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REARRANGEMENTS THAT PRESERVE RATES OF DIVERGENCE

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1. Introduction. Let $\sum a_k$ be an infinite series of real numbers and let π be a permutation of $\mathbf{N}$, the set of positive integers. The series $\sum a_{\pi(k)}$ is then called a rearrangement of $\sum a_k$. A classical theorem of Riemann states that if $\sum a_k$ is a conditionally convergent series and s is any fixed real number (or $\pm\infty$), then there is a permuation π such that $\sum a_{\pi(k)} = s$. The problem of determining those permutations that convert any conditionally convergent series into a convergent rearrangement (such permuations are called convergence preserving) has received wide attention (see, for example [**6**]). Of special interest is a paper by P. A. B. Pleasants [**5**] in which is shown that the set of convergence preserving permutations do not form a group.

In this paper we consider questions similar to those above, but for rearrangements of divergent series of positive terms. We establish some notation before stating the precise problem.

Definition 1. Let $\sum a_k$ and $\sum b_k$ be divergent series of positive terms. For positive integer n, let $A_n = \sum_{k=1}^{n} a_k$ be the nth partial sum of $\sum a_k$. We say that $\sum a_k$ and $\sum b_k$ *diverge at the same rate* if

$$(1) \quad 0 < \alpha = \liminf_{n\to\infty} \frac{B_n}{A_n} \leqq \limsup_{n\to\infty} \frac{B_n}{A_n} = \beta < +\infty.$$

If $\alpha = \beta = 1$ in (1), we say the two series are *asymtotic* and write $\sum a_k \sim \sum b_k$.

In (1), (2) and (3), Diananda, assuming $na_n \to 0$, found conditions on π that guarantee $\sum a_k \sim \sum a_{\pi(k)}$. Stenberg [**7**] studied rearrangements of divergent series and the divergent subseries of these rearrangements. In [**4**], the author showed that divergent series of positive terms can be rearranged to give some predesignated rates of divergence. More precisely:

THEOREM 2. *Let $\sum a_k$ be a divergent series of positive real numbers with $a_k \to 0$. Let $f(x)$, defined for $x \geqq 0$, be a positive, strictly increasing, concave function with*

(i) $\lim_{x\to\infty} f(x) = +\infty$
(ii) $\lim_{x\to\infty} \{f(x+1) - f(x)\} = 0$
(iii) $\operatorname{limsup}_{n\to\infty} f(n)/A_n \leqq 1$.

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Then there is a permutation π *such that* $A_n^\pi \sim f(n)$ *(as* $n \to \infty$*) where*

$$A_n^\pi = \sum_{k=1}^{n} a_{\pi(k)}.$$

In view of the above it is natural to ask for a characterization of these permutations that do not affect the rate of divergence of any divergent series of positive terms. In this paper we give a combinatoric characterization of such permutations and show that this collection of permutations forms a subgroup (though not a normal subgroup) of the group of all permutations on **N**. We will assume that all series are divergent series of positive, bounded terms.

2. Results. The collection of permutations that we wish to characterize are called divergence preserving (DP) and defined as follows.

Definition 3. A permutation π on the positive integers is called *divergence preserving* if for each series $\sum a_k$, the two series $\sum a_k$ and $\sum a_{\pi(k)}$ diverge at the same rate. Let DP denote the collection of all divergence preserving permutations.

THEOREM 4. *The permutations in* DP *form a group.*

Proof. It is clear that i (the identity permutation) is in DP and that if $\pi, \rho \in$ DP, then $\pi\rho \in$ DP. Finally, if $\pi \in$ DP, then $\sum a_{\pi^{-1}(k)}$ diverges. But then $\sum a_{\pi^{-1}(k)}$ and $\sum a_{\pi\pi^{-1}(k)} = \sum a_k$ diverge at the same rate. Hence $\pi^{-1} \in$ DP.

We will shortly show that DP is not a normal subgroup of the group of all permutations on **N**. It is convenient to first establish a combinatorial condition that is necessary and sufficient for a permutation to be in DP. In what follows, I will denote an interval in **N**; that is

$$\mathrm{I} = \{a, a+1, \ldots, b\} \subseteq \mathbf{N}$$

for some $a, b \in \mathbf{N}$ $(a \leqq b)$. If S, T are subsets of **N**, then $S < T$ will mean

$$\max\{k\colon k \in S\} < \min\{k\colon k \in T\}.$$

THEOREM 5. *Let* π *be a permuation on* **N**. *Then* $\pi \in$ DP *if and only if there is a positive integer* M $(= M(\pi))$ *such that*

$$\#\{n\colon n \in I \backslash \pi(I)\} \leqq M \tag{2}$$

for every interval I *in* **N**.

Proof. (Sufficiency) Suppose π satisfies condition (2) and $\sum a_k$ is a divergent series of positive, bounded real numbers; say $a_k \leqq B$ for all k. Then for any positive integer N

$$|A_N - A_N^\pi| \leqq 2MB$$

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is bounded. Thus $A_N \sim A_N^{\pi}$, so $\sum a_k$ and $\sum a_{\pi(k)}$ certainly diverge at the same rate.

