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ON ADEQUATE LINKS AND HOMOGENEOUS LINKS

SANG YOUL LEE, CHAN-YOUNG PARK AND MYOUNGSOO SEO

In this paper, we give several inequalities concerning the genus and the degree of the Jones polynomial of an adequate link and of a homogeneous link and their applications.

1. INTRODUCTION

The Tait conjectures for alternating links have been proved to be true in several papers [**5, 11, 12, 19, 8**] by the aid of the Jones polynomial invariant and its generalisations. Since then many authors have generalised some results on alternating links to new classes of links [**2, 13, 7, 14, 17, 18**]. For examples, Lickorish and Thistlethwaite [**7**] introduced a certain class of links, called adequate links, that properly contains all alternating links and found some properties of the Kauffman bracket polynomials of such links, and generalised the Tait first and second conjectures to this class ([**20**]). Also, Cromwell [**2**] introduced the class of homogeneous links which contains all alternating links and positive links and gave inequalities for the degree of the skein polynomial of a homogeneous link and tried to classify homogeneous links and their diagrams.

The purpose of this paper is to prove some inequalities for adequate links and for homogeneous links and give some applications. Section 2 contains preliminaries. In Section 3, we give some inequalities concerning the lowest and highest degrees of the Jones polynomials of adequate links. In Section 4, we generalise the Bennequin's inequality ([**1**]) for the genus, writhe, and the number of Seifert circles of a braid diagram of a knot to that of a diagram of a homogeneous link. As applications we show that any adequate diagram of a positive adequate link is a positive minimal genus diagram and any minimal crossing diagram of a positive adequate link is a positive diagram.

Unless otherwise stated, all links are tame in $\mathbb{R}^3$ or $S^3 = \mathbb{R}^3 \cup \infty$ and links and their diagrams under consideration are assumed to be oriented and non-split.

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2. Preliminaries

Let D be a diagram of a link L and let $c^+(D)$ and $c^-(D)$ denote the number of positive and negative crossings of D, respectively. Then the *crossing number* $c(D)$ and the *writhe* $w(D)$ of D are defined to be $c(D) = c^+(D)+c^-(D)$ and $w(D) = c^+(D)-c^-(D)$, respectively. Let $c(L)$ denote the minimal crossing number of L over all diagrams of L.

The nullifying rules at each crossing of a link diagram as shown in Figure 2.1 are called the *rule P* and the *rule N*.

the rule P the rule N

Figure 2.1

The Kauffman bracket polynomial ([4]) of a diagram D is an element $\langle D\rangle \in \mathbb{Z}[A^{\pm 1}]$ defined as follows. Let $c_1, c_2, \ldots, c_n$ denote the crossings of D. A *state* for D is a function $s : \{1, 2, \ldots, n\} \to \{-1, 1\}$. For a given state s for D, let sD denote the diagram obtained from D by nullifying all crossing $c_i (i = 1, 2, \ldots, n)$ of D according to the rule P if $s(i) = 1$ and the rule N if $s(i) = -1$. We call it the *state diagram* corresponding to the state s for D. Let $\mathcal{S}(D)$ denote the set of all states for D. Note that each state diagram sD consists of disjoint simple closed curves. Let $|sD|$ denote the number of all components of sD. Then

$$\langle D\rangle = \sum_{s\in\mathcal{S}(D)} A^{\sum s(i)}(-A^{-2} - A^2)^{|sD|-1}.$$

If D is a diagram of a link L with writhe $w(D)$, then the Jones polynomial $V_L(t)$ of L is given by

$$V_L(t) = (-t^{-3/4})^{-w(D)}\langle D\rangle|_A = t^{-1/4}.$$

Let $\min\deg V_L$ and $\max\deg V_L$ denote the lowest and the highest degrees of $V_L(t)$, respectively.

A *spanning surface* for a link L is an oriented and connected surface F in S^3 such that F spans the link L and the orientation on F induces the given orientation on L. The *genus* $g(L)$ of a link L is the least genus of all spanning surfaces for L ([3]). Let D be a diagram of a link L. Then it is well known that a spanning surface for L can be constructed from D by using the Seifert's algorithm ([16]). This spanning surface is called the *Seifert surface* associated to the diagram D. The genus of the Seifert surface associated to D is denoted by $g(D)$. So $g(L) \leqslant g(D)$. A diagram D of a link L is called *a minimal genus diagram* if the associated Seifert surface is a minimal genus spanning surface for L, that is, $g(D) = g(L)$.

