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ON THE VARIATIONAL CONSTANT ASSOCIATED TO THE L_p-HARDY INEQUALITY

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Abstract

Let Ω be a domain in $\mathbb{R}^m$ with nonempty boundary. In Ward ['On essential self-adjointness, confining potentials and the L_p-Hardy inequality', PhD Thesis, NZIAS Massey University, New Zealand, 2014] and ['The essential self-adjointness of Schrödinger operators on domains with non-empty boundary', *Manuscripta Math.* **150**(3) (2016), 357–370] it was shown that the Schrödinger operator $H = -\Delta + V$, with domain of definition $D(H) = C_0^\infty(\Omega)$ and $V \in L_\infty^{\mathrm{loc}}(\Omega)$, is essentially self-adjoint provided that $V(x) \geq (1 - \mu_2(\Omega))/d(x)^2$. Here $d(x)$ is the Euclidean distance to the boundary and $\mu_2(\Omega)$ is the nonnegative constant associated to the L_2-Hardy inequality. The conditions required for a domain to admit an L_2-Hardy inequality are well known and depend intimately on the Hausdorff or Aikawa/Assouad dimension of the boundary. However, there are only a handful of domains where the value of $\mu_2(\Omega)$ is known explicitly. By obtaining upper and lower bounds on the number of cubes appearing in the kth generation of the Whitney decomposition of Ω, we derive an upper bound on $\mu_p(\Omega)$, for $p > 1$, in terms of the inner Minkowski dimension of the boundary.

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1. Introduction

The problem of the essential self-adjointness of Schrödinger operators has a long and distinguished history (see [4, 16, 18] for an overview). In [18, 19] it was shown that if Ω is a domain in $\mathbb{R}^m$ with nonempty boundary, then the Schrödinger operator $H = -\Delta + V$, with domain of definition $D(H) = C_0^\infty(\Omega)$ and $V \in L_\infty^{\mathrm{loc}}(\Omega)$, is essentially self-adjoint provided that, near to the boundary,

$$V(x) \geq \frac{1 - \mu_2(\Omega)}{d(x)^2}. \tag{1.1}$$

Furthermore, this condition can be shown to be optimal on certain geometrically simple domains (see [18, Ch. 6]). Here $d(x)$ is the Euclidean distance to the boundary and $\mu_2(\Omega)$ is the nonnegative constant associated to the L_2-Hardy inequality.

Heuristically, such a Schrödinger operator is essentially self-adjoint provided that a particle under the influence of the potential V is unable to come into contact with the boundary of the domain. Indeed, Equation (1.1) casts the problem of the essential self-adjointness of Schrödinger operators in terms of a balancing act between the quantum tunneling effect and the uncertainty principle. Suppose that the domain Ω does not admit an L_2-Hardy inequality so that $\mu_2(\Omega) = 0$. Then (1.1) implies that the potential V must inflate like $d(x)^{-2}$ in order to ensure that the *probability* of finding a particle under its influence at the boundary is zero. In other words, if the potential inflates at this rate, then this is sufficient to ensure that the particle does not tunnel through any classically forbidden region and reach the boundary.

On the other hand, suppose that $\mu_2(\Omega) > 0$, so that Ω admits an L_2-Hardy inequality. This obviously relaxes the criteria for essential self-adjointness. The physical reason for this is that the value of $\mu_2(\Omega)$ places limits on the certainty with which we can say that a particle is located at the boundary. To see this, suppose that the state of the particle is described by the (unit) wavefunction $\omega \in W^1_{2,0}(\Omega)$. An application of Hölder's inequality followed by an application of the L_2-Hardy inequality yields

$$\begin{aligned} 1 = \int_\Omega \frac{d(x)\,|\omega(x)|^2}{d(x)}\,dx &\le \Big(\int_\Omega d(x)^2\,|\omega(x)|^2\,dx\Big)^{1/2}\Big(\int_\Omega \frac{|\omega(x)|^2}{d(x)^2}\,dx\Big)^{1/2} \\ &\le \mu_2(\Omega)^{-1/2}\Big(\int_\Omega d(x)^2\,|\omega(x)|^2\,dx\Big)^{1/2}\cdot\Big(\int_\Omega |\nabla\omega(x)|^2\,dx\Big)^{1/2}. \end{aligned}$$

By squaring both sides of the last equation and rearranging, we obtain that

$$\mu_2(\Omega) \le \Big(\int_\Omega d(x)^2\,|\omega(x)|^2\,dx\Big)\cdot\Big(\int_\Omega |\nabla\omega(x)|^2\,dx\Big). \tag{1.2}$$

The second integral on the right-hand side of Equation (1.2) describes the total momentum of the particle, while the first integral gives a measure of the particle's proximity to the boundary. Hence, if $\mu_2(\Omega) > 0$, we see that one cannot confine a particle to a smaller neighborhood of the boundary without producing a corresponding increase in the particle's total momentum.

However, the condition described by Equation (1.1) is not explicit as, for a given domain Ω with nonempty boundary, we do not know the *value* of the variational constant $\mu_2(\Omega)$.

The necessary and sufficient conditions required for a domain to admit an L_p-Hardy inequality, for $p > 1$, are well known and have an intimate dependence on the dimension of the boundary, or thickness of the complement of a domain. In [10] Koskela and Zhong showed that if a domain admits an L_p-Hardy inequality, then either $\dim_{\mathcal{H}}(\partial\Omega) > m - p$ or $\dim_A(\partial\Omega) < m - p$. Here $\dim_{\mathcal{H}}(E)$ and $\dim_A(E)$ represent the Hausdorff and Aikawa/Assouad dimension of the set $E \subseteq \mathbb{R}^m$, respectively. As such, if the boundary of a domain is $(m - p)$-dimensional, then that domain cannot admit an L_p-Hardy inequality.

Furthermore, the necessary conditions described by this dimensional dichotomy are almost sufficient conditions. It is known that if $\dim_A(\partial\Omega) < m - p$, then this is

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enough to guarantee the existence of an L_p-Hardy inequality (see [12, Theorem 1.2]). On the other hand, it is also known that if $\mathbb{R}^m \backslash \Omega$ is uniformly p-fat, then Ω admits an L_p-Hardy inequality (see [14, Theorem 2] and [9, Theorem 3]) and that in this case $\dim_{\mathcal{H}}(\partial\Omega) > m - p$ (see [13, page 2194]). Indeed, there are many equivalent conditions for the uniform p-fatness of the complement of a domain (see [7, Sections 3 and 4]). In particular, in the 'borderline' case where $m = p$ the uniform p-fatness of the complement turns out to be equivalent to the unboundedness and uniform perfectness of the complement. Rather surprisingly, therefore, the condition $\dim_{\mathcal{H}}(\partial\Omega \cap B(x, r)) > m - p$ for all $x \in \partial\Omega$ and $r > 0$ is not sufficient for the existence of an L_p-Hardy inequality, as can be proven with a simple example. If $\Omega = \mathbb{R}^2 \backslash C$, where C is the middle thirds Cantor set, then despite the fact that $\dim_{\mathcal{H}}(\partial\Omega \cap B(x, r)) = \log 2/\log 3 > 0$, for all $x \in \partial\Omega$ and $r > 0$, Ω does not admit an L_2-Hardy inequality since the complement of the domain is bounded.

