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BLOW-UP OF SMOOTH SOLUTIONS TO THE NAVIER–STOKES EQUATIONS OF COMPRESSIBLE VISCOUS HEAT-CONDUCTING FLUIDS

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Abstract

We give a simpler and refined proof of some blow-up results of smooth solutions to the Cauchy problem for the Navier–Stokes equations of compressible, viscous and heat-conducting fluids in arbitrary space dimensions. Our main results reveal that smooth solutions with compactly supported initial density will blow up in finite time, and that if the initial density decays at infinity in space, then there is no global solution for which the velocity decays as the reciprocal of the elapsed time.

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1. Introduction

The compressible Navier–Stokes equations which govern the motion of a compressible viscous and heat-conducting fluid, representing the conservation of mass, the balance of momentum and energy, for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ $(n \geq 1)$, are as follows:

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.1}$$

$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \operatorname{div} \mathbb{S}, \tag{1.2}$$

$$(\rho E)_t + \operatorname{div}(\rho \mathbf{u} E + p\mathbf{u}) - k\Delta\theta = \operatorname{div}(\mathbf{u}\mathbb{S}), \tag{1.3}$$

where the unknown functions $\rho = \rho(t, x)$, $\mathbf{u} = (u^1, \ldots, u^n)$, p and θ denote the density, velocity, pressure and absolute temperature of the fluid, respectively. We write E for $\frac{1}{2}|\mathbf{u}|^2 + e$, the total energy per unit mass, and e for the specific internal energy per unit mass. The viscous stress tensor $\mathbb{S}$ is given by the Newtonian viscosity formula

$$\mathbb{S} = \mu(\nabla \mathbf{u} + \nabla^t \mathbf{u}) + \lambda(\operatorname{div} \mathbf{u})\mathbb{I}, \tag{1.4}$$

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where the constant viscosity coefficients satisfy

$$\mu > 0, \quad \lambda + \frac{2}{n}\mu \geq 0, \tag{1.5}$$

$\mathbb{I}$ is the identity matrix and $\nabla^t \mathbf{u}$ is the transpose of $\nabla \mathbf{u}$. We denote by $k \geq 0$ the constant heat conductivity coefficient.

In this paper we will consider only polytropic ideal fluids, that is, the equations of state for the fluid are given by

$$p = R\rho\theta, \quad e = c_v\theta, \quad p = Ae^{S/C_v}\rho^\gamma, \tag{1.6}$$

where $R > 0$, $A > 0$ are the fluid constants, $\gamma > 1$ the ratio of specific heat, $c_v = R/(\gamma - 1)$ the specific heat at constant volume and S is the entropy. Through the algebraic relations (1.6), it is convenient to denote the solution of (1.1)–(1.3) by the triplet $(\rho, \mathbf{u}, S)$. We complement the system (1.1)–(1.3), (1.6) with the initial data

$$(\rho, u, S)(t = 0, x) \equiv (\rho_0(x), \mathbf{u}_0(x), S_0(x)) \in H^m(\mathbb{R}^n), \tag{1.7}$$

where $m > \lfloor \frac{n}{2} \rfloor + 4$. We refer to the initial value problem (1.1)–(1.3), (1.6), (1.7) as (NS).

When the initial data are sufficiently close to a constant state, Matsumura and Nishida [2] proved that there exists a global smooth solution to (NS) in $H^m(\mathbb{R}^n)$ and that the velocity decays as $(1 + t)^{-\beta}$, $\beta = \beta(n)$. The essential point in [2] is that under the assumptions the initial density is bounded far away from vacuum. Hence, a natural question to ask is whether we can still obtain the global existence of small solutions when there is a vacuum initially. Such a problem was considered by Xin [4], Cho and Jin [1] for the case where the initial density has compact support. In particular, Xin [4] proved that smooth solutions to (NS) with $k = 0$ will blow up in finite time. The key points of his proof are the lower bound of the entropy and the time decay of total pressure, but his proof seems hard to apply for the case $k > 0$. For the case $k > 0$, but in a different way by estimating the quantity $\int_{\mathbb{R}^n} \rho(t, x)|x|^2\, dx$, Cho and Jin [1] also showed that with the initial data compactly supported, the life span of smooth solutions is finite. In this paper, we shall give a simpler and refined proof, by estimating the time evolution of the quantity $\int_{\mathbb{R}^n} \rho(t, x)x_i\, dx$, to obtain the blow-up phenomena for smooth solutions to (NS) with $k \geq 0$ and the initial data compactly supported; the precise results are stated in Theorem 1.1. Theorem 1.1 implies in particular that (NS) is not well posed in the presence of a vacuum, but compared with [2] it raises another interesting question: whether we can obtain the global smooth solutions when there is no vacuum but the density is not bounded away from vacuum. We discuss this in Theorem 1.2.

