ON BANACH LIMIT OF FOURIER SERIES AND CONJUGATE SERIES I

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1. Let (x_n) be a sequence of real numbers. (x_n) corresponds to a number $\operatorname{Lim} x_n$ called the Banach limit of (x_n) satisfying the following conditions:

(1) $\operatorname{Lim}(a x_n + b y_n) = a \operatorname{Lim} x_n + b \operatorname{Lim} y_n$

(2) If $x_n \geq 0$ for every n, then $\operatorname{Lim} x_n \geq 0$

(3) $\operatorname{Lim} x_{n+1} = \operatorname{Lim} x_n$

(4) If $x_n = 1$ for every n, then $\operatorname{Lim} x_n = 1$

The existence of such limits is proved by Banach [1].

The object of this paper is to obtain certain criteria for the existence of unique Banach limit of Fourier series and conjugate series which gives a new criterion for the convergence of Fourier series and conjugate series.

Definitions: A sequence (s_n) is said to be almost convergent to a limit s if

$$(1.1) \qquad \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=p}^{n+p} s_k = s$$

uniformly with respect to p.

The following functions are frequently used:

$$g_x(t) = \{f(x+t) + f(x-t) - 2f(x)\},$$

$$\psi_x(t) = \{f(x+t) - f(x-t)\},$$

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$$G_x(t) = \int_0^t |g_x(u)|\, du,$$

$$\Psi_x(t) = \int_0^t \psi_x(u)\, du.$$

Lorentz [2] has proved the following theorems:

THEOREM A. *A sequence* (x_n) *has unique Banach limit if and only if it is almost convergent.*

THEOREM B. $a_n = O(c_n)$ *is a Taubarian condition for an almost convergent series* Σa_n *if and only if, for every* $\epsilon > 0$, *there exists a lacunary sequence* $\{n_\nu\}$ *with* $c_n < \epsilon$ for $n \neq n_\nu$, $\nu = 1, 2, 3, \ldots$.

We prove the following theorems:

THEOREM 1. *Let* $f(x)$ *be a L-integrable and* 2π *periodic function. The associated Fourier series*

$$(1.2) \qquad 1/2\, a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

has unique Banach limit $f(x)$ *provided the following conditions hold*

$$(1.3) \qquad \begin{aligned} &(1) \quad \int_0^\tau |g_x(t)|\, dt = o(\tau) \quad \text{as } \tau \to 0+ \,, \\ &(2) \quad \int_{\frac{1}{n+p}}^{\frac{1}{n+1}} |g_x(t)|/t\, dt = o(1) \quad \text{as } n \text{ tends to } \infty \,, \end{aligned}$$

uniformly with respect to p.

As the terms of the Fourier series tend to zero, theorems A and B imply that (1.3) are the sufficient conditions for the convergence of Fourier series.

It is important to note that the above conditions (1.3) do not imply the Dini condition of convergence of Fourier series. The following example illustrates the above fact:

Consider the function

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$$f(x) = x^2 \sin 1/x^2 - \cos 1/x^2 , \quad x \neq 0.$$

$$f(0) = 0 ,$$

$$g_0(t) = \{ f(0+t) + g(0-t) - 2f(0)\}$$

$$= 2t^2 \sin 1/t^2 - 2 \cos 1/t^2 \geq 0.$$

Evidently (1.3) (1) holds for above $g_0(t)$, and for (1.3) (2) we have

$$\int_{\frac{1}{n+p+1}}^{\frac{1}{n+1}} |g_0(t)|/t \, dt = [t^2 \sin 1/t^2]_{\frac{1}{n+p+1}}^{\frac{1}{n+1}}$$

which tends to zero as n tends to infinity uniformly with respect to p, whereas

$$\int_0^1 |g_0(t)|/t \, dt = \infty. \quad (*)$$

This shows that Dini's condition does not hold.