Whereas the necessary and sufficient conditions for a domain to admit an L_p-Hardy inequality have been extensively studied, there are only a handful of domains where the value of the associated variational constant $\mu_p(\Omega)$ is known explicitly. Given that $\mu_p(\Omega) > 0$ if and only if Ω admits an L_p-Hardy inequality, we should expect the value of this constant to share some dependence on the dimension of the boundary. Indeed, this belief is born out by example. If $\Omega = \mathbb{R}^m \backslash E$, where E is an affine set of dimension $0 \le k \le m - 1$, then $\mu_p(\Omega) = |(m - k - p)/p|^p$. Similarly, if Ω is a bounded, convex domain with smooth boundary of codimension one, then $\mu_p(\Omega) = |(m - (m - 1) - p)/p|^p = |(1 - p)/p|^p$ (see [1, 2] and [18, Section 5.2]). This suggests that if Ω is a domain in $\mathbb{R}^m$ with Ahlfors λ-regular boundary, then

$$\mu_p(\Omega) = \left| \frac{m - \lambda - p}{p} \right|^p. \tag{1.3}$$

We recall that for any set $E \subseteq \mathbb{R}^m$, $\dim_{\mathcal{H}}(E) \le \dim_A(E)$ and that if E is bounded then $\dim_{\mathcal{H}}(E) \le \underline{\dim}_M(E) \le \overline{\dim}_M(E) \le \dim_A(E)$ where the middle inequalities represent the lower and upper Minkowski dimensions of the set E. The condition that E is Ahlfors λ-regular ensures that $\dim_{\mathcal{H}}(E) = \dim_A(E)$ so that all these notions of dimension agree (see [12, Lemma 2.1]).

Indeed, whenever Equation (1.3) does hold many of the intricacies of the L_p-Hardy inequality follow immediately. For example, if (1.3) holds it is trivially true that whenever the boundary is $(m - p)$-dimensional the domain Ω does not admit an L_p-Hardy inequality. Furthermore, the continuity of the right-hand side of (1.3) in p, for $p > 0$, would also explain why a domain which admits an L_{p_0}-Hardy inequality also admits an L_p-Hardy inequality for all p sufficiently close to p_0 (see [10, Theorem 1.2]). Finally, if (1.3) holds and Ω^c is uniformly p_0-fat, then the fact that $\lambda > m - p_0$ automatically implies that Ω admits an L_p-Hardy inequality for all $p \ge p_0$ (see the remarks following [8, Theorem 3.9]).

Unfortunately, the situation is not that simple. The fact that the domain $\Omega = \mathbb{R}^2 \backslash C$ does not admit an L_2-Hardy inequality implies that Equation (1.3) does not hold in general. Nevertheless, it would be extremely interesting to produce an example of a domain with *fractal* Ahlfors λ-regular boundary upon which Equation (1.3) holds in the case $p = 2$. Then the corresponding value for $\mu_2(\Omega)$ could be put back into Equation (1.2) or Equation (1.1) to obtain a quantum mechanical result that depends on the fractal dimension of the boundary, that is, on a scale infinitely smaller than the wavelength of an electron. This would be interesting as quantum mechanics does not presuppose interactions at infinitesimally small scales.

This paper goes some way to achieving that aim. However, we are not the first to consider the relationship between the value of the variational constant $\mu_p(\Omega)$ and the dimension of the boundary. The main theorem of this paper, which is stated below, can be considered to be an extension of a result of Davies in [3].

THEOREM 1.1. *Let Ω be an inner γ-domain in $\mathbb{R}^m$ with nonempty, compact boundary. Let the inner Minkowski dimension of $\partial\Omega$ be well defined and equal to λ_M. If $\partial\Omega$ is inner Minkowski measurable so that $0 < \hat{M}_l^{\lambda_M}(\partial\Omega) \le \hat{M}_u^{\lambda_M}(\partial\Omega) < \infty$, then for all $p > 1$ we have $\mu_p(\Omega) \le |m - \lambda_M - p/p|^p$. In particular, if $\lambda_M = m - p$, then Ω does not admit an L_p-Hardy inequality.*

We have decided to structure this paper as follows. In Section 2 we present relevant definitions and notation. Here we also introduce a new concept; that of the inner γ-domain. This concept is introduced purely for technical convenience. In Section 3 we develop some new results connecting the inner Minkowski dimension of the boundary to the number of cubes appearing in the kth generation of the Whitney decomposition of a domain. In particular, we are able to obtain lower bounds for the number of such cubes in terms of the inner Minkowski content of the boundary. Next, in Section 4, we use the results of the previous section to characterize the inner Minkowski dimension of the boundary of an inner γ-domain in terms of the integrability of powers of the Euclidean distance function. The proof of Theorem 1.1 is given in Section 5. Finally, in Section 6 we construct a domain with fractal boundary that satisfies the conditions of Theorem 1.1.

2. Definitions and notation

Throughout this paper Ω denotes a domain (connected, open set) in $\mathbb{R}^m$ with nonempty boundary $\partial\Omega = \bar{\Omega}\backslash\Omega$. The Euclidean distance between two sets $A, B \subseteq \mathbb{R}^m$ is denoted by $d(A, B) = \operatorname{dist}(A, B) = \inf\{|x - y| \mid x \in A, y \in B\}$. The notation $d(x)$ is reserved to mean $\operatorname{dist}(x, \partial\Omega)$. Given a set $A \subseteq \mathbb{R}^m$, $|A|$ is its m-dimensional Lebesgue measure and $A_\delta = \{x \in \mathbb{R}^m \mid d(x, A) < \delta\}$ is a δ-neighborhood of A.

2.1. The L_p-Hardy inequality. We begin by recalling precisely what it means for a domain to admit an L_p-Hardy inequality.

DEFINITION 2.1. Let Ω be a domain in $\mathbb{R}^m$ with nonempty boundary. Ω is said to admit an L_p-Hardy inequality if there exists a finite, uniform constant $C > 0$ so that the estimate

$$\int_\Omega \frac{|\omega(x)|^p}{d(x)^p}\, dx \le C \int_\Omega |\nabla\omega(x)|^p\, dx$$

holds for all $\omega(x) \in W^1_{p,0}(\Omega)$.

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There is a natural variational problem associated to the existence of an L_p-Hardy inequality on a given domain. To make this idea precise we make the following definition.

Definition 2.2. Let Ω be a domain in $\mathbb{R}^m$ with nonempty boundary. Define the variational constant $\mu_p(\Omega)$ by

$$\mu_p(\Omega) = \inf_{\omega \in W^1_{p,0}(\Omega)} \left\{ \frac{\int_\Omega |\nabla \omega(x)|^p \, dx}{\int_\Omega \frac{|\omega(x)|^p}{d(x)^p} \, dx} \right\}.$$

Then $\mu_p(\Omega) > 0$ if and only if Ω admits an L_p-Hardy inequality.

