conclusion from (2) is a straightforward process.

3. Functional interpretation

A careful analysis of the proof procedure in Section 2 leads us to a certain construction principle, based on which the accessibility of $(\mathbf{E}, \prec)$ can be established. We first present an informal account of construction in order to get the general idea, the formulation of which will be given in the next two sections.

DEFINITION 3.1. Finite types are defined as usual (see Gödel (1958), Hindley, Lercher and Seldin (1972), Yasugi (1963)): 0 is the ground type (representing natural numbers); $s \to t$ is a type if s and t are, representing functionals which map the objects of type s to those of type t. $s_1 \to (s_2 \to (\cdots (s_n \to t) \cdots))$ will be abbreviated to $s_1, s_2, \ldots, s_n \to t$.

A type function is a primitive recursive function whose values are finite types. We list some useful type functions below.

$$\begin{aligned}
[0] &= 1 \to 1, \quad \text{where } 1 = 0 \to 0.\\
[i+1] &= 0 \to ([i] \to [i]), \quad \text{or } 0, [i] \to [i].\\
\langle\langle i \rangle\rangle &= 0 \to [i].\\
\{i\} &= 0 \to (\langle\langle i \rangle\rangle \to [i]), \quad \text{or } 0, \langle\langle i \rangle\rangle \to [i].\\
\langle i \rangle &= \{i\} \to \{i+1\}.\\
\|i\| &= \{i\}, \quad ([i+1] \to [i+1]), \quad [i] \to [i].
\end{aligned}$$

$\phi, \psi, \chi, \ldots$ will be used for functional variables; a functional variable of type t will be denoted by ϕ^t, ψ^t, etc., but we omit the type symbol whenever possible.

DEFINITION 3.2. $G_i(\phi^{[i]}, a)$ will be defined for each i, $i = 0, 1, 2, \ldots$.

$G_0(\phi^{[0]}, a)$: Given a function f, if f is a decreasing sequence from $\mathbf{E}$ led by an $x \preccurlyeq a$, then $\phi^{[0]}(f)$ is a decreasing sequence from $\mathbf{E}$ satisfying that $\phi^{[0]}(f)(j)$ is the homogeneous term of the highest power of $f(n_j)$ for some n_j, where $\{n_j\}_j$ is an increasing sequence and $n_0 = 0$, and such that if $\phi^{[0]}(f)$ is finite, then so is f. (See Section 1 for homogeneous terms, etc.)

This is the way I would like to describe G_0. Dissatisfaction has been expressed to me, however, as to the informal manner as it stands, hence an attempt of a formulation of G_0 in a semi-formalism.

Let f stand for a function from natural numbers to $\mathbf{E}$. We call such a function a sequence from $\mathbf{E}$. We include finite sequences here. f is said to be decreasing if $f(n) \prec f(m)$ when $n > m$. f is said to be led by x an element of $\mathbf{E}$ if $f(0) = x$. Now $G_0(\phi^{[0]}, a)$ can be expressed as follows.

$$\begin{aligned}\forall f\ \{[(&f \text{ is a decreasing sequence from } \mathbf{E})\\ &\wedge \exists x(x \preccurlyeq a \wedge f(0) = x)\\ &\wedge \exists \{n_j\}_j\ ((\{n_j\}_j \text{ is an increasing sequence of natural numbers } \wedge n_0 = 0)\\ &\wedge \forall j(\phi^{[0]}(f)(j) = \mathrm{hpt}(f(n_j))))\\ &\wedge (\phi^{[0]}(f) \text{ is a finite sequence})] \supset f \text{ is finite}\},\end{aligned}$$

where hpt is seen in Definition 1.4.

Suppose $G_i(\phi^{[i]}, x)$ has been defined.

$$G_{i+1}(\phi^{[i+1]}, a)\colon \forall x \forall \phi^{[i]}\big(G_i(\phi^{[i]}, x) \supset G_i\big(\phi^{[i+1]}(x, \phi^{[i]}), x + \omega^a\big)\big).$$

Note. If we define $G_i^*(\phi^{[i]}, a)$ to be

$$\forall x\big(x \preccurlyeq a \supset G_i(\phi^{[i]}, x)\big),$$

then it can be easily shown that

$$G_i^*(\phi, a) \leftrightarrow G_i(\phi, a).$$

Thus, G_i can be replaced by G_i^* everywhere.

As in Section 2, we shall write $G(i; \phi, x)$ for $G_i(\phi, x)$ for the notational reason.

DEFINITION 3.3. For each i, $i = 0, 1, 2, \ldots,$ we define the following.

$$Pr_i(\psi^{\{i\}})\colon \forall z \forall \phi^{\langle\langle i\rangle\rangle}\big(\forall x\big(x \prec z \supset G(i; \phi^{\langle\langle i\rangle\rangle}(x), x)\big) \supset G\big(i; \psi^{\{i\}}(z, \phi^{\langle\langle i\rangle\rangle}), z\big)\big).$$

$$P_i(\Psi^{\langle i\rangle})\colon \forall \psi^{\{i\}}\big(Pr_i(\psi^{\{i\}}) \supset Pr_{i+1}\big(\Psi^{\langle i\rangle}(\psi^{\{i\}})\big)\big).$$

Once again we write $Pr(i; \psi)$ for $Pr_i(\psi)$ and $P(i; \Psi)$ for $P_i(\Psi)$.

PROPOSITION 3.1. *There is a uniform* (*in i*) *method to construct primitive recursive functionals of appropriate types for* $1 \sim 4$ *below.* (*See Gödel* (1958), *Hinata* (1967), *Hindley and others* (1972) *and Yasugi* (1963) *for primitive recursive functionals of finite type.*)

1. *$Pr(0; \alpha_0)$, where α_0 is of type $\{0\}$.*
2. *$P(i; \beta_i)$, where β_i is of type $\langle i\rangle$.*
3. *$Pr(i; \psi) \to G(i+1; \gamma_i(\psi), 0)$, where γ_i is of type $\{i\} \to [i+1]$.*
4. *$Pr(i; \psi)$ and*

$$\forall \phi\big(G(i+1; \phi, 0) \supset G(i+1; \Phi(\phi), a)\big)$$

imply

$$\forall \chi\big(G(i; \chi, 0) \supset G(i; \delta_i(\psi, \Phi, \chi), \omega^a)\big),$$

where δ_i is of type $\|i\|$.

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PROOF. 1. Assume

$$\forall x(x \prec a \supset G(0; \phi(x), x)),$$

where ϕ is of type $\langle\langle 0\rangle\rangle$. Let α^* be the functional satisfying

$$\alpha^*(f)(i) = f(i+1)$$

for every f of type 1. In particular $\alpha^*(f)(0) = f(1)$ and, if f is a decreasing sequence led by a, then $\alpha^*(f)$ is a decreasing sequence led by $f(1)$ ($\prec a$). So, from the assumption,

$$G(0; \phi(f(1)), f(1)),$$

or

$$G(0; \phi(\alpha^*(f)(0)), \alpha^*(f)(0)),$$

and hence

$$g: \phi(\alpha^*(f)(0))(\alpha^*(f))$$

is a decreasing sequence of homogeneous terms satisfying the appropriate condition. Let ξ be the operation which augments g with the highest homogeneous term of a, hpt(a) (see Definition 1.2), as the initial entry, in case $g(0) \prec \text{hpt}(a)$ (and leaves g unchanged otherwise). Thus, define α_0 to be

$$\text{(i)} \quad \lambda a \lambda \phi \xi(a, \phi(f(1))(\alpha^*(f))),$$

where λ denotes the usual λ-notation. From the construction,

$$G(0; \alpha_0(a, \phi), a),$$

which, together with the assumption, implies $Pr(0; \alpha_0)$.