Let D be a diagram of a link. Nullifying each crossing of D as shown in Figure 2.2, we have a collection of disjoint simple closed curves. These curves are called *Seifert circles* of D and let $s(D)$ denote the number of all Seifert circles of D. Then for a diagram D of a link with μ components, we have the well known formula for the genus $g(D)$ of the Seifert surface associated to D:

$$g(D) = \frac{c(D) - s(D) + 2 - \mu}{2} \tag{2.1}$$

Figure 2.2

3. Inequalities for adequate links

Let s_+ be the state for which $s_+(i) = 1$ for all i and let s_- be the state for which $s_-(i) = -1$ for all i. A link diagram D is said to be $+$ *adequate* if, when s_+D is created from D by nullifying each crossing according to the rule P, the two segments of the image part of the rule P belong to different components of s_+D. Similarly, a link diagram D is said to be $-$ *adequate* if, when s_-D is created from D by nullifying each crossing according to the rule N, the two segments of the image part of the rule N belong to different components of s_-D. A link diagram is said to be *adequate* if it is both $+$ adequate and $-$ adequate and a link is called an *adequate link* if it admits at least one adequate diagram.

Theorem 3.1. ([7]) *Let D be an adequate diagram with $c(D)$ crossings of an adequate link. Then the terms of the lowest degree and the highest degree in $\langle D\rangle$ are*

$$(-1)^{|s_-D|-1}A^{-c(D)-2|s_-D|+2}, \quad (-1)^{|s_+D|-1}A^{c(D)+2|s_+D|-2}.$$

That is,

$$\min\deg\langle D\rangle = -c(D) - 2|s_-D| + 2, \tag{3.2}$$

$$\max\deg\langle D\rangle = c(D) + 2|s_+D| - 2. \tag{3.3}$$

Theorem 3.2. *Let D be an adequate diagram of an adequate link.*

(1) *If $c^-(D) \geqslant 1$, then*

$$|s_+D| \geqslant s(D) - c^-(D) + 2. \tag{3.4}$$

(2) *If $c^+(D) \geqslant 1$, then*

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(3.5) $$|s_-D| \geqslant s(D) - c^+(D) + 2.$$

PROOF: Let $\{c_1, c_2, \dots, c_m\}$ be the set of all negative crossings of D. Put $s_0D = s_+D$ and for each $i = 1, \dots, m$, let s_iD be the state diagram obtained from D by nullifying the negative crossings $c_1, \dots, c_i$ according to the rule N and the other crossings of D according to the rule P. Then the difference of $s_{i-1}D$ and s_iD appears at the negative crossing c_i only and the components of s_mD are the Seifert circles of D. So, $|s_iD| - |s_{i-1}D| = \pm 1$ and $|s_mD| = s(D)$. Since D is an adequate diagram, the two segments in the image part of the rule P belong to different components of s_+D. So, $|s_1D| - |s_+D| + 1 = 0$. Thus we have

$$\begin{aligned} c^-(D) = m &= \sum_{i=1}^{m} \big||s_iD| - |s_{i-1}D|\big| \\ &= \big||s_1D| - |s_0D|\big| + \sum_{i=2}^{m} \big||s_iD| - |s_{i-1}D|\big| \\ &\geqslant 1 + \left|\sum_{i=2}^{m}\big(|s_iD| - |s_{i-1}D|\big)\right| \\ &= 1 + \big||s_mD| - |s_1D|\big| \\ &\geqslant 1 + |s_mD| - |s_1D| \\ &= 1 + |s_mD| - |s_1D| + |s_1D| - |s_+D| + 1 \\ &= |s_mD| - |s_+D| + 2 \\ &= s(D) - |s_+D| + 2. \end{aligned}$$

Hence $|s_+D| \geqslant s(D) - c^-(D) + 2$.

By the similar argument we also obtain the inequality (3.5). ∎

A diagram of a link is said to be *positive(negative)* if $c^-(D) = 0(c^+(D) = 0$, respectively) and a link is called a *positive(negative) link* if it has a positive(negative, respectively) diagram.