We will consider the solutions in the space $C^1([0, T]; H^m(\mathbb{R}^n))$ for $T > 0$, so if the initial density $\rho_0(x)$ has compact support we can deduce from the mass equation (1.1) that the density $\rho(t, x)$ always has compact support in space, thus

$$R(t) \equiv \inf\{r \mid \text{supp}\ \rho(t, x) \subseteq B_r\}$$

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is well defined and finite for all $t \in [0, T]$. Here we denote by $B_r = B_r(0)$ the ball in $\mathbb{R}^n$ centered at the origin with radius r. Our analysis is based on some average quantities, similarly to those in Sideris [3], Xin [4], Cho and Jin [1], given by

$$\begin{cases} m(t) = \int_{\mathbb{R}^n} \rho(t, x)\, dx & \text{‘total mass’}, \\ E^i(t) = \int_{\mathbb{R}^n} \rho(t, x) x_i \, dx & \text{‘}i\text{th component of expectation’}, \\ M(t) = \int_{\mathbb{R}^n} \rho(t, x)|x|\, dx & \text{‘first moment’}, \\ A^i(t) = \int_{\mathbb{R}^n} \rho(t, x) u^i(t, x)\, dx & \text{‘}i\text{th component of momentum’}. \end{cases} \tag{1.8}$$

We are now in a position to state the main results of this paper.

THEOREM 1.1. *Let $T > 0$, assume that μ, λ, k satisfy*

$$\mu > 0, \quad \lambda + \frac{2}{n}\mu \geq 0, \quad k \geq 0, \tag{1.9}$$

and let $(\rho, \mathbf{u}, S)(t, x) \in C^1([0, T]; H^m(\mathbb{R}^n))$ be a solution to (NS). We also suppose that the initial density $\rho_0(x)$ has compact support so that there exists a positive constant R_0 such that

$$\operatorname{supp} \rho_0(x) \subseteq B_{R_0},$$

and assume that

$$A^l(0) \neq 0 \quad \textit{for some } l \in \{1, 2, \ldots, n\}. \tag{1.10}$$

If either (i) the support of the density grows sublinearly in time, that is, there exist constants C_1 $(C_1 > 0)$ and α $(0 \leq \alpha < 1)$, independent of T, such that

$$R(t) \leq C_1(1 + t)^{\alpha}, \quad \forall t \in [0, T], \tag{1.11}$$

or (ii)

$$\lambda + \frac{2}{n}\mu > 0, \tag{1.12}$$

then the life span T of the solution $(\rho, \mathbf{u}, S)$ is finite.

The key step in the proof of Theorem 1.1 is to establish inequalities (2.5) and (2.6). These two inequalities may suggest that one expect the life span of the smooth solution $T = \infty$ for the case where the density is positive on the whole space, the positive result is obtained in Matsumura and Nishida [2] for the density bounded far away from vacuum. However, if the initial density decays at infinity in the sense that $M(0) < \infty$ (this implies that the initial density is not bounded away from the vacuum but is allowed to be positive everywhere), then there are no global solutions for which $\mathbf{u}$ decays as the reciprocal of the time as time increases.