2. Assume $Pr(i; \psi)$ where ψ is of type $\{i\}$, namely,

$$\text{(0)} \quad \forall z \forall \chi(\forall x(x \prec z \supset G(i; \chi(x), x)) \supset G(i; \psi(z, \chi), z)),$$

where χ is of type $\langle\langle i\rangle\rangle$. Abbreviate

$$\forall y(y \prec a \supset G(i+1; \Phi(y), y))$$

to $\mathbf{C}(\Phi)$, where Φ is of type $\langle\langle i+1\rangle\rangle$. Recall that $G(i+1, \Phi(y), y)$ stands for

$$\forall x \forall \phi(G(i; \phi, x) \supset G(i; \Phi(y)(x, \phi), x + \omega^y)),$$

where ϕ is of type $[i]$. As before, let d be $fu_1(b, c, a)$ and let n be $fu_2(b, c, a)$, so $b \prec c + \omega^a$ implies $b \preccurlyeq c + \omega^d n$, where $d \prec a$ and $n \prec \omega$. Thus, assuming $b \prec c + \omega^a$,

$$\mathbf{C}(\Phi) \to G(i+1; \Phi(d), d),$$

or

$$\mathbf{C}(\Phi), G(i; \phi, c) \to G(i; \Phi(d)(c, \phi), c + \omega^d).$$

Define

$$\text{(ii)} \quad \Phi^{\sim}(0, a, b, c, \Phi, \phi) = \phi;$$

$$\Phi^{\sim}(l+1, a, b, c, \Phi, \phi) = \Phi(d, c + \omega^d l, \Phi^{\sim}(l, a, b, c, \Phi, \phi)).$$

$\Phi^{\sim}$ is of type 0, 0, 0, $\langle\langle i+1\rangle\rangle$, $[i] \to [i]$. The recursion in (ii) is "uniform" in i without assuming any particular functionals. Then, more generally,

$$\mathbf{C}(\Phi),\ G(i;\ \Phi^{\sim}(l, a, b, c, \Phi, \phi), c + \omega^d l) \\ \to G(i;\ \Phi^{\sim}(l+1, a, b, c, \Phi, \phi), c + \omega^d(l+1)).$$

Letting l be 0, 1, 2, . . . , we obtain

$$\mathbf{C}(\Phi),\ G(i;\ \phi, c) \to G(i;\ \Phi^{\sim}(l, a, b, c, \Phi, \phi), c + \omega^d l)$$

for every l. Put $l = n = n(a, b, c) = fu_2(b, c, a)$ and $d = d(a, b, c) = fu_1(b, c, a)$. Then

$$\mathbf{C}(\Phi),\ G(i;\ \phi, c) \to G(i;\ \phi^{\sim}(n(a, b, c), a, b, c, \Phi, \phi), c + \omega^{d(a,b,c)} n(a, b, c)).$$

Thus,

$$\mathbf{C}(\Phi),\ G(i;\ \phi, c) \to \forall x(x \prec c + \omega^a \supset G(i;\ \Phi^{\sim}(n(a, x, c), a, x, c, \Phi, \phi), x)).$$

Rewriting

$$\text{(iii)} \quad \Phi^*(a, x, c, \Phi, \phi) = \Phi^{\sim}(n(a, x, c), a, x, c, \Phi, \phi),$$

we obtain

$$(1) \qquad \mathbf{C}(\Phi),\ G(i;\ \phi, c) \to \forall x(x \prec c + \omega^a \supset G(i;\ \Phi^*(a, x, c, \Phi, \phi), x)).$$

In (0), let z be $c + \omega^a$ and let χ be $\lambda x \Phi^*(a, x, c, \Phi, \phi)$. Then

$$(2) \qquad \forall x(x \prec c + \omega^a \supset G(i;\ \Phi^*(a, x, c, \Phi, \phi), x)) \\ \to G(i;\ \psi(c + \omega^a, \lambda x \Phi^*(a, x, c, \Phi, \phi)), c + \omega^a).$$

(1) and (2) imply

$$\mathbf{C}(\Phi),\ G(i;\ \phi, c) \to G(i;\ \psi(c + \omega^a, \lambda x \Phi^*(a, x, c, \Phi, \phi)), c + \omega^a),$$

or

$$(3) \quad \mathbf{C}(\Phi) \to \forall\phi\forall c(G(i;\ \phi, c) \supset G(i;\ \psi(c + \omega^a, \lambda x \Phi^*(a, x, c, \Phi, \phi)), c + \omega^a)).$$

Define β_i to be

$$\text{(iv)} \quad \lambda\psi\lambda a\lambda\Phi\lambda c\lambda\phi\psi(c + \omega^a, \lambda x \Phi^*(a, x, c, \Phi, \phi)).$$

(3) will then turn to

$$\mathbf{C}(\Phi) \to \forall\phi\forall c(G(i;\ \phi, c) \supset G(i;\ \beta_i(\psi)(a, \Phi)(c, \phi), c + \omega^a)),$$

or, recalling what $\mathbf{C}(\Phi)$ is,

$$\forall z \forall \Phi(\forall y(y \prec z \supset G(i+1;\ \Phi(y), y)) \supset G(i+1;\ \beta_i(\psi)(a, \Phi), z)),$$

which is $Pr(i+1;\ \beta_i(\psi))$. Thus, we have established

$$Pr(i;\ \psi) \to Pr(i+1;\ \beta_i(\psi)),$$

hence $P(i;\ \beta_i)$.

Notice that β_i is defined uniformly in i and no induction on i is involved in the course of the proof.

3. Assume $Pr(i; \psi)$. Then, in particular,

(1) $\forall x(x \prec z + 1 \supset G(i; \chi(x), x)) \rightarrow G(i; \psi(z + 1, \chi), z + 1)$

for any χ. Suppose $G(i; \phi, z)$ holds. Then,

(2) $\forall x(x \preccurlyeq z \supset G(i; \phi, x))$.

Let χ be $\lambda x\phi$, so (2) becomes

$$\forall x(x \prec z + 1 \supset G(i; \chi(x), x)),$$

hence by (1), where χ is $\lambda x\phi$,

(3) $G(i; \psi(z + 1, \chi), z + \omega^0)$.

Define γ_i by

$$\text{(v)} \quad \lambda\psi\lambda z\lambda\phi\psi(z + 1, \lambda x\phi).$$

Then

$$\gamma_i(\psi)(z, \phi) = \psi(z + 1, \lambda x\phi).$$

So, (3) under the assumption $G(i; \phi, z)$ yields

$$G(i; \gamma_i(\psi)(z, \phi), z + \omega^0),$$

and hence

$$\forall z\forall\phi(G(i; \phi, z) \supset G(i; \gamma_i(\psi)(z, \phi), z + \omega^0)),$$

which is $G(i + 1; \gamma_i(\psi), 0)$.

γ_i is defined uniformly in i without assuming any particular functionals.

4. Assume $Pr(i; \psi)$ and $\forall\phi(G(i + 1; \phi, 0) \supset G(i + 1; \Phi(\phi), a))$. The two assumptions together with 3 yield

$$G(i + 1; \Phi(\gamma_i(\psi)), a),$$

or

$$\forall x\forall\chi(G(i; \chi, x) \supset G(i; \Phi(\gamma_i(\psi))(x, \chi), x + \omega^a)).$$

Letting x be 0, we obtain

$$\forall\chi(G(i; \chi, 0) \supset G(i; \Phi(\gamma_i(\psi))(0, \chi), \omega^a)).$$

Define δ_i to be

$$\text{(vi)} \quad \lambda\psi\lambda\Phi\lambda\chi\Phi(\gamma_i(\psi))(0, \chi).$$

This will do.

δ_i is defined uniformly in i, assuming the existence of γ_i, whose uniform definition has been given in (v).