THEOREM 2. Let $f(x)$ be a L-integrable and periodic 2π function. The associated conjugate series of (1.2)

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \tag{1.4}$$

has unique Banach limit $\tilde{f}(x)$ provided it exists and the following conditions hold:

$$\int_0^{\tau} |\psi_x(t)| dt = o(\tau) \quad \text{as } \tau \text{ tends to } 0+$$

$$\int_{\frac{1}{n+p+1}}^{\frac{1}{n+1}} |\psi_x(t)|/t \, dt = o(1) \quad \text{as } n \text{ tends to } \infty \tag{1.5}$$

uniformly with respect to p.

2. Before proving the theorems we prove the following

(*) Titchmarsh, E. C. : Theory of functions p.342

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LEMMA

If

$$N_n^p(t) = \frac{1}{n+1} \sum_{k=p}^{n+p} \frac{\sin(k+1/2)t}{\sin t/2}$$

then

$$\text{(i)} \quad N_n^p(t) = 0\,(1/nt^2)\,, \quad 0 < t < \pi\,,$$

(2.1) $$\text{(ii)} \quad N_n^p(t) = 0\,(n+p+1)\,,$$

$$\text{(iii)} \quad N_n^p(t) = 0(1/t) \qquad 0 < t < \pi\,,$$

$$\text{(iv)} \quad \int_{-\pi}^{+\pi} N_n^p(t)dt = \pi$$

Proof of the lemma.

$$\text{(i)} \quad N_n^p(t) = \frac{1}{n+1} \frac{\{\cos pt - \cos(p+n+1)t\}}{(2 \sin t/2)^2}$$

$$N_n^p(t) = 0(1/nt^2)\,, \qquad 0 < t < \pi$$

$$\text{(ii) Since } D_k(t) = \frac{\sin(k+1/2)t}{2 \sin t/2}$$

$$= 0\,(k+1)$$

we have $N_n^p(t) = 0\,(n+p+1)$:

$$\text{(iii)} \quad N_n^p(t) = \frac{2 \sin(p+(n+1)/2)t \; \sin(n+1)t/2}{(n+1)(2 \sin t/2)^2}$$

$$= 0(1/t) \qquad 0 < t < \pi$$

$$\text{(iv)} \quad 1/\pi \int_{-\pi}^{+\pi} N_n^p(t)\,dt = \frac{1}{n+1} \sum_{k=p}^{n+p} \frac{1}{\pi} \int_{-\pi}^{\pi} D_k(t)\,dt$$

$$= \frac{1}{n+1} \sum_{k=p}^{n+p} 1 = 1$$

This proves the lemma.

Proof of the theorem 1.

Let $s_n(x)$ be the nth partial sum of the series (1.2). It is easy to show that

$$s_n(x) - f(x) = 1/2\pi \int_0^{\pi} g_x(t) \frac{\sin(n+1/2)t}{\sin t/2} dt$$

$$t_n^p(x) - f(x) = \frac{1}{n+1} \sum_{k=p}^{n+p} \{s_k(x) - f(x)\}$$

$$= 1/2\pi \int_0^{\pi} g_x(t) \; N_n^p(t) \; dt$$

$$= 1/2\pi \left(\int_0^{\frac{1}{n+p+1}} + \int_{\frac{1}{n+p+1}}^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right) g_x(t) \; N_n^p(t) \; dt$$

$$= P + Q + R .$$

$P = o(1)$ uniformly with respect to p, by the hypothesis (1.3) (1) and (2.1) (ii).

Using (2.1) (i)

$$R = 0 \left(1/n \int_{\frac{1}{n+1}}^{\pi} \frac{|g_x(t)|}{t^2} dt\right) .$$

Applying integration by parts and (1.3) (1),

$$R = 0\left(1/n \left[\frac{G_x(t)}{t^2}\right]_{\frac{1}{n+1}}^{\pi} + 2/n \int_{\frac{1}{n+1}}^{\pi} \frac{G_x(t)}{t^3} dt\right)$$

$= o(1)$ uniformly with respect to p, and

$$Q = \frac{1}{2(n+1)\pi} \int_{\frac{1}{n+p+1}}^{\frac{1}{n+1}} g_x(t) \frac{\sin(p+(n+1)/2)t \sin (n+1)t/2}{2 (\sin t/2)^2} dt.$$

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Using the well known inequality $|\sin(n+1)t| \le (n+1)|\sin t|$,

$$Q = O\left(1/2\pi \int_{\frac{1}{n+p+1}}^{\frac{1}{n+1}} \frac{|g_x(t)|}{t}\, dt\right)$$

$= o(1)$ uniformly with respect to p, by (1.3)(2).

