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Ideal Structure of Multiplier Algebras of Simple C^*-algebras With Real Rank Zero

To my wife Montserrat

Francesc Perera

Abstract. We give a description of the monoid of Murray-von Neumann equivalence classes of projections for multiplier algebras of a wide class of σ-unital simple C^*-algebras A with real rank zero and stable rank one. The lattice of ideals of this monoid, which is known to be crucial for understanding the ideal structure of the multiplier algebra $\mathcal{M}(A)$, is therefore analyzed. In important cases it is shown that, if A has finite scale then the quotient of $\mathcal{M}(A)$ modulo any closed ideal I that properly contains A has stable rank one. The intricacy of the ideal structure of $\mathcal{M}(A)$ is reflected in the fact that $\mathcal{M}(A)$ can have uncountably many different quotients, each one having uncountably many closed ideals forming a chain with respect to inclusion.

Introduction

Let A be a non-unital C^*-algebra with a faithful and non-degenerate action on a Hilbert space $\mathcal{H}$. An element x in $B(\mathcal{H})$ is a *multiplier* for A if $xA \subset A$ and $Ax \subset A$. The *multiplier algebra* of A is defined as

$$\mathcal{M}(A) = \{x \in B(\mathcal{H}) \mid x \text{ is a multiplier for } A\}.$$

The representation of $\mathcal{M}(A)$ as the C^*-algebra of double centralizers shows that the construction of the multiplier algebra is independent of the particular Hilbert space $\mathcal{H}$ (see [42, Proposition 3.12.3]).

Denote by $K(\mathcal{H})$ the algebra of compact operators on $\mathcal{H}$. Then it is well-known that $\mathcal{M}\big(K(\mathcal{H})\big) = B(\mathcal{H})$ and that the Calkin algebra $B(\mathcal{H})/K(\mathcal{H})$ is simple, and therefore it is natural to ask when the corona algebra $\mathcal{M}(A)/A$ is simple. This question has been considered in different instances (see *e.g.* [34], [56], [35]), being completely solved in case A is separable and simple: $\mathcal{M}(A)/A$ is simple if and only if A is elementary or A has continuous scale ([35, Theorem 2.10]).

Many papers have been concerned with $\mathcal{M}(A)$ and the ideal structure of $\mathcal{M}(A)/A$. To cite a few examples, see [20], [10], [34], [54], [55], [56], [35], [36], [58], [59], [60], [30]. Our main objective is to analyze the lattice of closed ideals of $\mathcal{M}(A)/A$ when the scale is not continuous and under certain assumptions on A which include the case of AF algebras. We therefore work within the class of σ-unital simple

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C^*-algebras A with real rank zero (in the sense of [11]) and stable rank one. We also assume that $V(A)$, the monoid of Murray-von Neumann equivalence classes of projections in $M_\infty(A)$, is strictly unperforated (equivalently, $K_0(A)$ is weakly unperforated). Such a class has been considered successfully in [30], where the additive structure of $K_0\big(\mathcal{M}(A)\big)$ is completely described, as is the order structure in certain cases. As a consequence, several results concerning stably cofinite closed ideals follow. For expository reasons, we shall denote by $\mathcal{N}$ the class of non-elementary C^*-algebras satisfying the abovementioned conditions. It is remarkable that there are many simple C^*-algebras that fall in this class, as noted by Lin in [36]. Among them there are the C^*-algebras classified by Elliott in [24] and also stabilizations of the ones considered in [28]. It is important to note that there are no examples known, at least at present, of C^*-algebras with real rank zero and stable rank one whose K_0's are not weakly unperforated (though there are examples of simple C^*-algebras with real rank one and stable rank one whose K_0's have perforation, see [50]). As a matter of fact, if A is a unital simple C^*-algebra with real rank zero, then $V(A)$ is strictly unperforated if and only if the well-known open question (FCQ) has a positive answer for A (see [5]), as shown in [43, Corollary 3.10].