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PROPOSITION 3.2. *There is a uniform (in i) method to construct functionals ε_i and ν_i which satisfy the following.*

1. *$Pr(i;\ \varepsilon_i)$, where ε_i is of type $\{i\}$.*
2. *$G(0;\ \nu_i, \omega_i)$, where ν_i is of type $[0]$ (for every i).*

PROOF. 1. Define ε_i by

$$\text{(vii)} \quad \varepsilon_0 = \alpha_0, \qquad \varepsilon_{i+1} = \beta_i(\varepsilon_i),$$

where β_i was defined in (iv).

$$(1) \qquad Pr(0, \varepsilon_0)$$

by 1 of Proposition 3.1, and

$$Pr(i;\ \varepsilon_i) \to Pr(i+1;\ \beta_i(\varepsilon_i))$$

by 2 there, or

$$(2) \qquad Pr(i;\ \varepsilon_i) \to Pr(i+1;\ \varepsilon_{i+1}).$$

(1) and repeated applications of (2) yield $Pr(i;\ \varepsilon_i)$ for every i.

Notice that, according to (vii), β_j, which was defined identically for every j, supplies with the mechanism to produce ε_i for every i.

2. Define $\tau_{(i,k)}$ for $k = i+1, i, \ldots, 0$ as follows.

$$\text{(viii)} \quad \tau_{(i,i+1)} = I_i, \quad \tau_{(i,k)} = \lambda\chi\delta_k(\varepsilon_k, \tau_{(i,k+1)}, \chi),$$

where I_i is the identity of type $[i+1] \to [i+1]$, $\tau_{(i,k)}$ is of type $[k] \to [k]$. $\tau_{(i,k)}$ is defined uniformly in i by the reversed recursion on k, $k = i+1, i, \ldots, 0$, assuming the recursion mechanism of δ_k and ε_k. δ_k was defined in (vi) uniformly in k; ε_k was defined in (vii) by recursion on k. Now,

$$(1) \qquad G(i+1;\ \phi, 0) \to G(i+1;\ \tau_{(i,i+1)}(\phi), \omega_0)$$

is a tautology. Suppose $\tau_{(i,k+1)}$ satisfies

$$(2) \qquad G(k+1;\ \psi, 0) \to G(k+1;\ \tau_{(i,k+1)}(\psi), \omega_{i-k}).$$

1 above with $i = k$, 4 of Proposition 3.1 with $i = k$, $\psi = \varepsilon_k$, $\Phi = \tau_{(i,k+1)}$ and $a = \omega_{i-k}$, and (2) above yield

$$G(k;\ \chi, 0) \to G(k;\ \delta_k(\varepsilon_k, \tau_{(i,k+1)}, \chi), \omega_{i-k+1}),$$

or

$$(3) \qquad G(k;\ \chi, 0) \to G(k;\ \tau_{(i,k)}(\chi), \omega_{i-k+1}).$$

(1) through (3) ensure that (3) holds for every k, $k = i+1, i, \ldots, 0$. In particular,

$$(4) \qquad G(0;\ \chi, 0) \to G(0;\ \tau_{(i,0)}(\chi), \omega_{i+1}),$$

where χ is of type $[0]$. On the other hand,

$$(5) \qquad G(0;\ I^{[0]}, 0)$$

trivially holds, where $I^{[0]}$ is the identity of type [0]. (5) and (4) with $\chi = I^{[0]}$ yield

(6) $$G(0; \tau_{(i,0)}(I^{[0]}), \omega_{i+1}).$$

Define ν_i by

$$\text{(ix)} \quad \nu_0 = I^{[0]}, \qquad \nu_{i+1} = \tau_{(i,0)}(I^{[0]}).$$

Then (5) and (6) yield

$$G(0, \nu_i, \omega_i)$$

for every i.

ν_i is defined from $I^{[0]}$ and $\tau_{(j,0)}$ with a simple mechanism.

PROPOSITION 3.3. *There are functionals which work as described below.*

ht(f): *See Definition* 1.3.

hp(a): *See Definition* 1.4.

μ: *For any decreasing sequence f of homogeneous terms, $\mu(f)$ is a decreasing sequence of monomials such that $\mu(f)(i)$ is a monomial in $f(m_i)$ for some m_i, where $\{m_i\}_i$ is an increasing sequence and, if $\mu(f)$ is finite, then so is f.*

M_h *for each h a natural number: For any decreasing sequence f from E such that* $\mathrm{ht}(f) = h$, *$M_h(f)$ is a decreasing sequence of the highest powers of some entries of f, hence* $\mathrm{ht}(M_h(f)) = h - 1$, *and the finiteness of $M_h(f)$ implies the same of f.*

The types of those functionals are as follows.

$$\mathrm{ht}(f)\colon 1 \to 0, \qquad \mathrm{hp}\colon 0 \to 0,$$
$$\mu\colon 1 \to 1 \quad \textit{and} \quad M_h\colon 1 \to 1 \quad \textit{for all } h.$$

PROOF. ht and hp are easily defined.

$(x)^0$ $\quad \mu(f)(0) =$ the monomial of $f(0)$ if $f(0)$ is a homogeneous term.

Suppose $\mu(f)(0), \mu(f)(1), \ldots, \mu(f)(j)$ and $0, m_1, \ldots, m_j$ have been defined in a manner that

$(x)^j$ $$\mu(f)(j) = \text{the monomial of } f(m_j).$$

Check $f(m_j + 1)$ and see if it is a homogeneous term. If so, and if $f(m_j) = \exp(\omega, a_1)n_1$ and $f(m_j + 1) = \exp(\omega, a_2)n_2$, then either $a_1 \succ a_2$, or $a_1 = a_2$ and $n_1 > n_2$. If the former is the case, let $m_{j+1} = m_j + 1$ and

$$\mu(f)(j + 1) = \exp(\omega, a_2).$$

If the latter is the case, check $f(m_j + 2)$. $n_1 > n_2 > \ldots$ will stop within at most n_1 steps. Then $f(m_j + p) = \exp(\omega, a_{p+1})n_{p+1}$ and $a_1 \succ a_{p+1}$. Put $m_{j+1} = m_j + p$ and

$(x)^{j+1}$ $$\mu(f)(j + 1) = \exp(\omega, a_{p+1}).$$

$(x)^0$, $(x)^j$ and $(x)^{j+1}$ define μ, which is primitive recursive in f.

In case one hits a q such that $f(m_j + q)$ is not a homogeneous term satisfying $f(m_j + q) \prec f(m_j + q - 1)$ while searching for the p as above, define $\mu(f)(l) = 0$ for every $l \geqslant m_j + q$.

Next, let ν_h be the functional defined in 2 of Proposition 3.2. Define M_h to be

$$\text{(xi)} \quad \lambda f \lambda j(\mathrm{hp}(\mu(\nu_{h+1}(f))(j))).$$

Then by Proposition 3.2 and the properties of μ and hp, M_h satisfies the condition. The construction of M_h is uniform in h inasmuch as that of ν_h is.

THEOREM 2. *There is a uniform method to establish that every element of* **E** *is accessible with regards to* $\prec$.

PROOF. The construction of M_h in Proposition 3.3 is uniform in h as mentioned above. Thus, define N by

$$\text{(xii)} \quad N(0; f) = f, \quad N(i + 1; f) = M_{\mathrm{ht}(f) \dot{-} i}(N(i; f)).$$

N is of type $0, 1 \to 1$. If f is any decreasing sequence from **E**, then $N(\mathrm{ht}(f); f)$ is a decreasing sequence of height 0, which is infallibly finite; according to the definition of M_h, the finiteness of $N(\mathrm{ht}(f); f)$ implies the same of f. Thus, f must be finite. This proves that every element of **E** is $\prec$-accessible.

$N(i; f)$ is defined by recursion on i, whose mechanism is provided by the construction of M_h, which in turn depends on ν_h.