THEOREM 1.2. *Let* $(\rho, \mathbf{u}, S)(t, x) \in C^1([0, T]; H^m(\mathbb{R}^n))$ *be a solution to (NS), and assume that*

$$A^l(0) \neq 0 \quad \text{for some } l \text{ and } M(0) < \infty. \tag{1.13}$$

Then there are no global solutions with $T = \infty$ *such that*

$$\limsup_{t\to\infty} \left\| \frac{t}{1+|x|}\mathbf{u}(t, x) \right\|_{L^\infty(\mathbb{R}^n)} < 1. \tag{1.14}$$

The rest of this paper is devoted to proving our results. Our proof is based on fairly elementary calculus, and the key point is to control the lower bound of the time evolution of the 'ith component of expectation' $E^i(t)$ or the 'first moment' $M(t)$ by their initial value and the initial momentum. Finally, some generalizations will be stated as remarks.

2. Proof of theorems

2.1. Proof of Theorem 1.1. We first prove part (i). For this, in view of the mass equation (1.1) and the assumptions of the theorem, we can simply compute

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^n} \rho(t, x)x_i\,dx &= \int_{\mathbb{R}^n} \rho_t(t, x)x_i\,dx = -\int_{\mathbb{R}^n} \operatorname{div}(\rho\mathbf{u})x_i\,dx \\ &= \int_{\mathbb{R}^n} \rho(t, x)u^i(t, x)\,dx. \end{aligned}$$

Thus by the definitions in (1.8), we obtain

$$\frac{d}{dt}E^i(t) = A^i(t). \tag{2.1}$$

Integrating (1.1) and (1.2) respectively over $\mathbb{R}^n$, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^n} \rho(t, x)\,dx = 0 = \frac{d}{dt}\int_{\mathbb{R}^n} \rho(t, x)u^i(t, x)\,dx.$$

Thus

$$m(t) = m(0), \qquad A^i(t) = A^i(0). \tag{2.2}$$

Then integrating (2.1) directly over $(0, t)$ and using (2.2), we deduce that

$$E^i(t) = E^i(0) + \int_0^t A^i(s)\,ds = E^i(0) + A^i(0)t. \tag{2.3}$$

On the other hand, by (1.11) we can estimate $E^i(t)$ as

$$\begin{aligned} |E^i(t)| &= \left| \int_{\mathbb{R}^n} \rho(t, x)x_i\,dx \right| \\ &= \left| \int_{R(t)} \rho(t, x)x_i\,dx \right| \le m(t)R(t) \le m(0)C_1(1+t)^\alpha. \end{aligned} \tag{2.4}$$

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Therefore, (2.3) and (2.4), together with the triangle inequality, yield

$$|A^i(0)|t - |E^i(0)| \le |E^i(0) + A^i(0)t| = |E^i(t)| \le m(0)C_1(1+t)^\alpha,$$

hence

$$|A^i(0)|t \le |E^i(0)| + m(0)C_1(1+t)^\alpha \quad \forall i = 1, 2, \ldots, n \text{ and } t > 0. \tag{2.5}$$

Consequently, (2.5), (1.10) and the inequalities $C_1 > 0$ and $0 \le \alpha < 1$ together yield that the life span T of the solution $(\rho, \mathbf{u}, S)$ should be finite. More precisely,

$$T \le \min\{t_l \mid A^l(0) \ne 0, l \in \{1, 2, \ldots, n\}\},$$

where t_l is the maximum time satisfying (2.5) for i replaced by l with l satisfying (1.10). Here we have used the fact that $|E^i(0)|$, $m(0)$ are finite since the initial density has compact support. This proves part (i).

To prove part (ii), we only have to prove that the support of the density will not grow in time. Indeed, we shall prove that

$$R(t) = R(0) = \inf\{r \mid \operatorname{supp} \rho_0 \subseteq B_r\};$$

thus (ii) follows from (i) by taking (1.11) with $C_1 = R(0)$, $\alpha = 0$. And the relationship (2.5) between the size of initial density and the life span of the solutions becomes

$$|A^i(0)|t \le |E^i(0)| + m(0)R(0) \quad \forall i = 1, 2, \ldots, n \text{ and } t > 0. \tag{2.6}$$

To this end, we denote by $\mathbf{X}(t; x_0)$ the particle path starting at x_0 when $t = 0$, that is,

$$\frac{d}{dt}\mathbf{X}(t; x_0) = \mathbf{u}(t, \mathbf{X}(t; x_0)), \qquad \mathbf{X}(t = 0;\ x_0) = x_0,$$

and set

$$\Omega(0) = \operatorname{supp} \rho_0(x), \qquad \Omega(t) = \{x = \mathbf{X}(t; x_0) \mid x_0 \in \Omega(0)\}.$$