COROLLARY 3.3. *Let K be a non-alternating adequate genus one knot which is neither positive nor negative. Then K has no adequate minimal genus diagram.*

PROOF: Let D be an adequate diagram of K. Then D is neither positive nor negative. So, $c^-(D) \geqslant 1$ and $c^+(D) \geqslant 1$. Thus the equality (2.1) and the inequalities (3.4) and (3.5) imply that

$$c(D) - 4g(D) - 2\mu + 8 \leqslant |s_+D| + |s_-D|.$$

Now suppose that the knot K has an adequate minimal genus diagram D. Then $c(D) + 2 \leqslant |s_+D| + |s_-D|$. It is well known that every genus one knot is a prime knot and that $|s_+D| + |s_-D| < c(D) + 2$ for any connected non alternating prime diagram D. This is a contradiction. ∎

It is known that the Seifert surface associated to a positive diagram is a minimal genus spanning surface [**2**]. Using this fact and some properties of the Jones polynomial, Stoimenow ([**18**]) gave the following equality for the lowest degree of the Jones polynomial of a positive link of μ components:

$$\min\deg V_L = g(L) + \frac{\mu - 1}{2}. \tag{3.6}$$

For an alternating diagram D of an alternating link L with μ components, Stoimenow[**18**] gave the inequality:

$$\min\deg V_L \leqslant g(L) + \frac{\mu - 1}{2} - c^-(D). \tag{3.7}$$

For an adequate diagram of an adequate link, we have

THEOREM 3.4. *Let D be an adequate diagram of an adequate link L of μ components such that $c(D) \neq |w(D)|$. Then*

$$\min\deg V_L \leqslant g(D) + \frac{\mu - 1}{2} - c^-(D) - 1, \tag{3.8}$$

$$\max\deg V_L \geqslant -g(D) - \frac{\mu - 1}{2} + c^+(D) + 1. \tag{3.9}$$

Furthermore, $\min\deg V_L = g(D) + (\mu - 1)/2$ *if* $c^-(D) = 0$ *and* $\min\deg V_{L^*} = g(D) + (\mu - 1)/2$ *if* $c^+(D) = 0$.

PROOF: Suppose that $c^-(D) \geqslant 1$. From the equation (2.1), Theorem 3.1 and Theorem 3.2, we have

$$\begin{aligned}
\min\deg V_L(t) &= \frac{3}{4}w(D) + \min\deg\big(\langle D\rangle|_{A=t^{-1/4}}\big)\\
&= \frac{3}{4}w(D) - \frac{1}{4}\max\deg\langle D\rangle\\
&= \frac{3}{4}\big(c(D) - 2c^-(D)\big) - \frac{1}{4}\big(c(D) + 2|s_+D| - 2\big)\\
&= \frac{1}{2}c(D) - \frac{1}{2}|s_+D| - \frac{3}{2}c^-(D) + \frac{1}{2}\\
&\leqslant \frac{1}{2}c(D) - \frac{1}{2}(s(D) - c^-(D) + 2) - \frac{3}{2}c^-(D) + \frac{1}{2}\\
&= \frac{c(D) - s(D) + 1}{2} - c^-(D) - 1\\
&= g(D) + \frac{\mu - 1}{2} - c^-(D) - 1.
\end{aligned}$$

If $c^-(D) = 0$, then D is a positive diagram and so $g(D) = g(L)$, the result follows from the equation (3.6). By the similar argument, we also obtain the inequality (3.9). ∎

Let $\sigma(L)$ denote the signature of a link L ([**10**]).

Theorem 3.5. *Let L be an adequate link with μ components which is neither positive nor negative. If D is an adequate diagram of L, then*

$$\left|\max\deg V_L + \min\deg V_L - w(D) + \frac{1}{2}\sigma(L)\right| \leqslant g(D) + \frac{\mu-1}{2} - 1. \tag{3.10}$$

In particular, if L has an adequate minimal genus diagram, then

$$\left|\max\deg V_L + \min\deg V_L - w(D) + \frac{1}{2}\sigma(L)\right| \leqslant g(L) + \frac{\mu-1}{2} - 1. \tag{3.11}$$

Proof: From [**13**, Theorem 13.3] and Theorem 3.4, we have

$$-c^-(D) - \frac{1}{2}\sigma(L) \leqslant \min\deg V_L \leqslant g(D) + \frac{\mu-1}{2} - c^-(D) - 1, \tag{3.12}$$

$$-g(D) - \frac{\mu-1}{2} + c^+(D) + 1 \leqslant \max\deg V_L \leqslant c^+(D) - \frac{1}{2}\sigma(L). \tag{3.13}$$

Adding two inequalities (3.12) and (3.13), we obtain the inequality (3.10). If L has an adequate minimal genus diagram D, then $g(D) = g(L)$. By (3.10), we obtain the inequality (3.11). ∎

Corollary 3.6. *Let K be an adequate knot of genus one which is neither positive nor negative. If K has an adequate minimal genus diagram D, then*

$$\max\deg V_K + \min\deg V_K = w(D) - \frac{1}{2}\sigma(K). \tag{3.14}$$

If K is an alternating knot which has an adequate minimal genus diagram, then $\sigma(K) = 0$.