SPECULATION. In (i) through (xii), we have defined functionals for each set of values of parameters, $i, k, \ldots$. Thus, let S_i denote the functional for i. We can regard $\{S_i\}_i$ as an enumeration of some functionals, which we write as S. From a reversed view point, S can be regarded as an object, or a construction principle, which produces individual S_i's as instances of it. It is such an S we are to characterize in the subsequent sections.

4. Construction principle

Here we shall formulate the construction principle, which was announced in the preceding section.

DEFINITION 4.1. 1) Symbols.

1.1) $\mathfrak{A}, \mathfrak{B}, \ldots$: parameters.

1.2) f, g, . . . : a finite list of unary function symbols.

1.3) p, q, . . . : a finite list of function symbols (distinct from those of 1.2)).

1.4) P, Q, . . . : a finite list of predicate symbols.

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2) Pre-types.

2.1) A function symbol (in 1.2)) accompanied by a parameter is a pre-type.

2.2) A finite type (denoted by 0, s, t, etc.) is a pre-type (see Definition 3.1).

2.3) If $\mathfrak{s}$ and $\mathfrak{t}$ are pre-types, then so is $\mathfrak{s} \to \mathfrak{t}$.

3) Pre-variables. For each pre-type $\mathfrak{s}$, there are assumed to be denumerably many pre-variables $\phi^{\mathfrak{s}}, \psi^{\mathfrak{s}}, \ldots$.

4) Pre-terms and free and bound occurrences of parameters.

4.1) Pre-variables are pre-terms; the parameters in their superscripts (pre-types) are free.

4.2) Primitive recursive terms of finite type (see Gödel (1958), Hinata (1967), Hindley and others (1972), Yasugi (1963)) are pre-terms; there are no parameters here.

4.3) If X and Y are pre-terms, then so is $X\langle Y\rangle$; the free occurrences of parameters in $X\langle Y\rangle$ are exactly those in X or Y, and the bound occurrences likewise.

4.4) If X is a pre-term and ϕ is a pre-variable which is free in X, then $\lambda\phi X$ is a pre-term; the free occurrences of parameters in it are exactly those in ϕ or X, and the bound occurrences likewise.

4.5) If $\mathrm{P}_1, \ldots, \mathrm{P}_m$ are predicate symbols accompanied by appropriate numbers of parameters and $X_1, \ldots, X_m, X_{m+1}$ are pre-terms, then $C[\mathrm{P}_1, \ldots, \mathrm{P}_m, X_1, \ldots, X_m, X_{m+1}]$ is a pre-term; the free occurrences of parameters in it are those in any of $\mathrm{P}_1, \ldots, \mathrm{P}_m, X_1, \ldots, X_m, X_{m+1}$, and the bound ones likewise.

4.6) If X and Y are pre-terms which are free of R, then $\rho[X, Y]$ is a pre-term. The free parameters are those in one of X and Y, and the bound ones likewise.

4.7) If X and Y are pre-terms, where $\mathcal{Q}$ does not occur free in X, then $R[\mathcal{Q}; X, Y]$ is a pre-term. The free occurrences of parameters are $\mathcal{Q}$ and those in X or Y; the bound ones are those in X or Y.

4.8) If X is a pre-term, $\mathcal{C}_1, \ldots, \mathcal{C}_l$ are some of the free parameters in X and $\mathrm{P}_1, \ldots, \mathrm{P}_l$ are auxiliary function symbols accompanied by some parameters which are not among $\mathcal{C}_1, \ldots, \mathcal{C}_l$ and not bound in X, then $\mathrm{Sub}(X; {\mathcal{C}_1 \cdots \mathcal{C}_l \atop \mathbf{P}_1 \cdots \mathbf{P}_l})$ is a pre-term. The free parameters in X which are distinct from $\mathcal{C}_1, \ldots, \mathcal{C}_l$ and the parameters in $\mathrm{P}_1, \ldots, \mathrm{P}_m$ are free in the new pre-term. The bound ones are those in X and $\mathcal{C}_1, \ldots, \mathcal{C}_l$.

Definition 4.2. Functional specification.

1) A type function is a primitive recursive function (of one or several arguments) which enumerates some finite types.

2) Let $\mathfrak{F}$ be a finite list of unary type functions corresponding to function symbols in 1.2) of Definition 4.1.

Let $\mathfrak{p}$ be a finite list of primitive recursive functions corresponding to auxiliary function symbols in Definition 4.1. Those will be called auxiliary functions.

Let $\mathfrak{P}$ be a finite set of primitive recursive predicates corresponding to predicate symbols in Definition 4.1. Those will be called case-predicates.

3) The functional specification (with regards to ($\mathfrak{F}$, $\mathfrak{p}$, $\mathfrak{P}$)) of any symbol defined in 1.2) $\sim$ 1.4) of Definition 4.1 is obtained from it by specifying it by the corresponding object from $\mathfrak{F}$, $\mathfrak{p}$ and $\mathfrak{P}$.

4) For a pre-type $\mathfrak{s}$, the functional specification of $\mathfrak{s}$ is defined as follows. If $\mathfrak{s}$ is a function symbol accompanied by a parameter, then its specification is the corresponding function from $\mathfrak{F}$, regarding the parameter as the variable. If $\mathfrak{s}$ is a finite type, then $\mathfrak{s}$ is itself its own specification. Suppose the specifications of $\mathfrak{s}$ and $\mathfrak{t}$ are respectively primitive recursive functions f and g of several parameters, then the specification of $\mathfrak{s} \to \mathfrak{t}$ is a function of m variables h such that for each set of values of parameters, say $\mathbf{k}$, $h(\mathbf{k}) = f(\mathbf{k}) \to g(\mathbf{k})$. We write $f \to g$ for h.

Note. Various properties of primitive recursive functions and predicates which are presumed in the definitions are supposed to be provable in primitive recursive arithmetic.

Definition 4.3. 1). Let F be the functional specification of any symbol from 1.2) $\sim$ 1.4) of Definition 4.1 accompanied by some parameters (see 3) of Definition 4.2), and let $\mathfrak{B}_1, \ldots, \mathfrak{B}_m$ be some parameters. Then

$$\mathrm{Spec}\left(F; \begin{smallmatrix} \mathfrak{B}_1 & \cdots & \mathfrak{B}_m \\ k_1 & \cdots & k_m \end{smallmatrix}\right)$$

will denote the substitution in F of numbers $k_1, \ldots, k_m$ for $\mathfrak{B}_1, \ldots, \mathfrak{B}_m$ respectively. We call this a numerical specification of F. If $\mathfrak{B}_1, \ldots, \mathfrak{B}_m$ exhaust all the parameters in F, then we say that it is a complete (numerical) specification of F.

2) The numerical specification defined in 1) above induces the numerical specification of the functional specification of a pre-type, which represents a finite type. It will be denoted as in 1).

3) Let $\phi^{\mathfrak{s}}$ be a pre-variable and let F be the functional specification of $\mathfrak{s}$. Then ϕ^F will be called a variable of the type function F. If $\mathfrak{B}_1, \ldots, \mathfrak{B}_m$ exhaust the parameters occurring in F, then $G \equiv \mathrm{Spec}(F; \begin{smallmatrix} \mathfrak{B}_1 & \cdots & \mathfrak{B}_m \\ k_1 & \cdots & k_m \end{smallmatrix})$, a complete (numerical) specification of ϕ^F, is defined to be ϕ^G. Notice that G is a finite type, hence there is a pre-variable ϕ^G, which is regarded also as a variable of type G.

4) Let X be the functional specification of a pre-term, where $\mathfrak{B}_1, \ldots, \mathfrak{B}_m$ exhaust the free parameters in X. A complete numerical specification of X, $\mathrm{Spec}(X; \begin{smallmatrix} \mathfrak{B}_1 & \cdots & \mathfrak{B}_m \\ k_1 & \cdots & k_m \end{smallmatrix})$, is defined according to the construction in 4) of Definition 4.1.