Proof of the theorem 2. Let $\tilde{s}_n(x)$ be the n^{th} partial sum of (1.4).

We have $$\tilde{s}_n(x) = -\frac{2}{\pi} \int_0^{\pi} \psi_x(t)\, \tilde{D}_k(t)\, dt$$

where $\tilde{D}_k(t)$ is Dirichlet's conjugate kernel.

$$\tilde{t}_n^p = \frac{1}{n+1} \sum_{k=p}^{n+p} \tilde{s}_k(x)$$

$$\tilde{t}_n^p = -\frac{2}{\pi} \int_0^{\pi} \psi_x(t) \frac{1}{(n+1)} \sum_{k=p}^{n+p} \left\{ \frac{\cos t/2}{\sin t/2} - \frac{\cos(k+1/2)t}{\sin t/2} \right\} dt$$

$$\tilde{t}_n^p - \left(-\frac{2}{\pi} \int_{\frac{1}{n+p+1}}^{\pi} \psi_x(t) \cot t/2\, dt \right)$$

$$= -\frac{2}{\pi(n+1)} \int_0^{\frac{1}{n+p+1}} \psi_x(t) \sum_{k=p}^{n+p} \tilde{D}_k(t)$$

$$+ \frac{2}{\pi(n+1)} \int_{\frac{1}{n+p+1}}^{\pi} \psi_x(t) \sum_{k=p}^{n+p} \frac{\cos(k+1/2)}{\sin t/2}\, dt$$

$= A + B.$

$$A = O\left(\frac{2}{\pi} \int_0^{\frac{1}{n+p+1}} |\psi_x(t)|\, (n+p+1)\, dt\right),$$

$= o(1)$ uniformly with respect to p, by (1.5);

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$$B = \frac{2}{\pi(n+1)} \left\{ \int_{\frac{1}{n+p+1}}^{\frac{1}{n+1}} \psi_x(t) \frac{\cos(p+\frac{n}{2}+1)t \sin nt/2}{(\sin t/2)^2} dt \right.$$

$$\left. + \int_{\frac{1}{n+1}}^{\pi} \psi_x(t) \frac{\cos(p+\frac{n}{2}+1)t \sin t/2}{(\sin t/2)^2} \right\}$$

$$= B_1 + B_2 .$$

$$B_1 = 0 \left(\int_{\frac{1}{n+p+1}}^{\frac{1}{n+1}} |\psi_x(t)| /t\, dt \right) = o(1)$$ uniformly with respect to p, by (1.5).

$$B_2 = 0 \left(1/n \int_{\frac{1}{n+1}}^{\pi} |\psi_x(t)| / t^2 \, dt\right)$$

$$= 0 \left\{ \frac{1}{n} \left(\left[\frac{\psi_x(t)}{t^2} \right]_{\frac{1}{n+1}}^{\pi} + 2 \int_{\frac{1}{n+1}}^{\pi} \frac{\psi_x(t)}{t^3} dt \right) \right\} .$$

$B_2 = o(1)$ uniformly with respect to p, using (1.5).

$$\left| - \frac{2}{\pi} \int_{\frac{1}{n+1}}^{\pi} \psi_x(t) \cot t/2 \, dt + \frac{2}{\pi} \int_{\frac{1}{n+p+1}}^{\pi} \psi_x(t) \cot t/2 \, dt \right|$$

$$\leq \frac{2}{\pi} \int_{\frac{1}{n+p+1}}^{\frac{1}{n+1}} |\psi_x(t)| / t \, dt = o(1) \text{ as } n \text{ tends to } \infty$$

uniformly with respect to p, which proves the theorem.

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