Proof: Let D be an adequate minimal genus diagram of K. By (3.11), we get (3.14). If K is an alternating knot, then $\max\deg V_K + \min\deg V_K = w(D) - \sigma(K)$ for any reduced alternating diagram D of K ([**12**]). Since $w(D)$ is an invariant of K over all adequate diagrams of L, it follows from (3.14) that $\sigma(K) = 0$. ∎

4. Inequalities for homogeneous links

Let D be a diagram of a knot K, then

$$|w(D)| + 1 \leqslant s(D) + 2g(K). \tag{4.15}$$

This inequality (4.15) was first proved by Bennequin [**1**] for braid diagrams and then it was improved by Stoimenow [**17**] for knot diagrams by combining the Vogel's result and the Bennequin's inequality. In general, it does not hold for links. The Hopf link which is a positive link as shown in Figure 3.1 is a counterexample.

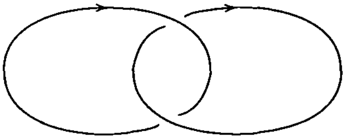


Figure 3.1

In [2], Cromwell defined the class of homogeneous links which contains all alternating links and positive links as extreme cases. We give a generalisation of (4.15) for the class of homogeneous links.

THEOREM 4.1. *If D is a diagram of a homogeneous link L with μ components, then*

$$|w(D)| + 1 \leqslant s(D) + 2g(L) + \mu - 1.$$

In particular, the equality holds if and only if D is a positive diagram.

PROOF: Let L be a link with μ components and let D be a diagram of L. By Theorem 1 in [9], $w(D) - s(D) + 1 \leqslant \min \deg_v P_L(v, z)$, where $\min \deg_v P_L(v, z)$ denotes the lowest degree of the variable v in the skein polynomial $P_L(v, z) = \sum_k a_k(z) v^k$ of L.

Now let $\chi(L)$ denote the maximal Euler characteristic over all orientable surfaces spanning L and $\chi(D)$ the Euler characteristic of the Seifert surface associated to D. Then $\chi(L) \geqslant \chi(D)$ and $\chi(L) = 2 - 2g(L) - \mu$. By Theorem 4 in [2], for a homogeneous link L,

$$\min \deg_v P_L(v, z) \leqslant 1 - \chi(L).$$

So

$$\begin{aligned} w(D) - s(D) + 1 &\leqslant \min \deg_v P_L(v, z) \\ &\leqslant 1 - \chi(L) \\ &= 1 - \big(2 - 2g(L) - \mu\big) \\ &= 2g(L) + \mu - 1. \end{aligned}$$

Now let D^* denote the mirror image of the diagram D. Then it is clear that D^* is a diagram of the mirror image L^* of L which is a homogeneous link. By the same argument as above, $w(D^*) + 1 \leqslant s(D^*) + 2g(L^*) + \mu - 1$. But $w(D^*) = -w(D), s(D^*) = s(D)$, and $g(L^*) = g(L)$. This implies the result. ∎

COROLLARY 4.2. *If D is a diagram of a homogeneous link L, then*

$$g(D) - c^-(D) \leqslant g(L) \leqslant g(D).$$

PROOF: By definition $g(L) \leqslant g(D)$. Recall that $w(D) = c(D) - 2c^-(D)$ and $2g(D) = c(D) - s(D) + 2 - \mu$. By Theorem 4.1, $c(D) - 2c^-(D) + 1 \leqslant s(D) + 2g(L) + \mu - 1$. So

$$g(D) - c^-(D) = \frac{c(D) - s(D) + 2 - \mu}{2} - c^-(D) \leqslant g(L). \qquad ∎$$

A link is called an *positive adequate link* if it is both an adequate link and a positive link.

The diagrams of the knots 10_{154} and the mirror image 10^*_{152} of the knot 10_{152} in the Rolfsen's table [**15**] are adequate, positive and non-alternating. The diagram of the knot 10_{162} in the Rolfsen's table [**15**] is a positive, non-adequate and minimal crossing diagram. In fact, the knot 10_{162} is not adequate. So, the class of all adequate links doesn't contain all positive links. The diagrams of the knot 10_{153} in the Rolfsen's table [**15**] is adequate. It is known that the knot 10_{153} is non-homogeneous ([**2**]). In general, adequate links aren't contained in the class of homogeneous links. Note that alternating links and positive links are homogeneous links.