2.2. Inner Minkowski dimension. As we have already mentioned, the necessary and sufficient conditions for a domain to admit an L_p-Hardy inequality have an intimate dependence on the dimension of the boundary. In this paper we have occasion only to consider the inner Minkowski dimension of the boundary of a domain.

Definition 2.3. Let Ω be a domain in $\mathbb{R}^m$ with nonempty, compact boundary. For $\lambda \geq 0$, the upper and lower inner λ-Minkowski content of $\partial\Omega$ are respectively given by

$$\hat{M}^\lambda_u(\partial\Omega) = \limsup_{r \to 0^+} \frac{|(\partial\Omega)_r \cap \Omega|}{r^{m-\lambda}}, \quad \hat{M}^\lambda_l(\partial\Omega) = \liminf_{r \to 0^+} \frac{|(\partial\Omega)_r \cap \Omega|}{r^{m-\lambda}}.$$

It is easy to see that there exists a unique value of $\lambda \in [0, m]$ at which the upper (lower) inner Minkowski content jumps from infinity to zero. This motivates the definition below.

Definition 2.4. Let Ω be a domain in $\mathbb{R}^m$ with nonempty, compact boundary. We define the upper and lower inner Minkowski dimension of $\partial\Omega$ respectively by

$$\overline{\dim}_{\hat{M}}(\partial\Omega) = \sup\{\lambda \geq 0 \,|\, \hat{M}^\lambda_u(\partial\Omega) = \infty\} = \inf\{\lambda \geq 0 \,|\, \hat{M}^\lambda_u(\partial\Omega) = 0\},$$
$$\underline{\dim}_{\hat{M}}(\partial\Omega) = \sup\{\lambda \geq 0 \,|\, \hat{M}^\lambda_l(\partial\Omega) = \infty\} = \inf\{\lambda \geq 0 \,|\, \hat{M}^\lambda_l(\partial\Omega) = 0\}.$$

When $\overline{\dim}_{\hat{M}}(\partial\Omega) = \underline{\dim}_{\hat{M}}(\partial\Omega)$ we say that the inner Minkowski dimension of $\partial\Omega$ is well defined, equal to this common value and denoted by $\dim_{\hat{M}}(\partial\Omega)$. If a set is known to have inner Minkowski dimension d, then we say that the set is inner Minkowski measurable if $0 < \hat{M}^d_l(\partial\Omega) \leq \hat{M}^d_u(\partial\Omega) < \infty$.

2.3. Whitney decompositions and inner γ-domains. If Ω is a domain in $\mathbb{R}^m$ with nonempty boundary, then it is always possible to decompose Ω into a collection of closed cubes $\mathcal{W} = \mathcal{W}(\Omega)$ so that $\Omega = \bigcup_{Q \in \mathcal{W}} Q$. Such a construction is known as a Whitney decomposition and has the following properties. The interiors of the cubes are pairwise disjoint, have edges parallel to the coordinate axes and have diameter proportional to the distance to the boundary, such that for each cube $Q \in \mathcal{W}$ we have $\operatorname{diam}(Q) \leq d(Q, \partial\Omega) \leq 4 \operatorname{diam}(Q)$. Furthermore, if $Q \in \mathcal{W}$, then $\operatorname{diam}(Q) \in \{\sqrt{m}2^{-k} \,|\, k \in \mathbb{Z}\}$. As such, it is natural to refer to the set of cubes

$$\mathcal{W}_k = \{Q \in \mathcal{W} \,|\, \operatorname{diam}(Q) = \sqrt{m}2^{-k}\} = \{Q \in \mathcal{W} \,|\, |Q| = 2^{-mk}\}$$

as the kth generation of cubes in the decomposition $\mathcal{W}$. We will denote an arbitrary cube in $\mathcal{W}_k$ by Q_k, and $N_k = \sharp\,\mathcal{W}_k$ will represent the number of cubes in $\mathcal{W}_k$. It is plain to see that if $x \in Q_k$ then $\sqrt{m}2^{-k} \leq d(x) \leq 5\sqrt{m}2^{-k}$, so that we have the inclusion

$$\mathcal{W}_k \subseteq \{x \in \Omega \mid \sqrt{m}2^{-k} \leq d(x) \leq 5\sqrt{m}2^{-k}\}. \tag{2.1}$$

For further information on the construction and properties of Whitney decompositions the reader is directed to [17, Ch. 1].

In the next section our objective is to establish a relationship between the inner Minkowski dimension of the boundary and the number of cubes appearing in the kth generation of the Whitney decomposition of the domain. To do so we will restrict our analysis to domains which admit the following geometric property.

Definition 2.5. Let Ω be a domain in $\mathbb{R}^m$ and let $\gamma > 1$. Ω is said to be an inner γ-domain if there exist a finite uniform constant $C > 0$ and $\hat{r} > 0$ so that the estimate

$$|(\partial\Omega)_r \cap \Omega| \leq C\,|[(\partial\Omega)_{\gamma r} \backslash (\partial\Omega)_r] \cap \Omega|$$

holds for all $0 < r < \hat{r}$. We call the value of $\hat{r}$ the upper radial limit of the domain.

While a full characterization of γ-domains is beyond the remit of the current paper, preliminary investigation suggests that domains admitting the interior cone condition and John domains are γ-domains.

3. Whitney decompositions and inner Minkowski content

Loosely speaking, the bigger the dimension of $\partial\Omega$ the bigger the volume of the region $\{x \in \Omega \mid \sqrt{m}2^{-k} \leq d(x) \leq 5\sqrt{m}2^{-k}\}$, so that more cubes in $\mathcal{W}_k$ are required to fill this space. This intuition is reflected in the next two results. Indeed, the following lemma was originally proven by Martio and Vuorinen in [15] before being stated in its present context by Edmunds and Evans [5].

Lemma 3.1 [5, Lemma 4.3.7]. *Let Ω be a domain in $\mathbb{R}^m$ with nonempty, compact boundary. Let $\mathcal{W}$ be a Whitney decomposition of Ω and $N_k = \sharp\,\mathcal{W}_k$. If $\hat{M}^{\lambda}_u(\partial\Omega) < \infty$, then there exist $K_1 > 0$ and $k_1 \in \mathbb{N}$ so that for all integers $k \geq k_1$,*

$$N_k \leq K_1 2^{\lambda k}.$$

The above result bounds the number of cubes appearing in $\mathcal{W}_k$ from above in terms of λ when $\hat{M}^{\lambda}_u(\partial\Omega) < \infty$. The next lemma bounds the number of cubes appearing in sequential generations of the decomposition from below in terms of λ, under the assumption that $\hat{M}^{\lambda}_l(\partial\Omega) > 0$.

Lemma 3.2. *Let Ω be an inner γ-domain in $\mathbb{R}^m$ with nonempty, compact boundary. Let $\mathcal{W}$ be a Whitney decomposition of Ω and $N_k = \sharp\,\mathcal{W}_k$. If $\hat{M}^{\lambda}_l(\partial\Omega) > 0$, then there exist $K_2 > 0$ and $k_2 \in \mathbb{N}$ so that for all integers $k \geq k_2$,*

$$\sum_{j=k-n(\gamma)}^{k+2} N_j \geq K_2 2^{\lambda k}.$$

Here $n(\gamma)$ is a nonnegative integer that varies proportionally with γ.