4.1) $\mathrm{Spec}(\phi^F;\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m}_{k_1 \cdots k_m})$ has been defined in 3) above.

4.2) Terms of finite type are not affected by specifications.

4.3) $\mathrm{Spec}(X\langle Y\rangle;\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m}_{k_1 \cdots k_m}) \equiv \mathrm{Spec}(X;\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m}_{k_1 \cdots k_m})\langle \mathrm{Spec}(Y;\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m}_{k_1 \cdots k_m})\rangle$.

4.4) $\mathrm{Spec}(\lambda\phi X;\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m}_{k_1 \cdots k_m}) \equiv \lambda \mathrm{Spec}(\phi;\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m}_{k_1 \cdots k_m})\mathrm{Spec}(X;\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m}_{k_1 \cdots k_m})$.

4.5) $\mathrm{Spec}(C[P_1, \ldots, P_l, X_1, \ldots, X_l, X_{l+1}];\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m}_{k_1 \cdots k_m})$

$$\equiv C[\sigma P_1, \ldots, \sigma P_l, \sigma X_1, \ldots, \sigma X_l, \sigma X_{l+1}],$$

where σY abbreviates $\mathrm{Spec}(Y;\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m}_{k_1 \cdots k_m})$.

4.6) $\mathrm{Spec}(\rho[X, Y];\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m}_{k_1 \cdots k_m}) \equiv \rho[\mathrm{Spec}(X;\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m}_{k_1 \cdots k_m}), \mathrm{Spec}(Y;\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m}_{k_1 \cdots k_m})]$.

4.7) $\mathrm{Spec}(R[\mathcal{Q}, X, Y];\ {}^{\mathcal{Q}\ \mathcal{B}_1 \cdots \mathcal{B}_m}_{0\ k_1 \cdots k_m})$

$$\equiv \mathrm{Spec}\big(X;\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m}_{k_1 \cdots k_m}\big);$$

$$\mathrm{Spec}\big(R[\mathcal{Q}, X, Y];{}^{\mathcal{Q}}_{k+1}\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m}_{k_1 \cdots k_m}\big)$$

$$\equiv \mathrm{Spec}\big(Y;{}^{\mathcal{Q}}_{k}\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m}_{k_1 \cdots k_m}\big)\big\langle \mathrm{Spec}\big(R[\mathcal{Q}, X, Y];{}^{\mathcal{Q}}_{k}\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m}_{k_1 \cdots k_m}\big)\big\rangle.$$

4.8) $\mathrm{Spec}(\mathrm{Sub}(X;\ {}^{\mathcal{C}_1 \cdots \mathcal{C}_l}_{p_1 \cdots p_l});\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m}_{k_1 \cdots k_m})$

$$\equiv \mathrm{Spec}\big(X;\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m\ \mathcal{C}_1 \cdots \mathcal{C}_l}_{k_1 \cdots k_m\ \sigma p_1 \cdots \sigma p_l}\big),$$

where p_i denotes the functional specification of an auxiliary function symbol and $\sigma p_i \equiv \mathrm{Spec}(p_i;\ {}^{\mathcal{B}_1 \cdots \mathcal{B}_m}_{k_1 \cdots k_m})$ (see 2) above).

Since the parameters in $p_1, \ldots, p_l$ are not among $\mathcal{C}_1, \ldots, \mathcal{C}_l$, the definition is consistent.

PROPOSITION 4.1. *The definition immediately above is complete.*

PROOF. By induction on the construction of pre-terms, within which by induction on k in 4.7).

DEFINITION 4.4. Term-forms and the associated type functions.

The functional specification of a pre-term is a term form in the subsequent circumstances. We quote 4) of Definition 4.1.

4.1) A variable is a term-form; the associated type function is the type function in 3) of Definition 4.3.

4.2) A term is a special case of the term-form and its type is a special case of the type function.

4.3) The type function of X is of the form $f \to g$ and that of Y is f. The type function of $X\langle Y\rangle$ is g.

4.4) If ϕ is a variable of type function f and X is a term-form of g, then $\lambda\phi X$ is a term-form of $f \to g$.

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4.5) If $P_1, \ldots, P_m$ are mutually exclusive predicates and X_i is a term-form of type function f_i, $i = 1, \ldots, m+1$, then $C[P_1, \ldots, P_m, X_1, \ldots, X_m, X_{m+1}]$ is a term-form of type function

$$n(\chi_1)f_1 + \cdots + n(\chi_m)f_m + \chi_1 \cdots \chi_m f_{m+1},$$

where χ_i is the characteristic function of P_i and $n(\chi_i)$ represents its reciprocal signum.

4.6) If X is a term-form of f, then Y is one of $0 \to (f \to f)$. $\rho[X, Y]$ is a term-form of $0 \to f$.

4.7) Let f and g be type functions of term-forms X and Y respectively, where $\mathfrak{A}, \mathfrak{B}_1, \ldots, \mathfrak{B}_m$ exhaust all the free parameters of f and g. Suppose for a type function h with parameters $\mathfrak{A}, \mathfrak{B}_1, \ldots, \mathfrak{B}_m$,

$$h(0, k_1, \ldots, k_m) = f(k_1, \ldots, k_m),$$

$$g(k, k_1, \ldots, k_m) = h(k, k_1, \ldots, k_m) \to h(k+1, k_1, \ldots, k_m)$$

for every set of $k, k_1, \ldots, k_m$ (values of $\mathfrak{A}, \mathfrak{B}_1, \ldots, \mathfrak{B}_m$). Then $R[\mathfrak{A}, X, Y]$ is a term-form of type function h.

4.8) Suppose X is a term-form of type function $f(\mathfrak{D}_1, \ldots, \mathfrak{D}_n, \mathfrak{C}_1, \ldots, \mathfrak{C}_l)$. Then $\mathrm{Sub}(X; {}^{\mathfrak{C}_1 \cdots \mathfrak{C}_l}_{p_1 \cdots p_l})$ is a term-form of type function

$$f(\mathfrak{D}_1, \ldots, \mathfrak{D}_n, p_1, \ldots, p_l)$$

whose parameters are $\mathfrak{D}_1, \ldots, \mathfrak{D}_n$ and those in $p_1, \ldots, p_l$.

COROLLARY. *A term-form and its associated type function share the same free parameters, and a finite type is associated with a complete specification of a term-form.*

PRINCIPLE. The term-form is what we claim to be a construction principle, since it is designed to enumerate some primitive recursive functionals, possibly of different types.

DEFINITION 4.5. Reduction. For a term-form X of associated type function $f(\mathfrak{B}_1, \ldots, \mathfrak{B}_m)$,

$$\sigma X \equiv \mathrm{Spec}\left(X; {}^{\mathfrak{B}_1 \cdots \mathfrak{B}_m}_{k_1 \cdots k_m}\right)$$

(a complete numerical specification) can be said of type $f(k_1, \ldots, k_m)$. X, $Y, \ldots$ will denote term-forms, $\sigma X, \sigma Y, \ldots$ their complete specifications (without explicitly specifying the numbers unless necessary) and $U, V, \ldots$ some primitive recursive terms of finite type.

$\sigma X \Rightarrow U$ will express the fact that U is the reduct of X, where σX and U share a common finite type. Reducts will be defined so that, if a variable $\phi_i^{f(\mathfrak{B}_1, \ldots, \mathfrak{B}_m)}$

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occurs in X, then $\phi_i^{f(k_1, \ldots, k_m)}$ occurs in U, and no new variables are introduced. The definition of reducts will be given according to the construction of pre-terms in 4) of Definition 4.1.

4.1) Consider a pre-variable $\phi_i^{\mathfrak{s}}$, where $\mathfrak{s}$ is a pre-type, and let s be a complete specification of $\mathfrak{s}$. More precisely, s is the result of computation of a specification. s represents a finite type, hence ϕ_i^s exists among the variables of finite type. Now

$$\sigma\phi \Rightarrow \phi_i^s.$$

4.2) $X \Rightarrow X$ if X is a primitive recursive term of finite type, since $\sigma X \equiv X$.