(Necessity) Suppose π does not satisfy condition (2). We will produce a series $\sum a_k$ (with, in fact, $a_k \to 0$) whose rate of divergence is different from $\sum a_{\pi^{-1}(k)}$. Since π does not satisfy (2), we can find a sequence $\{I_k\}$ of intervals in $\mathbf{N}$ with

$$(3) \quad I_k \cup \pi(I_k) < I_{k+1} \cup \pi(I_{k+1})$$

and

$$(4) \quad \#\{n: n \in (I_k \backslash \pi(I_k))\} \geqq 2(k+1)!$$

We consider two similar cases. In the first case assume that instead of (4) we actually have

$$(5) \quad \#\{n: n \in I_k, \pi(n) > I_k\} = M_k \geqq (k+1)!$$

Let $J_k = \{n \in I_k: \pi(n) > I_k\}$, so $\#J_k = M_k \geqq (k+1)!$ We then define

$$a_j = \begin{cases} \dfrac{k! - (k-1)!}{M_k} & (j \in J_k, k \geqq 2) \\ 2^{-j} & \left(j \notin \bigcup_{k=2}^{\infty} J_k\right). \end{cases}$$

Then $a_j \to 0$ as $j \to \infty$. Also, if $N_k = \max\{j: j \in I_k\}$ then

$$\sum_{j \leqq N_k} a_j \geqq k! - 1.$$

Furthermore, since $\pi(j) > N_k$ for $j \in J_n$ $(n \geqq k)$ it follows that

$$\sum_{j \leqq N_k} a_{\pi^{-1}(j)} \leqq (k-1)!$$

Thus

$$\limsup_{N \to \infty} \frac{A_N}{A_N^{\pi^{-1}}} \geqq \limsup_{k \to \infty} \frac{k! - 1}{(k-1)!} = +\infty.$$

Hence $\sum a_k$ and $\sum a_{\pi-1(k)}$ do not diverge at the same rate. Thus $\pi^{-1} \notin \mathrm{DP}$ and it follows from Theorem 4 that $\pi \notin \mathrm{DP}$.

If a sequence $\{I_k\}$ of intervals satisfying (3) and (5) is not possible, then we can find a sequence $\{\mathrm{I}_k\}$ of intervals satisfying (3) with

$$\#\{n: n \in I_k, \pi(n) < I_k\} = M_k' \geqq (k+1)!$$

Let $J_k' = \{n \in I_k: \pi(n) < I_k\}$, so $\#J_k' = M_k' \geqq (k+1)!$ We then define

$$a_j = \begin{cases} \dfrac{k! - (k-1)!}{M_k'} & (j \in J_k', k \geqq 2) \\ 2^{-j} & \left(j \notin \bigcup_{k=2}^{\infty} J_k'\right). \end{cases}$$

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Let $N_k' = \min\{j: j \in I_k\} - 1$. Then

$$\liminf_{N\to\infty} \frac{A_N}{A_N^{\pi-1}} \leqq \liminf_{k\to\infty} \frac{A_{N_k}}{A_{N_k}^{\pi-1}} \leqq \liminf_{n\to\infty} \frac{(k-1)!}{k!-1} = 0$$

again showing $\pi \notin$ DP.

As an immediate consequence of the proof of Theorem 5 we have

COROLLARY 6. *If* $\pi \in$ DP, *then* $\sum a_k \sim \sum a_{\pi(k)}$.

Remark 7. We now present an example showing that DP is not a normal subgroup of the group of all permutations on $\mathbf{N}$. Take $\pi \in$ DP with

$$\pi(n) = \begin{cases} n+1 & (n \text{ odd}) \\ n-1 & (n \text{ even}) \end{cases}$$

and let σ be defined by

$$\sigma(n) = \begin{cases} \frac{2}{3}n & (n \equiv 0 \pmod 3) \\ \frac{4n-1}{3} & (n \equiv 1 \pmod 3) \\ \frac{4n+1}{3} & (n \equiv 2 \pmod 3). \end{cases}$$

Then for any positive integer of the form $4k$ ($k \in \mathbf{N}$) we have

$$\sigma\pi\sigma^{-1}(4k) = 8k - 1.$$

But then if I_{4k} is the interval $\{1, 2, \ldots, 4k\} \subseteq \mathbf{N}$, we see that

$$\#\{n: n \in I_{4k} \backslash \sigma\pi\sigma^{-1}(I_{4k})\} \geqq k/2.$$

Hence $\sigma\pi\sigma^{-1} \notin$ DP, showing DP is not normal.

Remark 8. Any finite set S of positive integers can be written as a union of disjoint, nonadjacent intervals. Let $v(S)$ denote the number of such intervals. In [**5**], Pleasants showed that a permutation π preserves the sum of all conditionally convergent series if and only if there is a constant C $(=C(\pi))$ such that $v(\pi^{-1}(I)) \leqq C$ for all intervals I in $\mathbf{N}$. The set of such permutations is denoted by CP (for convergence preserving).

Now let $\pi \in$ DP. By Theorems 4 and 5 there is a constant $M_{\pi^{-1}}$ such that

$$\#\{n: n \in I \backslash \pi^{-1}(I)\} \leqq M_{\pi^{-1}}$$

for all intervals I in $\mathbf{N}$. It follows immediately that

$$v(\pi^{-1}(I)) \leqq 2M_{\pi^{-1}} + 1 \quad \text{for all } I.$$

Thus DP $\subset$CP, and the inclusion is proper since DP is a group but CP is not (see [**5**]). Thus there is a permutation $\rho \in$ CP\DP. This permutation will preserve the sum of all conditionally convergent series but will alter the rate of divergence of some divergent series of positive terms. Thus in one sense divergent series of positive terms are "more delicate" than conditionally convergent series.

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