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Proof. Since $\hat{M}_l^\lambda(\partial\Omega) = \liminf_{r\to 0^+} (|(\partial\Omega)_r \cap \Omega|/r^{m-\lambda}) > 0$ there exist $\kappa > 0$ and $r_0 > 0$ so that $|(\partial\Omega)_r \cap \Omega| \geq \kappa r^{m-\lambda}$ for all $r \leq r_0$. Without loss of generality we may assume that r_0 is less than the upper radial limit of the domain. Set $k_2 = \lceil \ln(\sqrt{m}/r_0)/\ln 2 \rceil$ and consider any integer $k \geq k_2$. We will prove that the desired inequality is true for the chosen value of k. First, set $r = \sqrt{m}/2^k$ so that $r \leq r_0$. Since Ω is an inner γ-domain, and since $r \leq r_0$ is less than the upper radial limit of Ω, we have that

$$|(\partial\Omega)_r \cap \Omega| \leq C|[(\partial\Omega)_{\gamma r} \backslash (\partial\Omega)_r] \cap \Omega|. \tag{3.1}$$

Now, from Equation (2.1) we have the inclusion $[(\partial\Omega)_{\gamma r} \backslash (\partial\Omega)_r] \cap \Omega \subseteq \bigcup_{j=k-n(\gamma)}^{k+2} \mathcal{W}_j$, where $n(\gamma)$ is an appropriately chosen nonnegative integer. The bigger γ is the bigger the value of $n(\gamma)$ as more generations of the decomposition are needed to obtain the desired inclusion. Continuing from Equation (3.1),

$$\begin{aligned} |(\partial\Omega)_r \cap \Omega| &\leq C\Big|\bigcup_{j=k-n(\gamma)}^{k+2} \mathcal{W}_j\Big| \leq C \sum_{j=k-n(\gamma)}^{k+2} N_j\,|Q_j| \\ &\leq \frac{C}{2^{m(k-n(\gamma))}} \sum_{j=k-n(\gamma)}^{k+2} N_j, \end{aligned}$$

and since $r \leq r_0$ we have

$$\sum_{j=k-n(\gamma)}^{k+2} N_j \geq C2^{mk}\,|(\partial\Omega)_r \cap \Omega| \geq C2^{mk}\kappa r^{m-\lambda} = K_2\, 2^{\lambda k}.$$

This completes the proof. □

4. Inner Minkowski dimension and the integrability of the distance to the boundary

In this section we characterize the inner Minkowski dimension of the boundary of an inner γ-domain in terms of the integrability of powers of the distance function. Although results of a similar nature have been obtained in [11, 20, 21], we include the following theorem within this paper as the proof of Theorem 1.1 essentially hinges upon exploiting a dimensional dichotomy concerning the integrability of the Euclidean distance function.

Theorem 4.1. *Let Ω be an inner γ-domain in $\mathbb{R}^m$ with nonempty, compact boundary whose inner Minkowski dimension is well defined. Then for any $\delta > 0$,*

$$\begin{aligned} \dim_{\hat{\mathcal{M}}}(\partial\Omega) &= \sup\Big\{\lambda \geq 0 \,\Big|\, \int_{(\partial\Omega)_\delta \cap \Omega} \frac{1}{d(x)^{m-\lambda}}\,dx = \infty\Big\} \\ &= \inf\Big\{\lambda \geq 0 \,\Big|\, \int_{(\partial\Omega)_\delta \cap \Omega} \frac{1}{d(x)^{m-\lambda}}\,dx < \infty\Big\}. \end{aligned}$$

Proof. Let $\mathcal{W}$ be a Whitney decomposition of Ω and let $N_k = \sharp\mathcal{W}_k$. For notational convenience we will set $\dim_{\hat{\mathcal{M}}}(\partial\Omega) = \lambda_M$. First, let us suppose that $\lambda > \lambda_M$ and let us choose some $\lambda_1 \in (\lambda_M, \lambda)$. Consequently, it must be the case that $\hat{M}_u^{\lambda_1}(\partial\Omega) < \infty$.

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By Lemma 3.1, there must exist some $K_1 > 0$ and $k_1 \in \mathbb{N}$ so that for all integers $k \geq k_1$ we have $N_k \leq K_1 2^{\lambda_1 k}$. Now let us partition the region $(\partial\Omega)_\delta \cap \Omega$ in the following way:

$$(\partial\Omega)_\delta \cap \Omega \subseteq \bigcup_{k=k_1}^{\infty} \mathcal{W}_k \cup \Big[[(\partial\Omega)_\delta \cap \Omega] \Big\backslash \bigcup_{k=k_1}^{\infty} \mathcal{W}_k\Big] \equiv \bigcup_{k=k_1}^{\infty} \mathcal{W}_k \cup \mathcal{B}.$$

We note that the region $\mathcal{B}$ may be empty if δ is chosen sufficiently small, and that on this region $d(x) \geq (5/8)(\sqrt{m}/2^{k_1})$. Furthermore, since $\partial\Omega$ is compact it must also be the case that $|(\partial\Omega)_\delta \cap \Omega| < \infty$. Therefore, we have that

$$\begin{aligned}\int_{(\partial\Omega)_\delta\cap\Omega} \frac{1}{d(x)^{m-\lambda}}\,dx &\leq \int_{\cup_{k=k_1}^{\infty}\mathcal{W}_k} \frac{1}{d(x)^{m-\lambda}}\,dx + \int_{\mathcal{B}} \frac{1}{d(x)^{m-\lambda}}\,dx \\ &\leq \int_{\cup_{k=k_1}^{\infty}\mathcal{W}_k} \frac{1}{d(x)^{m-\lambda}}\,dx + C. \end{aligned} \tag{4.1}$$

Utilizing the standard properties of Whitney decompositions expressed in Section 2.3, we can estimate the integral term in (4.1) in the following way:

$$\begin{aligned}\int_{\cup_{k=k_1}^{\infty}\mathcal{W}_k} \frac{dx}{d(x)^{m-\lambda}} &\leq \sum_{k=k_1}^{\infty} \sum_{Q_k\in\mathcal{W}_k} \int_{Q_k} \frac{dx}{d(x)^{m-\lambda}} \leq \sum_{k=k_1}^{\infty} \frac{CN_k 2^{(m-\lambda)k}}{2^{km}} \\ &\leq CK_1 \sum_{k=k_1}^{\infty} 2^{\lambda_1 k}\, 2^{-\lambda k} \leq CK_1 \sum_{k=k_1}^{\infty} \Big(\frac{1}{2^{\lambda-\lambda_1}}\Big)^k < \infty.\end{aligned}$$

We conclude that if $\lambda > \lambda_M$, then $\int_{(\partial\Omega)_\delta\cap\Omega} (1/d(x)^{m-\lambda})\,dx$ is finite.