Our work relies on focusing on the non-stable K-theory of $\mathcal{M}(A)$, that is, $V\big(\mathcal{M}(A)\big)$, which by a result of Zhang reflects the lattice of all closed ideals of $\mathcal{M}(A)$, instead of just the stably cofinite ones. We therefore describe $V\big(\mathcal{M}(A)\big)$ in terms of $V(A)$ and a certain semigroup of functions on the state space of $V(A)$. This description allows to systematize the study of the ideal structure of $\mathcal{M}(A)/A$ and provides a new insight into some of the previous work done for AF algebras. For example, if the algebra A has exactly n extremal infinite quasitraces, then there is a quotient of $\mathcal{M}(A)/A$ that has exactly 2^n closed ideals. This generalizes a result of Lin ([34]). Some of our results use the additional hypothesis that $\mathcal{M}(A)$ has also real rank zero. This occurs, for example, if A is σ-unital, with real rank zero and stable rank one, and $K_1(A) = 0$, as shown by Lin in [37, Theorem 10]. In general, the vanishing of K_1 is a necessary condition for $\mathcal{M}(A)$ to have real rank zero in case A is stable (see [57]). Since $\mathcal{M}(A)$ does not have cancellation, our method of proof leads to work with non-cancellative monoids; this language provides cleaner proofs and major intuition of how a "global picture" of the lattice of closed ideals of $\mathcal{M}(A)/A$ could be traced. Moreover, many of the results obtained concerning the ideal lattice of $\mathcal{M}(A)$ will be used in a subsequent paper ([44]) to analyze in an efficient way the extremal richness of $\mathcal{M}(A)$ and $\mathcal{M}(A)/A$ (see also [12] and [33]).

The paper is organized as follows. In the first Section we summarize the basic notions. In Sections 2 and 3, an isomorphism from $V\big(\mathcal{M}(A)\big)$ onto a disjoint union of $V(A)$ and a semigroup of lower semicontinuous affine functions over the state space of $V(A)$ is established. The algebras with finite scale are considered in Section 4. We provide a characterization in terms of a stable rank condition on a quotient of $\mathcal{M}(A)/A$ that in turn yields an answer to a question of Goodearl in [30]. Semifinite quasitraces are the main topic of Section 5. Using techniques from [43] we get a characterization of algebras with bounded scale that extends a result of Blackadar in [4]. The methods developed in previous sections are used in Section 6 to obtain results on multiplier algebras with unbounded scale. In particular, we construct a chain of uncountably many closed ideals if the algebra A has (at least) one infinite extremal quasitrace which is not isolated. In Section 7, we use the concept of stable rank of an element of a monoid introduced in [3] to compute the stable rank of $\mathcal{M}(A)$ which, for non-unital A, can only take the values 2 or ∞, depending on the scale of A.

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1 Notation and Preliminaries

In this section we recall some basic definitions on monoids and spaces of functions, and we fix the notation that will be used in our work. Since the literature concerning abelian monoids is quite fragmentary, we have given some references in order, we hope, to help the reader tracing back the history of some of the concepts.

All monoids in this paper will be abelian. We shall write them additively and we shall use 0 for their identity element. The operation on a monoid M defines a natural (translation-invariant) preordering by:

$$x \leq y \iff y = x + z \quad \text{for some } z \text{ in } M.$$

This preordering is called the *algebraic preordering*, and M is said to be *algebraically ordered* (see, for example, [6, 2.1.1]).

We say that a monoid M has *refinement* (or that M is a *refinement monoid*) if, for all x_1, x_2, y_1, y_2 in M that satisfy $x_1 + x_2 = y_1 + y_2$ there exist elements z_{ij} in M, for $i, j = 1, 2$, such that $\sum_{j=1}^{2} z_{ij} = x_i$ and $\sum_{i=1}^{2} z_{ij} = y_j$ (*e.g.* [18], [52]). We say that M is a *Riesz monoid* provided that M satisfies the *Riesz Decomposition Property*, that is, whenever x, y_1, y_2 in M satisfy $x \leq y_1 + y_2$, then there exist x_1, x_2 in M such that $x = x_1 + x_2$ with $x_i \leq y_i$ for all i (see, for example, [53]). In the presence of cancellation, these properties are well-known to be equivalent, as is seen in [26, Proposition 2.1]. However, in general, the Riesz Decomposition Property is weaker than the refinement property.

A non-zero element u in M is called an *order-unit* if for any x in M there exists a natural number n such that $x \leq nu$ (*cf.* [9]). A monoid M is said to be *conical* provided that the set M^* of non-zero elements is closed under addition (see, *e.g.* [2], [3]). For any C^*-algebra A, we denote by $V(A)$ the monoid of Murray-von Neumann equivalence classes of projections from $M_\infty(A)$. (As usual, if p and q are projections in $M_\infty(A)$, we use $p \sim q$ to indicate Murray-von Neumann equivalence, whereas $p \lesssim q$ means that $p \sim q'$ for some projection $q' \leq q$.) Note that $V(A)$ is conical and in case A is unital, then $[1_A]$ is an order-unit for $V(A)$.

A non-empty subset S of a monoid M which is a submonoid and order-hereditary (that is, if $x \leq y$ and $y \in S$ then $x \in S$) will be called an *order-ideal* ([3]). We say that a monoid M is *simple* ([3]) if M has precisely two order-ideals, namely the ideal generated by 0 and M. Note that, if M is conical, then M is simple if and only if M is non-zero and every non-zero element is an order-unit.