As an analogue of the Tait first conjecture, Stoimenow [**17**] asked the following question: "Does a positive alternating knot always have a (simultaneously) positive (and) alternating diagram?" [**17**, Question 8.5]. The positive answer for this question were given implicitly in Lickorish's Theorem 5.13 in [**6**] in more general fashion. Recently, Nakamura [**14**] and Stoimenow ([**18**]) independently gave another proof of the fact that any reduced alternating diagram of a positive alternating link is a positive diagram and any minimal crossing diagram of a positive alternating link is positive. The following Theorem 4.3 gives an alternative proof for more generalised form.

Theorem 4.3. *Let L be a positive adequate link. Then*

(1) *Any adequate diagram of L is a positive diagram.*

(2) *Any minimal crossing diagram of L is a positive diagram.*

(3) *The Seifert surface associated to an adequate diagram of L is a spanning surface of minimal genus.*

Proof: (1) Let D be an adequate diagram of a positive adequate link L. Suppose that D is not a positive diagram. Then $c^-(D) \geqslant 1$. Since L is a positive link, by (3.6),

$$\min\deg V_L = g(L) + \frac{\mu - 1}{2}.$$

Since D is an adequate diagram and $c^-(D) \geqslant 1$, by (3.8) of Theorem 3.4,

$$\min\deg V_L \leqslant g(D) + \frac{\mu - 1}{2} - c^-(D) - 1.$$

Hence

$$g(L) \leqslant g(D) - c^-(D) - 1.$$

By Corollary 4.2,

$$g(D) - c^-(D) \leqslant g(L) \leqslant g(D) - c^-(D) - 1.$$

This is a contradiction. Therefore D must be a positive diagram.

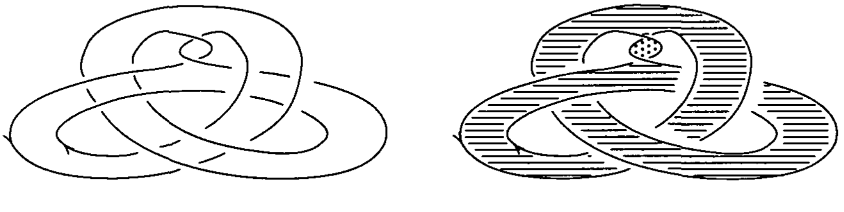
(2) Let D be a minimal crossing diagram of a positive adequate link. Since a minimal crossing diagram of an adequate link is adequate, D is adequate. So, by (1), D is a positive diagram.

(3) Let D be an adequate diagram of a positive adequate link. Then, by (1), D is a positive diagram. Since the Seifert surface associated to a positive diagram is a spanning surface of minimal genus [**2**], the result follows. ∎

Finally, we give an interesting example of a knot which has an adequate diagram whose Seifert surface isn't of minimal genus. So the positivity in Theorem 4.3 is essential.

EXAMPLE 4.4. Let D_T be the diagram as depicted in Figure 4.1 and let L_T be the knot represented by D_T. Then D_T is an adequate diagram and hence L_T is an adequate knot. Observe that $c(D_T) = 14$, $s(D_T) = 9$. From (2.1), $g(D_T) = 3$. But $g(L_T) = 1$ because the spanning surface F for L_T as depicted in Figure 4.1 is of genus 1. So the Seifert surface associated to an adequate diagram of an adequate knot is not a spanning surface of minimal genus in general. Moreover, it follows from Theorem 4.3 that L_T is not a positive knot.

On the other hand, by Theorem 2 in [**9**], for any diagram D of L, $\max\deg_z P_L(v,z) \leqslant c(D) - s(D) + 1$, where $\max\deg_z P_L(v,z)$ denotes the greatest degree of the variable z in the skein polynomial $P_L(v,z) = \sum_r b_r(v)z^r$ of L. From (2.1), $\max\deg_z P_{L_T}(v,z) \leqslant 2g(D)$ for any diagram D of L_T. Since $P_{L_T}(v,z) = (4v^{-2} - 8 + 7v^2 - 2v^4 - v^6 + v^8) + (4v^{-4} + 4v^{-2} - 11 + 10v^2 - 2v^4 - 2v^6)z^2 + (-7v^{-2} + 1 + 6v^2)z^4 + (v^2 - 1)z^6$, $\max\deg_z P_{L_T}(v,z) = 6$ and hence $3 \leqslant g(D)$. But $g(L_T) = 1$. So, any diagram of L_T cannot be a minimal genus diagram.



D_T F

Figure 4.1

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Department of Mathematics
Pusan National University
Pusan 609-735
Korea
e-mail: sangyoul@hyowon.pusan.ac.kr

Department of Mathematics
Kyungpook National University
Taegu 702-701
Korea
e-mail: chnypark@knu.ac.kr

Department of Mathematics
Kyungpook National University
Taegu 702-701
Korea
e-mail: myseo@kebi.com