Now we assume that $\lambda < \lambda_M$ and choose $\lambda_2 \in (\lambda, \lambda_M)$. As such we have that $\hat{M}_l^{\lambda_2}(\partial\Omega) > 0$. By Lemma 3.2, there must exist some $K_2 > 0$ and $k_2 \in \mathbb{N}$ so that for all integers $k \geq k_2$ we have the estimate

$$\sum_{j=k-n(\gamma)}^{k+2} N_j \geq K_2\, 2^{\lambda_2 k}. \tag{4.2}$$

Fix $k_3 > \max\{\lceil \ln(5\sqrt{m}/\delta)/\ln 2\rceil, k_2\}$ so that $\bigcup_{j=k_3}^{\infty} \mathcal{W}_j \subseteq (\partial\Omega)_\delta \cap \Omega$. Using Equation (4.2) and the properties of Whitney decompositions, we obtain the desired result in the following manner:

$$\begin{aligned}\int_{(\partial\Omega)_\delta\cap\Omega} \frac{1}{d(x)^{m-\lambda}}\,dx &\geq \sum_{j=k_3}^{\infty} \int_{\mathcal{W}_j} \frac{1}{d(x)^{m-\lambda}}\,dx \\ &= \sum_{j=k_3}^{\infty} \sum_{Q_j\in\mathcal{W}_j} \int_{Q_j} \frac{1}{d(x)^{m-\lambda}}\,dx \geq C(m) \sum_{j=k_3}^{\infty} N_j \frac{2^{(m-\lambda)j}}{2^m j} \\ &= C(m)\Big[\sum_{j=k_3}^{k_3+n(\gamma)+2} N_j 2^{-\lambda j} + \sum_{j=k_3+n(\gamma)+3}^{k_3+2n(\gamma)+5} N_j 2^{-\lambda j} + \cdots\Big]\end{aligned}$$

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$$
\begin{aligned}
&= C(m)\sum_{N=0}^{\infty}\Bigg(\sum_{j=k_3+Nn(\gamma)+3N}^{k_3+(N+1)n(\gamma)+3N+2} N_j 2^{-\lambda j}\Bigg)\\
&\geq C(m)\sum_{N=0}^{\infty}\Bigg(2^{-\lambda[k_3+(N+1)n(\gamma)+3N+2]}\sum_{j=k_3+Nn(\gamma)+3N}^{k_3+(N+1)\,n(\gamma)+3N+2} N_j\Bigg)\\
&\geq C(m)K_2\sum_{N=0}^{\infty}2^{-\lambda[k_3+(N+1)n(\gamma)+3N+2]}2^{\lambda_2[k_3+(N+1)n(\gamma)+3N]}\\
&\geq C(m)K_2 2^{-2\lambda}\sum_{N=0}^{\infty}2^{(\lambda_2-\lambda)[k_3+(N+1)n(\gamma)+3N]}.
\end{aligned}
$$

Since $\lambda_2 > \lambda$ this last series is divergent. The proof is now complete. □

5. Proof of main theorem

In [1], Barbatis *et al.* investigate the optimality of the constants appearing in their 'improved' L_p-Hardy inequalities. The following lemma is a simplification of results found in Section 5 of that paper. A short proof of the lemma can be found in Section 5.1 of [18].

LEMMA 5.1 ([1, Section 5]; see also [18, Lemma 5.1.1]). *Let Ω be a domain in $\mathbb{R}^m$ with nonempty boundary. Let $\phi(x) \in W^1_{\infty,0}(\Omega)$ be a real-valued function taking values in the interval* $[0, 1]$. *Then the inequality*

$$
\mu_p(\Omega) \leq \left|\frac{\alpha-p}{p}\right|^p + \left(\int_\Omega \frac{\phi^p(x)}{d(x)^\alpha}\,dx\right)^{-1} \cdot C_p\left[\left|\frac{\alpha-p}{p}\right|^{p-1}\int_\Omega \frac{|\nabla\phi(x)|}{d(x)^{\alpha-1}}\,dx + \int_\Omega \frac{|\nabla\phi(x)|^p}{d(x)^{\alpha-p}}\,dx\right]
$$

holds for all $\alpha \geq 0$ and $p > 1$. Here C_p is an absolute constant depending only on p.

We are now in a position to give a proof of Theorem 1.1.

PROOF OF THEOREM 1.1. Let $\mathcal{W}$ be a Whitney decomposition of Ω and let $N_k = \sharp\,\mathcal{W}_k$. For each $k \in \mathbb{N}$ define the smooth function $\hat{\phi}_k : \mathbb{R} \to [0, 1]$ by

$$
\hat{\phi}_k(t) = \begin{cases}
0 & \text{if } t \leq \dfrac{5/8\sqrt{m}}{2^k},\\
\sigma_k(t) & \text{if } \dfrac{5/8\sqrt{m}}{2^k} < t < \dfrac{\sqrt{m}}{2^k},\\
1 & \text{if } \dfrac{\sqrt{m}}{2^k} \leq t \leq \dfrac{5\sqrt{m}}{2},\\
\mu(t) & \text{if } \dfrac{5\sqrt{m}}{2} < t < \dfrac{8\sqrt{m}}{2},\\
0 & \text{if } t \geq \dfrac{8\sqrt{m}}{2},
\end{cases}
$$

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where the monotone functions $\sigma_k(t)$ and $\mu(t)$ are such that $|\sigma_k'(t)| \leq C_1 2^k$ and $|\mu'(t)| \leq C_2$. Here C_1 and C_2 are absolute constants independent of k. For each $k \in \mathbb{N}$ define the function $\phi_k : \Omega \to \mathbb{R}$ by $\phi_k(x) = \hat{\phi}_k(d(x))$ so that the sequence $\{\phi_k(x)\}_{k=1}^{\infty}$ belongs to $W^1_{\infty,0}(\Omega)$ and each $\phi_k(x)$ takes values in the interval $[0, 1]$. Therefore, if we set $\alpha = m - \lambda_M$, then it follows immediately from Lemma 5.1 that

$$\mu_p(\Omega) \leq \left|\frac{m-\lambda_M-p}{p}\right|^p + \left(\int_\Omega \frac{\phi_k^p(x)}{d(x)^{m-\lambda_M}}\,dx\right)^{-1} \cdot C_p\left[\left|\frac{m-\lambda_M-p}{p}\right|^{p-1} \int_\Omega \frac{|\nabla\phi_k(x)|}{d(x)^{m-\lambda_M-1}}\,dx + \int_\Omega \frac{|\nabla\phi_k(x)|^p}{d(x)^{m-\lambda_M-p}}\,dx\right]. \tag{5.1}$$

To prove the theorem we will show that in the limit as $k \to \infty$ the numerator in (5.1) stays bounded while the denominator tends to infinity. To do so it will be useful to define the following regions:

$$\mathcal{A}_k = \left\{x \in \Omega \,\middle|\, \frac{5/8\sqrt{m}}{2^k} < d(x) < \frac{\sqrt{m}}{2^k}\right\} \subseteq \mathcal{W}_{k+1} \cup \mathcal{W}_{k+2}$$
$$\mathcal{B} = \left\{x \in \Omega \,\middle|\, \frac{5\sqrt{m}}{2} < d(x) < \frac{8\sqrt{m}}{2}\right\}.$$

Since $\hat{M}_u^{\lambda_M}(\partial\Omega) < \infty$, by Lemma 3.1 there must exist some $K_1 > 0$ and $j_1 \in \mathbb{N}$ so that for all integers $j \geq j_1$ we have

$$N_j \leq K_1 2^{\lambda_M j}. \tag{5.2}$$

Similarly, given that $0 < \hat{M}_l^{\lambda_M}(\partial\Omega)$, the result of Lemma 3.2 implies that there exist $K_2 > 0$ and $j_2 \in \mathbb{N}$ so that for all integers $j \geq j_2$,

$$\sum_{s=j-n(\gamma)}^{j+2} N_s \geq K_2 2^{\lambda_M j}, \tag{5.3}$$

where $n(\gamma)$ is an appropriately chosen natural number. Without loss of generality we may assume that $k \geq \max\{j_1, j_2\}$ and $j_2 \geq 1$.