Let M be a monoid and let I be an order-ideal of M. Define a congruence on M as follows: if $x, y \in M$, write $x \sim y$ if and only if there exist z, w in I such that $x + z = y + w$. Denote by M/I the quotient of M modulo this congruence, and by $[x]$ the congruence class of an element x in M. Then $[x] + [y] = [x + y]$, and thus M/I becomes a monoid, referred as to the *quotient monoid* of M modulo I (see [3]). Observe that, if u in M is an order-unit, then $[u]$ is an order-unit for M/I.

Let M be a partially ordered monoid. A non-empty subset I of M is called an *interval* in M if I is upward directed and order-hereditary (see [27], [31]). We denote by $\Lambda(M)$ the set of all intervals in M. Note that $\Lambda(M)$ becomes an abelian monoid with operation defined by

$$I + J = \{z \in M \mid z \leq x + y \text{ for some } x \text{ in } I, y \text{ in } J\}.$$

If $I \in \Lambda(M)$, we say that I is *countably generated* ([43]) provided that I has a countable cofinal subset (*i.e.*, there is a sequence $\{x_n\}$ of elements in I such that, for any x in I, there exists n in $\mathbb{N}$ such that $x \leq x_n$). We denote by $\Lambda_\sigma(M)$ the set of all intervals in M that are countably generated. If D is a fixed interval in $\Lambda_\sigma(M)$, we denote by $\Lambda_{\sigma,D}(M)$ the submonoid of $\Lambda_\sigma(M)$ whose elements are those intervals I in $\Lambda_\sigma(M)$ such that $I \subseteq nD$ for some n.

Let K be a compact convex set. Following [1], we use $\mathrm{Aff}(K)$ to denote the group of all affine continuous real-valued functions on K. We denote by $\mathrm{LAff}(K)$ the monoid of all affine and lower semicontinuous functions on K with values on $\mathbb{R} \cup \{+\infty\}$. Let $\mathrm{LAff}_\sigma(K)$ be the submonoid of $\mathrm{LAff}(K)$ whose elements are pointwise suprema of increasing sequences of affine continuous functions on K. If $d \in \mathrm{LAff}_\sigma(K)$, we denote by $\mathrm{LAff}_{\sigma,d}(K)$ the submonoid of elements in $\mathrm{LAff}_\sigma(K)$ that are bounded by nd for some natural number n. The use of the superscript + (resp. ++) will always refer to positive (resp. strictly positive) functions.

2 The Ideals of $\mathcal{M}(A)$

In [56], Zhang established the exact relationship between the closed ideals of the multiplier algebra of a σ-unital C^*-algebra A with real rank zero and the order-ideals of its monoid $V\big(\mathcal{M}(A)\big)$ of equivalence classes of projections. We include below a slight re-statement of his result. The rest of the section is devoted to representing $V\big(\mathcal{M}(A)\big)$ (if A has furthermore stable rank one) as a certain monoid of intervals over $V(A)$, in analogy with [30].

If A is any C^*-algebra, denote by $L_c(A)$ the lattice of closed ideals of A, and by $L\big(V(A)\big)$ the lattice of order-ideals of $V(A)$.

Theorem 2.1 **([56, Theorem 2.3])** *Let A be a σ-unital C^*-algebra with real rank zero. Then the map $L_c\big(\mathcal{M}(A)\big) \to L\Big(V\big(\mathcal{M}(A)\big)\Big)$ given by $I \mapsto V(I)$ is a lattice isomorphism.* ■

Definition 2.2 (*cf.* [**21**]) Let A be a C^*-algebra. Define the *dimension range* of A as the set:

$$D(A) = \{[p] \in V(A) \mid p \text{ is a projection in } A\}.$$

If $\{e_n\}$ is an increasing approximate unit for A consisting of projections, then $D(A)$ can be described as a countably generated interval which has $\{[e_n]\}$ as a countable cofinal subset.