4.3) If $\sigma X \Rightarrow U$ and $\sigma Y \Rightarrow V$, then

$$\sigma(X\langle y\rangle) \equiv \sigma X\langle \sigma Y\rangle \Rightarrow U\langle V\rangle,$$

where $U\langle V\rangle$ represents the application of U to V in the usual sense.

4.4) If $\sigma\phi \Rightarrow \psi$ and $\sigma X \Rightarrow U$, then

$$\sigma(\lambda\phi X) \equiv \lambda\sigma\phi\sigma X \Rightarrow \lambda\psi U,$$

where the last λ represents the usual λ-notation.

4.5) Suppose $\sigma X_i \Rightarrow U_i$, $i = 1, \ldots, m + 1$. Then

$$\begin{aligned}\sigma C[&P_1, \ldots, P_m, X_1, \ldots, X_m, X_{m+1}]\\ &\equiv C[\sigma P_1, \ldots, \sigma P_m, \sigma X_1, \ldots, \sigma X_m, \sigma X_{m+1}]\\ &\Rightarrow n(\chi_1)U_1 + \cdots + n(\chi_m)U_m + \chi_1 \ldots \chi_m U_{m+1},\end{aligned}$$

where χ_i and $n(\chi_i)$ were defined in Definition 4.4 and the reduct is defined in the arithmetic of primitive recursive terms of finite type. χ_i here in fact represents either 0 or 1, since we are dealing with a complete specification.

4.6) If $\sigma X \Rightarrow U$ and $\sigma Y \Rightarrow V$, then

$$\sigma\rho[X, Y] \equiv \rho[\sigma X, \sigma Y] \Rightarrow \rho[U, V],$$

where the last ρ is interpreted to be the recursion operator.

4.7) Suppose Spec(X; $\begin{smallmatrix}\mathfrak{B}\\ r\end{smallmatrix}$) $\Rightarrow U$ and Spec(Y; $\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ k & r\end{smallmatrix}$) $\Rightarrow W$. (We have written out just one parameter $\mathfrak{B}$ for simplicity.) Then

$$\mathrm{Spec}\left(R[\mathfrak{A}, X, Y];\ \begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ 0 & r\end{smallmatrix}\right) \equiv \mathrm{Spec}\left(X;\ \begin{smallmatrix}\mathfrak{B}\\ r\end{smallmatrix}\right) \Rightarrow U,$$

$$\begin{aligned}\mathrm{Spec}\big(R[&\mathfrak{A}, X, Y];\ \begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ k+1 & k\end{smallmatrix}\big)\\ &\equiv \mathrm{Spec}\left(Y; \begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ k & r\end{smallmatrix}\right)\left\langle \mathrm{Spec}\left(R[\mathfrak{A}, X, Y];\ \begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ k & r\end{smallmatrix}\right)\right\rangle \Rightarrow W\langle V\rangle,\end{aligned}$$

where Spec($R[\mathfrak{A}, X, Y]$; $\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ k & r\end{smallmatrix}$) $\Rightarrow V$.

4.8) Let X be a term-form where $\mathfrak{B}_1, \ldots, \mathfrak{B}_m$ cover all the free parameters distinct from $\mathfrak{C}_1, \ldots, \mathfrak{C}_l$ and those in $p_1, \ldots, p_l$. Suppose

$$\mathrm{Spec}\left(X;\ \begin{smallmatrix}\mathfrak{B}_1 \cdots \mathfrak{B}_m & \mathfrak{C}_1 \cdots \mathfrak{C}_l\\ k_1 \cdots k_m & p_1 \cdots p_l\end{smallmatrix}\right) \Rightarrow U,$$

where $r_i = \mathrm{Spec}(p_i;\ \genfrac{}{}{0pt}{}{\mathfrak{B}_1 \cdots \mathfrak{B}_m}{k_1 \cdots k_m})$. Then

$$\mathrm{Spec}\left(\mathrm{Sub}\left(X;\ \genfrac{}{}{0pt}{}{\mathcal{C}_1 \cdots \mathcal{C}_l}{p_1 \cdots p_l}\right);\ \genfrac{}{}{0pt}{}{\mathfrak{B}_1 \cdots \mathfrak{B}_m}{k_1 \cdots k_m}\right) \Rightarrow U.$$

If $\sigma X \Rightarrow U$, then we say that σX reduces to U or σX is reducible to U. If every complete specification of a term-form X is reducible (to a primitive recursive term of finite type), then we say that X is reducible.

THEOREM 3. *Every term-form is reducible.*

PROOF. By induction on the construction of a term-form, within which by induction on l. We have only to examine each case in Definition 4.5; consult also Definitions 4.3 and 4.4.

CONCLUSION. Our construction principle—term-forms—is computable, in the sense that every term-form is reducible, namely every complete specification of a term-form is reduced to a primitive recursive term of finite type, which is considered to be computable.

Note. As for the computability of primitive recursive functionals of finite type, see for example Diller (1968), Hinata (1967) and Hindley and others (1972).

The demonstration of the computability of our construction is based on the soundness of the system of primitive recursive functionals of finite type and the double induction used in Theorem 3 above.

5. Functional interpretation concluded

Now we shall show that the functional construction in Section 3 can be interpreted in terms of the construction principle in Section 4. We first build up a term-form that corresponds to the functional N in Theorem 2 of Section 3, following the procedure in (i) $\sim$ (xii) there, and then demonstrate that it does the work.

Definition 5.1. 1) We specify the type functions, auxiliary functions and auxiliary predicates, $(\mathfrak{F}, \mathfrak{p}, \mathfrak{P})$, in Definition 4.2 (see also Definition 3.1).

$\mathfrak{F}$: $[\mathfrak{A}]$, $[\mathfrak{A} + 1]$, $\langle\langle\mathfrak{A}\rangle\rangle$, $\{\mathfrak{A}\}$, $\langle\mathfrak{A}\rangle$, $\|\mathfrak{A}\|$.

$\mathfrak{p}$: $\mathfrak{A} + 1, (\mathfrak{B} + 1) \dot{-} \mathfrak{A}, \mathfrak{B} \dot{-} \mathfrak{A}$.

$\mathfrak{P}$: $\mathfrak{A} = \mathfrak{B} + 1, (\mathfrak{B} \dot{-} 1) \dot{-} \mathfrak{A} \geqslant 0$.

2) Construction. For any object S which was defined in (i) $\sim$ (xii), Section 3, we let $T(S)$ denote the term-form which is to be defined corresponding to it. (Review the note at the end of Section 3.)

(i)* The α_0 in (i) is a primitive recursive functional of finite type. Let $T(\alpha_0)$ be α_0 itself.

(ii)* Let $T(\Phi^{\sim})$ be

$$\rho\left[\phi^{[\mathfrak{A}]}, \lambda l \Phi^{[\mathfrak{A}+1]}\langle d, c + \omega^{d} l\rangle\right],$$

where $d = d\langle a, b, c\rangle$ and a, b, c, ϕ, Φ are variables. Notice that R is not involved.

(iii)* $T(\Phi^{*}) \equiv T(\Phi^{\sim})\langle n\langle a, x, c\rangle\rangle$.

(iv)* $T(\beta) \equiv \lambda\psi\lambda a\lambda\Phi\lambda c\lambda\phi\psi\langle c + \omega^{a}, \lambda x T(\Phi^{*})\rangle$,

where the associated type functions are as follows.

ψ: $0, (0 \to [\mathfrak{A}]) \to [\mathfrak{A}]$,

Φ: $0 \to [\mathfrak{A} + 1]$,

ϕ: $[\mathfrak{A}]$.