We now investigate the behavior of the first integral term in the numerator of expression (5.1) as $k \to \infty$. Indeed,

$$\int_\Omega \frac{|\nabla\phi_k(x)|}{d(x)^{m-\lambda_M-1}}\,dx = \int_{\mathcal{A}_k} \frac{|\nabla\sigma_k(d(x))|}{d(x)^{m-\lambda_M-1}}\,dx + \int_{\mathcal{B}} \frac{|\nabla\mu(d(x))|}{d(x)^{m-\lambda_M-1}}\,dx$$
$$\leq \int_{\mathcal{A}_k} \frac{|\nabla\sigma_k(d(x))|}{d(x)^{m-\lambda_M-1}}\,dx + C. \tag{5.4}$$

Simply by using Equation (5.2), and the standard properties of Whitney decompositions expressed in Section 2.3, we can show that the remaining integral term in Equation (5.4) remains bounded above as $k \to \infty$ in the following way:

$$\begin{aligned}
\int_{\mathcal{A}_k} \frac{|\nabla \sigma_k(d(x))|}{d(x)^{m-\lambda_M-1}}\,dx &\le C_1 2^k \Big[\int_{\mathcal{W}_{k+1}} \frac{1}{d(x)^{m-\lambda_M-1}}\,dx + \int_{\mathcal{W}_{k+2}} \frac{1}{d(x)^{m-\lambda_M-1}}\,dx \Big] \\
&= C_1 2^k \Big[\sum_{Q_{k+1}\in\mathcal{W}_{k+1}} \int_{Q_{k+1}} \frac{dx}{d(x)^{m-\lambda_M-1}} + \sum_{Q_{k+2}\in\mathcal{W}_{k+2}} \int_{Q_{k+2}} \frac{dx}{d(x)^{m-\lambda_M-1}} \Big] \\
&\le C(m) 2^k \Big[N_{k+1} \frac{2^{(m-\lambda_M-1)k}}{2^{(k+1)m}} + N_{k+2} \frac{2^{(m-\lambda_M-1)k}}{2^{(k+2)m}} \Big] \\
&\le C(m) K_1 2^k [2^{\lambda_M(k+1)} 2^{-\lambda_M k} 2^{-k} + 2^{\lambda_M(k+2)} 2^{-\lambda_M k} 2^{-k}] \\
&\le C(m) K_1 2^{2\lambda_M} \le M_1.
\end{aligned}$$

A similar argument shows that the second term in the numerator of (5.1) also remains bounded above as $k \to \infty$. Therefore, to complete the proof it remains only to show that the denominator in expression (5.1) tends to infinity as $k \to \infty$. Using the fact that $\phi_k(x) = 1$ over the region $\bigcup_{j=1}^k \mathcal{W}_j$, we are led to the following set of inequalities:

$$\begin{aligned}
\int_\Omega \frac{\phi_k^p(x)}{d(x)^{m-\lambda_M}}\,dx &\ge \sum_{j=1}^k \int_{\mathcal{W}_j} \frac{dx}{d(x)^{m-\lambda_M}} \ge \sum_{j=j_2}^k \int_{\mathcal{W}_j} \frac{dx}{d(x)^{m-\lambda_M}} \\
&\ge \sum_{j=j_2}^k \sum_{Q_j\in\mathcal{W}_j} \int_{Q_j} \frac{1}{d(x)^{m-\lambda_M}}\,dx \ge C(m) \sum_{j=j_2}^k N_j \frac{2^{(m-\lambda_M)j}}{2^{mj}} \\
&\ge C(m) \Big[\sum_{j=j_2}^{j_2+n(\gamma)+2} N_j 2^{-\lambda_M j} + \sum_{j=j_2+n(\gamma)+3}^{j_2+2n(\gamma)+5} N_j 2^{-\lambda_M j} + \cdots \Big]. \qquad (5.5)
\end{aligned}$$

Now let $T(k) = \lfloor k - j_2 + 1/n(\gamma) + 3 \rfloor$, so that $T(k)$ denotes the number of 'complete' summation terms appearing in Equation (5.5). We note that $T(k) \to \infty$ as $k \to \infty$. Taking into account Equation (5.3), the bracketed terms within Equation (5.5) can be bounded from below as follows:

$$\begin{aligned}
&\sum_{j=j_2}^{j_2+n(\gamma)+2} N_j 2^{-\lambda_M j} + \sum_{j=j_2+n(\gamma)+3}^{j_2+2n(\gamma)+5} N_j 2^{-\lambda_M j} + \cdots \\
&\ge 2^{-\lambda_M(j_2+n(\gamma)+2)} \sum_{j=j_2}^{j_2+n(\gamma)+2} N_j + 2^{-\lambda_M(j_2+2n(\gamma)+5)} \sum_{j=j_2+n(\gamma)+3}^{j_2+2n(\gamma)+5} N_j + \cdots \\
&= K_2[2^{-2\lambda_M} + 2^{-2\lambda_M} + \cdots] = K_2\, 2^{-2\lambda_M} T(k).
\end{aligned}$$

Since $T(k) \to \infty$ as $k \to \infty$ the proof is complete. □

6. Construction of a fractal, inner Minkowski measurable, inner γ-domain

In this section, we produce an example of a fractal domain that satisfies the hypotheses of Theorem 1.1. Following the procedure outlined by Evans and Harris in [6, Section 6], we will construct a 'room and corridor' type domain that

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has well-defined inner Minkowski dimension and which is also inner Minkowski measurable. Furthermore, we will go on to show that this domain is also an inner γ-domain for $\gamma = 2$. We begin by choosing some $C \in (0, \frac{1}{2})$ and $\mu > 1$ so that $C^{\mu} + 2C^2 < 1$. Next we define the monotone decreasing sequences of real numbers $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ by the rules $\alpha_n = C^{\mu n}$ and $\beta_n = C^n$ for $n \geq 1$, where $\alpha_0 > \alpha_1, \beta_0 > \beta_1$ and $\beta_0 > \alpha_0$. We will also assume that the constants α_0 and β_0 have been chosen sufficiently large so that

$$4\beta_0 + 2\alpha_0 - 15/(1 - 2C) > 0. \tag{6.1}$$

It is easily checked that these sequences satisfy the following conditions:

(i) $\alpha_n < \beta_n$ for all $n = 0, 1, \ldots$.
(ii) There exists $t_1 > 0$ so that $0 < t_1 \leq \alpha_{n+1}/\alpha_n < 1$.
(iii) There exists $t_2 \in (0, 1)$ so that $0 < \beta_{n+1}/\beta_n \leq t_2 < 1$.
(iv) $2\beta_{n+2} < \beta_n - \alpha_{n+1}$.