It follows from the mass equation (1.1) that the density is simply transported along particle paths, so that

$$\operatorname{supp}_x \rho(t, x) \subseteq \Omega(t).$$

This means that

$$p(t, x) = \theta(t, x) = 0 \quad \text{if } x \in \Omega(t)^c.$$

This can be deduced from the equations of state (1.6), $\gamma > 1$ and the assumption that $(\rho, \mathbf{u}, S)(t, x) \in C^1([0, T]; H^m(\mathbb{R}^n))$, together with the Sobolev embedding theorem $H^m(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$; see Ziemer [5]. Therefore we can deduce from (1.2) and (1.3) that

$$\operatorname{div} \mathbb{S} = \operatorname{div}(\mathbf{u}\mathbb{S}) = 0 \quad \text{if } x \in \Omega(t)^c.$$

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Meanwhile we can compute

$$\operatorname{div}(\mathbf{u}\mathbb{S}) - \mathbf{u}\operatorname{div}\mathbb{S} = 2\mu\sum_{i=1}^{n}(\partial_i u^i)^2 + \lambda(\operatorname{div}\mathbf{u})^2 + \mu\sum_{i\neq j}^{n}(\partial_i u^j)^2 + 2\mu\sum_{i>j}^{n}(\partial_i u^j)(\partial_j u^i). \tag{2.7}$$

If $\lambda \le 0$, then

$$\begin{aligned} 0 &\ge (2\mu + n\lambda)\sum_{i=1}^{n}(\partial_i u^i)^2 + \mu\sum_{i\neq j}^{n}(\partial_i u^j)^2 + 2\mu\sum_{i>j}^{n}(\partial_i u^j)(\partial_j u^i) \\ &= (2\mu + n\lambda)\sum_{i=1}^{n}(\partial_i u^i)^2 + \mu\sum_{i>j}^{n}(\partial_i u^j + \partial_j u^i)^2. \end{aligned}$$

If $\lambda > 0$, then

$$\begin{aligned} 0 &\ge 2\mu\sum_{i=1}^{n}(\partial_i u^i)^2 + \mu\sum_{i\neq j}^{n}(\partial_i u^j)^2 + 2\mu\sum_{i>j}^{n}(\partial_i u^j)(\partial_j u^i) \\ &= 2\mu\sum_{i=1}^{n}(\partial_i u^i)^2 + \mu\sum_{i>j}^{n}(\partial_i u^j + \partial_j u^i)^2. \end{aligned}$$

Both cases, together with (1.9) and (1.12), imply that

$$\begin{cases} \partial_i u^i(t,x) = 0 \\ \partial_i u^j(t,x) = -\partial_j u^i(t,x) \quad (i \neq j) \end{cases} \qquad \text{on } \{t\}\times\Omega(t)^c. \tag{2.8}$$

This implies again that

$$\nabla^2\mathbf{u}(t,x) = 0 \quad \text{on } \{t\}\times\Omega(t)^c.$$

Hence the assumption $\mathbf{u} \in H^m(\mathbb{R}^n)$ immediately yields

$$\mathbf{u}(t,x) = 0 \quad \text{on } \{t\}\times\Omega(t)^c.$$

On the other hand, we observe from the definition of $\Omega(t)$ that

$$\mathbf{u}(t, \mathbf{X}(t; x_0)) = 0 \quad \text{if } x_0 \in \partial\Omega(0).$$

Thus

$$\mathbf{X}(t; x_0) = x_0 + \int_0^t \mathbf{u}(s, \mathbf{X}(s; x_0))\, ds = x_0 \quad \text{if } x_0 \in \partial\Omega(0),$$

hence

$$R(t) = R(0) \quad \forall t.$$

This proves assertion (ii) and the proof of Theorem 1.1 is complete. □

REMARK 2.1. The proof of part (i) can be applied to any equations of compressible fluids such as the Euler equations, shallow water equations and Korteweg type

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equations to show that if the support of the density grows sublinearly in time then the life span of smooth solutions is finite. Moreover, if C_1 is sufficiently small, then we can arrange for $\alpha \in [0, 1]$.