(v)* $T(\gamma) \equiv \lambda\psi\lambda z\lambda\phi\psi\langle z + 1, \lambda x\phi\rangle$,

where the associated type functions are $\{\mathfrak{A}\}$ for ψ and $[\mathfrak{A}]$ for ϕ.

(vi)* $T(\delta) \equiv \lambda\psi\lambda\Phi\lambda\chi\Phi\langle\gamma\langle\psi\rangle\rangle\langle 0, \chi\rangle$,

$\{\mathfrak{A}\}$ for ψ, $\{\mathfrak{A} + 1\}, 0, \{\mathfrak{A}\} \to [\mathfrak{A}]$ for Φ and $[\mathfrak{A}]$ for χ.

(vii)* $T(\varepsilon) \equiv R[\mathfrak{A}, T(\alpha_0), T(\beta)]$.

(viii)* Here we need some auxiliary steps.

Define $f(\mathfrak{B})$ to be $[\mathfrak{B} + 1] \to [\mathfrak{B} + 1]$ and $I^{f(\mathfrak{B})}$ to be $\lambda\eta \cdot \eta$, where η has the type function $[\mathfrak{B} + 1]$. Define further:

$$\begin{aligned}
\Theta &\equiv \lambda\phi 0, \quad \text{for } \phi \text{ of type } \|0\|,\\
Z &\equiv \mathrm{Sub}\left(T(\delta)\langle T(\varepsilon)\rangle;\ {}^{\mathfrak{A}}_{(\mathfrak{B}+1)\dot{-}\mathfrak{C}}\right),\\
Y &\equiv C\left[(\mathfrak{B} + 1) \dot{-} C > 0, Z, \Theta\right].
\end{aligned}$$

Then

$$T(\tau) \equiv R\left[\mathfrak{C}, I^{f(\mathfrak{B})}, Y\right].$$

(ix)* $T(\nu) \equiv C[\mathfrak{D} = 0, I^{[0]}, \mathrm{Sub}(\tau^{\sim};\ {}^{\mathfrak{B}}_{\mathfrak{D}-1})]$,

where $I^{[0]}$ is the identity of type [0], and

$$\tau^{\sim} \equiv \mathrm{Sub}\left(T(\tau);\ {}^{\mathfrak{C}}_{\mathfrak{B}+1}\right)\langle I^{[0]}\rangle.$$

(x)* μ is a term of finite type; $T(\mu) \equiv \mu$.

(xi)* $T(M) \equiv \lambda f\lambda j\langle \mathrm{hp}\langle \mu\langle \mathrm{Sub}(T(\nu);\ {}^{\mathfrak{D}}_{\mathfrak{E}+1})\rangle\langle f\rangle\rangle\langle j\rangle\rangle$.

(xii)* $T(N) \equiv R[\mathfrak{F}, f, M^{\sim}]$,

where

$$M^{\sim} \equiv \mathrm{Sub}\left(T(M);\ {}^{\mathfrak{E}}_{\mathfrak{G}\dot{-}\mathfrak{F}}\right).$$

PROPOSITION 5.1. *The expressions defined in* (i)* $\sim$ (xii)* *are legitimate term-forms.*

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Proposition 5.2. *For any f_0 a decreasing sequence from E and a number i,*

$$\mathrm{Spec}\big(T(N);\ \substack{\mathscr{G}\\ \mathrm{ht}(f_0)}\ \substack{\mathscr{F}\\ i}\big)$$

acts as the $N(i; f_0)$ in (xii).

Proof. We shall show that for each term-form $T(S)$ in one of (i)* $\sim$ (xii)* above, any of its complete specification is reduced to the term which is obtained from S by instantiating it by the same set of values of parameters. See Definitions 4.3 and and 4.5 for evaluation.

(i) Obvious.

(ii) $\mathrm{Spec}(T(\Phi^{\sim});\ \substack{\mathscr{A}\\ i}) \equiv \rho[\mathrm{Spec}(\phi^{[\mathscr{A}]};\ \substack{\mathscr{A}\\ i}), \mathrm{Spec}(\lambda l\,\Phi^{[\mathscr{A}+1]}\langle d, c+\omega^{d}l\rangle;\ \substack{\mathscr{A}\\ i})]$

$$\Rightarrow \rho\big[\phi^{\langle\langle i\rangle\rangle}, \lambda l\,\Phi^{[i+1]}\langle d, c+\omega^{d}l\rangle\big],$$

which represents the $\Phi_i^{\sim}$ in (ii) according to the interpretation of ρ as the recursion operator of functionals (see Yasugi (1963)).

(iii) $\sim$ (vi) are dealt with likewise. Let us consider (iv) as an example.

(iv) $\mathrm{Spec}(T(\beta);\ \substack{\mathscr{A}\\ i})$

$$\begin{aligned} &\equiv \lambda\,\mathrm{Spec}(\psi;\ \substack{\mathscr{A}\\ i})\lambda a\lambda\,\mathrm{Spec}(\Phi;\ \substack{\mathscr{A}\\ i})\lambda c\lambda\,\mathrm{Spec}(\phi;\ \substack{\mathscr{A}\\ i})\\ \mathrm{Spec}(\psi;\ \substack{\mathscr{A}\\ i})&\big\langle c+\omega^{a}, \lambda x\,\mathrm{Spec}\big(T(\Phi^{*});\ \substack{\mathscr{A}\\ i}\big)\big\rangle\\ &\Rightarrow \lambda\psi^{s}\lambda a\lambda\Phi^{t}\lambda c\lambda\phi^{[i]}\psi^{s}\langle c+\omega^{a}, \lambda x\Phi_i^{*}\rangle, \end{aligned}$$

where $\mathrm{Spec}(T(\Phi^{*});\ \substack{\mathscr{A}\\ i}) \Rightarrow \Phi_i^{*}$ by (iii), $s \equiv 0$, $(0 \to [i]) \to [i]$ and $t \equiv 0 \to [i+1]$.

(vii) We prove this case by induction on i.

$$\mathrm{Spec}\big(R[\mathscr{A}, T(\alpha_0), T(\beta)];\ \substack{\mathscr{A}\\ 0}\big) \Rightarrow \alpha_0 \equiv \varepsilon_0.$$

Suppose

$$\mathrm{Spec}\big(R[\mathscr{A}, T(\alpha_0), T(\beta)];\ \substack{\mathscr{A}\\ i}\big) \Rightarrow \varepsilon_i$$

has been established.

$$\begin{aligned} \mathrm{Spec}\big(R[\mathscr{A}, T(\alpha_0), T(\beta)];\ \substack{\mathscr{A}\\ i+1}\big)&\\ &\equiv \mathrm{Spec}\big(T(\beta);\ \substack{\mathscr{A}\\ i}\big)\big\langle \mathrm{Spec}\big(R[\mathscr{A}, T(\alpha_0), T(\beta)];\ \substack{\mathscr{A}\\ i}\big)\big\rangle\\ &\Rightarrow \beta_i\langle\varepsilon_i\rangle \equiv \varepsilon_{i+1} \end{aligned}$$

by (iv) and the induction hypothesis.

$$\begin{aligned} \text{(viii) } \mathrm{Spec}\big(Z;\ \substack{\mathscr{B}\\ l}\ \substack{\mathscr{C}\\ k}\big) &\equiv \mathrm{Spec}\big(T(\delta)\langle T(\varepsilon)\rangle;\ \substack{\mathscr{B}\\ l}\ \substack{\mathscr{C}\\ k}\ \substack{\mathscr{A}\\ (l+1)-k}\big)\\ &\equiv \mathrm{Spec}\big(T(\delta);\ \substack{\mathscr{B}\\ l}\ \substack{\mathscr{C}\\ k}\ \substack{\mathscr{A}\\ (l+1)-k}\big)\big\langle \mathrm{Spec}\big(T(\varepsilon);\ \substack{\mathscr{B}\\ l}\ \substack{\mathscr{C}\\ k}\ \substack{\mathscr{A}\\ (l+1)-k}\big)\big\rangle\\ &\equiv \mathrm{Spec}\big(T(\delta);\ \substack{\mathscr{A}\\ (l+1)\dot{-}k}\big)\big\langle \mathrm{Spec}\big(T(\varepsilon);\ \substack{\mathscr{A}\\ (l+1)\dot{-}k}\big)\big\rangle\\ &\Rightarrow \delta_{(l+1)\dot{-}k}\langle\varepsilon_{(l+1)\dot{-}k}\rangle \end{aligned}$$

by (vi) and (vii).