Now let Δ_0 consist of a rectangle $Q_{0,1}$ with edge lengths $2\alpha_0 \times 2\beta_0$, and let Δ_1 consist of a single rectangle $Q_{1,1}$ with edge lengths $2\alpha_1 \times 2\beta_1$. Attach a short edge of $Q_{1,1}$ to the middle portion of a long edge of $Q_{0,1}$. Similarly, for all $n \in \mathbb{N}$, let Δ_n consist of 2^{n-1} rectangles, denoted by $Q_{n,j}$, with edge lengths $2\alpha_n \times 2\beta_n$. For each $j = 1, \ldots, 2^{n-1}$ attach a short edge of $Q_{n,j}$ to the middle portion of a long edge of a rectangle in Δ_{n-1}. Finally, let

$$\Omega = \Big(Q_{0,1} \bigcup \Big[\bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{n-1}} Q_{n,j}\Big]\Big)^{o}$$

so that Ω is the 'room and corridor' type domain depicted in Figure 1. From condition (iv), the fact that $2\beta_{n+2} < \beta_n - \alpha_{n+1}$ ensures that the interiors of all the rectangles are disjoint. For instance, looking at Figure 1, in order to prevent the rectangle $Q_{3,1}$ from intersecting the rectangle $Q_{0,1}$ we require that $2\beta_3 < l = \beta_1 - \alpha_2$. Furthermore, this condition also ensures that $|\Omega| < \infty$, since Ω can evidently be contained in the rectangle with edge lengths $2\beta_0 \times (2\alpha_0 + 2\beta_1)$.

In order to proceed, let us note that since the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are monotone decreasing, then for all sufficiently small $\delta > 0$ there must exist some integers $M, \tilde{M}, N, \tilde{N} \in \mathbb{N}$ so that

$$\alpha_{M+1} < \delta \leq \alpha_M, \quad \beta_{N+1} < \delta \leq \beta_N,$$
$$\alpha_{\tilde{M}+1} < 2\delta \leq \alpha_{\tilde{M}}, \quad \beta_{\tilde{N}+1} < 2\delta \leq \beta_{\tilde{N}}.$$

Since δ can be taken to be arbitrarily small, we can always assume that we have the estimates $M + 1 \leq N$ and $\tilde{M} + 1 \leq \tilde{N}$. Furthermore, with the aid of some elementary algebra, it can be shown that the integer $M - \tilde{M} \in \{0, 1\}$ so that $\tilde{M} + 1 \geq M$. Calculating the volume of an inner δ-neighborhood of the boundary yields the equation

$$\begin{aligned}|(\partial\Omega)_\delta \cap \Omega| &= [4(\beta_0 + \alpha_0) - 2\alpha_1]\delta - (4 - \pi/2)\delta^2 \\ &\quad + \sum_{k=1}^{M} 2^{k-1}([4(\beta_k - \alpha_{k+1}) + 2\alpha_k]\delta + o(\delta^2)) + 2\sum_{k=M+1}^{\infty} 2^k \alpha_k \beta_k. \end{aligned} \tag{6.2}$$

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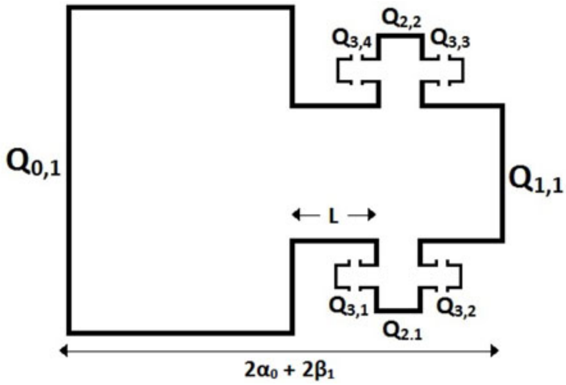

FIGURE 1. The 'room and corridor' type domain formed using the construction of Evans and Harris in [6] has inner Minkowski dimension equal to 1 and is inner Minkowski measurable. Moreover, the domain is also a γ-domain for $\gamma = 2$.

Here $o(\delta^2)$ is a positive constant with the property that $o(\delta^2) \asymp (\pi - 2)\delta^2$. Similarly, if one considers a 2δ-neighborhood of the boundary, then the following equation can easily be derived:

$$\begin{aligned} |(\partial\Omega)_{2\delta} \cap \Omega| &= 2[4(\beta_0 + \alpha_0) - 2\alpha_1]\delta - 4(4 - \pi/2)\delta^2 \\ &\quad + \sum_{k=1}^{\tilde{M}} 2^{k-1}(2[4(\beta_k - \alpha_{k+1}) + 2\alpha_k]\delta + 4o(\delta^2)) + 2\sum_{k=\tilde{M}+1}^{\infty} 2^k \alpha_k \beta_k. \end{aligned} \tag{6.3}$$

Our first task is to bound $|(\partial\Omega)_{2\delta} \cap \Omega|$ from below. Indeed, continuing from Equation (6.3), we obtain

$$\begin{aligned} |(\partial\Omega)_{2\delta} \cap \Omega| &\geq 2[4(\beta_0 + \alpha_0) - 2\alpha_1]\delta - 4(4 - \pi/2)\delta^2 + 4o(\delta^2)\sum_{k=1}^{\tilde{M}} 2^{k-1} \\ &\geq 2[4(\beta_0 + \alpha_0) - 2\alpha_1]\delta - 4(4 - \pi/2)\delta^2 + 2o(\delta^2)(2^M - 2), \end{aligned} \tag{6.4}$$

whereupon it follows that

$$\hat{M}_l^1(\partial\Omega) = \liminf_{d\to 0^+} \frac{|(\partial\Omega)_{2\delta} \cap \Omega|}{2\delta} \geq 4(\beta_0 + \alpha_0) - 2\alpha_1.$$

We can immediately conclude that $\hat{M}_l^1(\partial\Omega) > 0$ and $\underline{\dim}_{\hat{M}}(\partial\Omega) \geq 1$. Now let us estimate $|(\partial\Omega)_\delta \cap \Omega|$ from above. Following on from Equation (6.2), we have

$$\begin{aligned} |(\partial\Omega)_\delta \cap \Omega| &\leq [4(\beta_0 + \alpha_0) - 2\alpha_1]\delta + 3\delta\sum_{k=0}^{\infty} 2^k\beta_k + o(\delta^2)\sum_{k=1}^{M} 2^{k-1} + 2\delta\sum_{k=0}^{\infty} 2^k\beta_k \\ &= [4(\beta_0 + \alpha_0) - 2\alpha_1]\delta + \frac{5\delta}{1 - 2C} + o(\delta^2)(2^M - 1). \end{aligned} \tag{6.5}$$