2.2. Proof of Theorem 1.2. Suppose that $(\rho, \mathbf{u}, S) \in C^1([0, \infty); H^m(\mathbb{R}^n))$ is a solution to (NS) satisfying (1.14). Then there exist constants $t_0 > 0$ and $c < 1$ such that

$$\left\| \frac{\mathbf{u}(t, x)}{1 + |x|} \right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{c}{t} \quad \forall t \geq t_0. \tag{2.9}$$

Writing $\widetilde{M}(t) = \int_{\mathbb{R}^n} \rho(t, x)(1 + |x|)\, dx$, then it follows from (2.2) and (1.1) that

$$\begin{aligned} \frac{d}{dt}\widetilde{M}(t) = \int_{\mathbb{R}^n} \rho_t(t, x)|x|\, dx &= -\int_{\mathbb{R}^n} \operatorname{div}(\rho\mathbf{u})|x|\, dx \\ &= \int_{\mathbb{R}^n} \rho\mathbf{u} \cdot \frac{x}{|x|}\, dx \\ &= \int_{\mathbb{R}^n} \rho(1 + |x|)\frac{\mathbf{u}}{1 + |x|} \cdot \frac{x}{|x|}\, dx \\ &\leq \widetilde{M}(t) \left\| \frac{\mathbf{u}(t, x)}{1 + |x|} \right\|_{L^\infty(\mathbb{R}^n)}. \end{aligned} \tag{2.10}$$

Thus (2.9) and (2.10) imply that

$$\frac{d}{dt}\widetilde{M}(t) \leq \widetilde{M}(t)\frac{c}{t} \quad \forall t \geq t_0.$$

Integrating this inequality indirectly over $[t_0, t]$,

$$\widetilde{M}(t) \leq \widetilde{M}(t_0) + c\int_{t_0}^{t} \frac{\widetilde{M}(s)}{s}\, ds,$$

hence the Gronwall lemma yields

$$\widetilde{M}(t) \leq \widetilde{M}(t_0) \exp\left(c \log\left(\frac{t}{t_0}\right)\right) = \frac{\widetilde{M}(t_0)}{t_0^c} t^c. \tag{2.11}$$

On the other hand, by (2.2) and (2.3) we can estimate $\widetilde{M}(t)$ bounded below as

$$\begin{aligned} \widetilde{M}(t) &\geq \int_{\mathbb{R}^n} \rho(t, x)\, dx + \left| \int_{\mathbb{R}^n} \rho(t, x)x\, dx \right| \\ &\geq m(0) + |E^i(0) + A^i(0)t| \\ &\geq m(0) + |A^i(0)|t - M(0). \end{aligned} \tag{2.12}$$

Combining inequalities (2.11) and (2.12), we obtain

$$|A^i(0)|t \leq M(0) - m(0) + \frac{\widetilde{M}(t_0)}{t_0^c} t^c \quad \forall t \geq t_0 \text{ and all } i = 1, 2, \ldots, n. \tag{2.13}$$

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Meanwhile,

$$
\begin{aligned}
\widetilde{M}(t_0) &= \widetilde{M}(0) + \int_0^{t_0} \frac{d}{ds}\widetilde{M}(s)\,ds \\
&= m(0) + M(0) + \int_0^{t_0}\int_{\mathbb{R}^n} \rho\mathbf{u}\cdot\frac{x}{|x|}\,dx\,ds \\
&\le m(0) + M(0) + \sup_{0\le t\le t_0} \|\mathbf{u}(t,x)\|_{L^\infty(\mathbb{R}^n)} m(0) t_0.
\end{aligned} \tag{2.14}
$$

Thus (2.13) and (2.14), together with the assumptions (1.13) on the initial data and $\mathbf{u} \in C^1([0,\infty); H^m(\mathbb{R}^n))$, yield the contradiction to the hypothesis $c < 1$. This completes the proof of Theorem 1.2. □

REMARK 2.2. Similarly to Remark 2.1, Theorem 1.2 also holds for some other equations of compressible fluids.

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