$$\operatorname{Spec}\left(Y;\begin{smallmatrix}\mathfrak{B} & \mathcal{C}\\ l & k\end{smallmatrix}\right)\equiv C\left[\operatorname{Spec}\left((\mathfrak{B}+1)\dot{-}\mathcal{C}>0;\begin{smallmatrix}\mathfrak{B} & \mathcal{C}\\ l & k\end{smallmatrix}\right),\operatorname{Spec}\left(Z;\begin{smallmatrix}\mathfrak{B} & \mathcal{C}\\ l & k\end{smallmatrix}\right),\operatorname{Spec}\left(\Theta;\begin{smallmatrix}\mathfrak{B} & \mathcal{C}\\ l & k\end{smallmatrix}\right)\right]$$
$$\Rightarrow n(\chi(l,k))\delta_{(l+1)\dot{-}k}\langle\varepsilon_{(l+1)\dot{-}k}\rangle+\chi(l,k)\Theta,$$

where $\chi(l, k)$ is the characteristic function of $(l+1)\dot{-}k \geqslant 0$. Thus, if $k \leqslant l+1$, then the reduct is

$$\delta_{(l+1)\dot{-}k}\langle\varepsilon_{(l+1)\dot{-}k}\rangle.$$
$$\operatorname{Spec}\left(T(\tau);\begin{smallmatrix}\mathcal{C} & \mathfrak{B}\\ 0 & l\end{smallmatrix}\right)\equiv I^{f(0)}\Rightarrow I_l\equiv\tau_{(l,l+1)}.$$

Suppose $k \leqslant l$ and

$$\operatorname{Spec}\left(T(\tau);\begin{smallmatrix}\mathcal{C} & \mathfrak{B}\\ k & l\end{smallmatrix}\right)\Rightarrow\tau_{(l,(l+1)\dot{-}k)}.$$

Then

$$\begin{aligned}\operatorname{Spec}\left(T(\tau);\begin{smallmatrix}\mathcal{C} & \mathfrak{B}\\ k+1 & l\end{smallmatrix}\right)&\equiv\operatorname{Spec}\left(Y;\begin{smallmatrix}\mathcal{C} & \mathfrak{B}\\ k & l\end{smallmatrix}\right)\left\langle\operatorname{Spec}\left(T(\tau);\begin{smallmatrix}\mathcal{C} & \mathfrak{B}\\ k & l\end{smallmatrix}\right)\right\rangle\\ &\Rightarrow\delta_{(l+1)\dot{-}k}\langle\varepsilon_{(l+1)\dot{-}k}\rangle\langle\tau_{(l,(l+1)\dot{-}k)}\rangle\\ &\equiv\tau_{(l,(l+1)\dot{-}(k+1))}.\end{aligned}$$

(ix) $\operatorname{Spec}\left(T(\tau^{\sim});\begin{smallmatrix}\mathfrak{B}\\ l\end{smallmatrix}\right)\equiv\operatorname{Spec}\left(T(\tau);\begin{smallmatrix}\mathfrak{B} & \mathcal{C}\\ l & l+1\end{smallmatrix}\right)\left\langle\operatorname{Spec}\left(I^{[0]};\begin{smallmatrix}\mathfrak{B} & \mathcal{C}\\ l & l+1\end{smallmatrix}\right)\right\rangle\Rightarrow\tau_{(l,0)}\langle I^{[0]}\rangle$

by (viii) where $k = l + 1$.

(x) $$\begin{aligned}\operatorname{Spec}\left(T(\nu);\begin{smallmatrix}\mathfrak{D}\\ i\end{smallmatrix}\right)&\equiv C\left[\operatorname{Spec}\left(\mathfrak{D}=0;\begin{smallmatrix}\mathfrak{D}\\ i\end{smallmatrix}\right),I^{[0]},\operatorname{Spec}\left(T(\tau^{\sim});\begin{smallmatrix}\mathfrak{B} & \mathfrak{D}\\ i\dot{-}1 & i\end{smallmatrix}\right)\right]\\ &\Rightarrow n(\chi(i=0))I^{[0]}+\chi(i=1)\tau_{(i\dot{-}1,0)}\langle I^{[0]}\rangle\\ &\equiv\begin{cases}I^{[0]}\equiv\nu_0 & \text{if } i=0,\\ \tau_{(j,0)}(I^{[0]})\equiv\nu_i & \text{if } i=j+1.\end{cases}\end{aligned}$$

(xi) $$\begin{aligned}\operatorname{Spec}\left(T(M);\begin{smallmatrix}\mathfrak{E}\\ h\end{smallmatrix}\right)&\equiv\lambda f\lambda j\left\langle\operatorname{hp}\left\langle\mu\left\langle\operatorname{Spec}\left(T(\nu);\begin{smallmatrix}\mathfrak{D} & \mathfrak{E}\\ h+1 & h\end{smallmatrix}\right)\right\rangle\langle f\rangle\right\rangle\langle j\rangle\right\rangle\\ &\Rightarrow\lambda f\lambda j\langle\operatorname{hp}\langle\mu\langle\nu_{h+1}\rangle\langle f\rangle\rangle\langle j\rangle\rangle\end{aligned}$$

by (x).

(xii) Let $\operatorname{ht}(f_0)$ be j.

$$\begin{aligned}\operatorname{Spec}\left(M^{\sim};\begin{smallmatrix}\mathfrak{F} & \mathfrak{G}\\ i & j\end{smallmatrix}\right)&\equiv\operatorname{Spec}\left(T(M);\begin{smallmatrix}\mathfrak{F} & \mathfrak{G} & \mathfrak{E}\\ i & j & j\dot{-}1\end{smallmatrix}\right)\\ &\equiv\operatorname{Spec}\left(T(M):\begin{smallmatrix}\mathfrak{E}\\ j\dot{-}i\end{smallmatrix}\right)\\ &\Rightarrow M_{j\dot{-}i}\end{aligned}$$

by (xi).

$$\operatorname{Spec}\left(T(N);\begin{smallmatrix}\mathfrak{F} & \mathfrak{G}\\ 0 & j\end{smallmatrix}\right)\Rightarrow f_0.$$

Suppose $i < j$ and

$$\mathrm{Spec}\left(T(N);\, {}^{\mathcal{F}}_{i}\,{}^{\mathcal{G}}_{j}\right) \Rightarrow N(i; f_0).$$

Then

$$\begin{aligned}\mathrm{Spec}\left(T(N);\, {}^{\mathcal{F}}_{i+1}\,{}^{\mathcal{G}}_{j}\right) &\equiv \mathrm{Spec}\left(M^{\sim};\, {}^{\mathcal{F}}_{i}\,{}^{\mathcal{G}}_{j}\right)\left\langle \mathrm{Spec}\left(T(N);\, {}^{\mathcal{F}}_{i}\,{}^{\mathcal{G}}_{j}\right)\right\rangle \\ &\Rightarrow M_{j \dot{-} i}(N(i; f_0)) \equiv N(i+1; f_0).\end{aligned}$$

This completes the proof of the proposition.

CONCLUSION. The functional construction which the accessibility proof of (E, $\prec$) in Section 3 is based upon can be interpreted in the theory of term-forms.

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