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It follows that $\hat{M}^1_u(\partial\Omega)$ is finite and, therefore, that $\overline{\dim}_{\hat{M}}(\partial\Omega) \le 1$ since

$$\hat{M}^1_u(\partial\Omega) = \limsup_{d\to 0^+} \frac{|(\partial\Omega)_\delta \cap \Omega|}{\delta} \le 4(\beta_0 + \alpha_0) - 2\alpha_1 + \frac{5}{1-2C} < \infty.$$

As such we have succeeded in showing that $\dim_{\hat{M}}(\partial\Omega) = 1$ and that the boundary of the domain is inner Minkowski measurable given that $0 < \hat{M}^1_l(\partial\Omega) \le \hat{M}^1_u(\partial\Omega) < \infty$. We now wish to demonstrate that Ω is a γ-domain for $\gamma = 2$. In order to do this we must show that there exist a finite, uniform constant $K > 0$ and $\delta_0 > 0$ so that the estimate

$$|(\partial\Omega)_\delta \cap \Omega| \le K[|(\partial\Omega)_{2\delta} \cap \Omega| - |(\partial\Omega)_\delta \cap \Omega|]$$

holds for all $0 < \delta \le \delta_0$. We assert that the constant $K = 2$ is sufficient for this purpose. In other words, we will show that for sufficiently small δ,

$$3|(\partial\Omega)_\delta \cap \Omega| \le 2|(\partial\Omega)_{2\delta} \cap \Omega|. \tag{6.6}$$

From Equation (6.5) we have that

$$3|(\partial\Omega)_\delta \cap \Omega| \le 3[4(\beta_0 + \alpha_0) - 2\alpha_1]\delta + \frac{15\delta}{1-2C} + 3o(\delta^2)(2^M - 1),$$

while from Equation (6.4) it is also true that

$$2|(\partial\Omega)_{2\delta} \cap \Omega| \ge 4[4(\beta_0 + \alpha_0) - 2\alpha_1]\delta - 8(4 - \pi/2)\delta^2 + 4o(\delta^2)(2^M - 2).$$

The validity of Equation (6.6) then follows from the fact that it is always possible to find $\delta_0 > 0$ so that the expression

$$[4(\beta_0 + \alpha_0) - 2\alpha_1 - 15/(1-2C)]\delta - 8(4 - \pi/2)\delta^2 + (2^M - 5)o(\delta^2)$$

is positive for all $0 < \delta \le \delta_0$, given the assumption of Equation (6.1). In conclusion, we have shown that the constructed domain Ω satisfies all the requirements of Theorem 4.1 so that the following corollary is immediate.

Corollary 6.1. *Let Ω be the 'room and corridor' type domain described above and let $(\partial\Omega)_\delta$ be a tubular neighborhood of the boundary. Then the integral*

$$\int_{(\partial\Omega)_\delta \cap \Omega} \frac{1}{d(x)^{2-\lambda}}\, dx < \infty \quad \textit{if and only if } \lambda > 1.$$

Furthermore, Ω satisfies all the requirements of Theorem 1.1. Since the complement of the domain is unbounded and uniformly perfect (as a consequence of being connected), Ω must admit an L_2-Hardy inequality. Hence, $\mathbb{R}^m \backslash \Omega$ must be uniformly p-fat for all $p \ge 2$, so that we arrive at the following result.

Corollary 6.2. *Let Ω be the 'room and corridor' type domain described above. Then Ω admits an L_p-Hardy inequality for all $p \ge 2$ and the estimate $\mu_p(\Omega) \in (0, |1 - p/p|^p]$ holds. In particular, $\mu_2(\Omega) \in (0, 1/4]$.*

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References

[1] G. Barbatis, S. Filippas and A. Tertikas, 'A unified approach to improved L^p Hardy inequalities with best constants', *Trans. Amer. Math. Soc.* **356**(6) (2003), 2169–2196.

[2] G. Barbatis, S. Filippas and A. Tertikas, 'Series expansions for L^p Hardy inequalities', *Indiana Univ. Math. J.* **52**(1) (2003), 171–190.

[3] E. B. Davies, *A Review of Hardy Inequalities*, The Maz'ya Anniversary Collection (Birkhäuser, Basel, 1999).

[4] M. S. P. Eastham, W. D. Evans and J. B. Mcleod, 'The essential self-adjointness of Schrödinger type operators', *Arch. Ration. Mech. Anal.* **60**(2) (1976), 185–204.

[5] D. E. Edmunds and W. D. Evans, *Hardy Operators, Function Spaces and Embeddings* (Springer, Berlin, 2004).

[6] W. D. Evans and D. J. Harris, 'Fractals, trees and the Neumann laplacian', *Math. Ann.* **296** (1993), 493–527.

[7] J. Kinnunen and R. Korte, 'Characterizations for Hardys inequality', in: *Around the Research of Vladimir Mazya I* (Springer, New York, 2010), 239–254.

[8] J. Kinnunen and O. Martio, 'Hardy's inequalities for Sobolev functions', *Math. Res. Lett.* **4** (1997), 489–500.

[9] R. Korte, J. Lehrbäck and H. Tuominen, 'The equivalence between pointwise Hardy inequalities and uniform fatness', *Math. Ann.* **351**(3) (2011), 711–731.

[10] P. Koskela and X. Zhong, 'Hardy's inequality and the boundary size', *Proc. Amer. Math. Soc.* **131**(4) (2002), 1151–1158.

[11] M. L. Lapidus, J. A. Rock and D. Zubrinić, 'Box-counting fractal strings, zeta functions, and equivalent forms of Minkowski dimension', arXiv:1207.6681, 2012.

[12] J. Lehrbäck, 'Weighted Hardy inequalities and the size of the boundary', *Manuscripta Math.* **127** (2008), 249–273.

[13] J. Lehrbäck, 'Pointwise Hardy inequalities and uniformly fat sets', *Proc. Amer. Math. Soc.* **136**(6) (2008), 2193–2200.

[14] J. L. Lewis, 'Uniformly fat sets', *Trans. Amer. Math. Soc.* **308**(1) (1988), 177–196.

[15] O. Martio and M. Vuorinen, 'Whitney cubes, p-capacity and Minkowski content', *Exp. Math.* **5** (1987), 17–40.

[16] M. C. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. 2 Fourier Analysis and Self-Adjointness* (Academic Press, New York, 1975).

[17] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions* (Princeton University Press, Princeton, NJ, 1970).

[18] A. D. Ward, 'On essential self-adjointness, confining potentials and the L_p-Hardy inequality', PhD Thesis, NZIAS Massey University, New Zealand, 2014.

[19] A. D. Ward, 'The essential self-adjointness of Schrödinger operators on domains with non-empty boundary', *Manuscripta Math.* **150**(3) (2016), 357–370.

[20] D. Zubrinić, 'Minkowski content and singular integrals', *Chaos, Solitons and Fractals* **17** (2003), 169–177.

[21] D. Zubrinić, 'Analysis of Minkowski contents of fractal sets and applications', *Real Anal. Exchange* **31**(2) (2005), 315–354.

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