Let A be a σ-unital C^*-algebra with real rank zero. If e is a projection in $M_n\big(\mathcal{M}(A)\big)$, then $eM_n(A)e$ is a σ-unital C^*-subalgebra of $M_n(A)$ by [30, Lemma 1.3], and it has an approximate identity consisting of an increasing sequence of projections (see [57, Proposition 1.2]). Define, as in [30, 1.4],

$$\theta(e) = \{[p] \in V(A) \mid p \text{ is a projection in } eM_\infty(A)e\}.$$

Then $\theta(e)$ is an interval in $V(A)$. In fact,

$$\begin{aligned}\theta(e) &= \{[p] \in V(A) \mid p \text{ is a projection in } eM_n(A)e\}\\ &= \{[p] \in V(A) \mid p \lesssim e\}\\ &= \{[p] \in V(A) \mid p \lesssim e_k \text{ for some } k\},\end{aligned}$$

where $e_1 \leq e_2 \leq \cdots$ is an increasing approximate unit for the algebra $eM_n(A)e$ consisting of projections. On the other hand, it is clear that $\theta(e) \subseteq nD(A)$.

We will need the following result, which is essentially [30, Proposition 1.8]; for our purposes it needs only a minor modification.

Proposition 2.3 *Let A be a σ-unital C^*-algebra with real rank zero and stable rank one. Let g be a projection in $M_\infty\big(\mathcal{M}(A)\big)$ and let X and Y be countably generated intervals in $V(A)$ such that $X + Y = \theta(g)$. Then $g = e + f$ for some orthogonal projections e and f in $M_\infty\big(\mathcal{M}(A)\big)$ such that $\theta(e) = X$ and $\theta(f) = Y$.* ■

Let M and N be monoids and let u in M and v in N be order-units. A monoid morphism (that is, an additive map) $f\colon (M,u) \to (N,v)$ is called *normalized* provided that $f(u) = v$. The following is the monoid-theoretic version of [30, Theorem 1.10].

Theorem 2.4 *Let A be a σ-unital C^*-algebra with real rank zero and stable rank one. Then there exists a normalized monoid isomorphism from $\Big(V\big(\mathcal{M}(A)\big), [1_{\mathcal{M}(A)}]\Big)$ onto the abelian monoid $\Big(W_\sigma^D\big(V(A)\big), D\Big)$ whose elements are those countably generated intervals I in $V(A)$ for which there exist n in $\mathbb{N}$ and a countably generated interval J in $V(A)$ such that $I + J = nD$, where $D = D(A)$.*

Proof It is proved in [30, Proposition 1.6] that, if e, f are projections in $M_\infty\big(\mathcal{M}(A)\big)$, then $\theta(e \oplus f) = \theta(e) + \theta(f)$, and if $e \in M_n\big(\mathcal{M}(A)\big)$, then there exists a projection g in $M_n\big(\mathcal{M}(A)\big)$ such that $\theta(e) + \theta(g) = nD$. Also, in [30, Proposition 1.7], it is seen that $e \sim f$ if and only if $\theta(e) = \theta(f)$.

Define a map $\gamma\colon V\big(\mathcal{M}(A)\big) \to W_\sigma^D\big(V(A)\big)$ by $\gamma(v) = \theta(e)$, where $v = [e]$ for some projection e in $M_\infty\big(\mathcal{M}(A)\big)$. Using the observations above, it is clear that γ is an injective monoid morphism. Note that D is an order-unit for $W_\sigma^D\big(V(A)\big)$ and that $\gamma([1_{\mathcal{M}(A)}]) = D$. Hence γ is normalized. To see that γ is bijective, let $I \in W_\sigma^D\big(V(A)\big)$. There exists a countably generated interval J and a positive integer n such that $I + J = nD$. Observe that, in fact, $D = \theta(1_{\mathcal{M}(A)})$ and so $nD = \theta(1_{M_n(\mathcal{M}(A))})$. Hence, by Proposition 2.3, there exist orthogonal projections e, f in $M_n\big(\mathcal{M}(A)\big)$ such that $\theta(e) = I$ and $\theta(f) = J$. Thus $I = \gamma([e])$. ■

As a first application of the theorem, it is possible to relate the monoids of equivalence classes of projections of multiplier algebras of hereditary C^*-subalgebras of A to the monoid $S(A)$ of equivalence classes of positive elements in the sense of Cuntz (see [15], [8], [35], [39], [48] and [43]). We recall the definitions for the convenience of the reader. If $a, b \in A_+$, we write $a \lesssim b$ if there exist elements r_n, s_n in A such that $a = \lim_{n\to\infty} r_n b s_n$. When restricted to projections, the relation $\lesssim$ gives the usual Murray-von Neumann subequivalence (see [48, Proposition 2.1]). We write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$. If $x, y \in M_\infty(A)_+$, say that $x \sim y$ if and only if there exists a natural number n such that $x \sim y$ in $M_n(A)$. Denote by $\langle x\rangle$ the equivalence class of x with respect to $\sim$, and define $\langle x\rangle + \langle y\rangle = \big\langle \big(\begin{smallmatrix} x & 0 \\ 0 & y\end{smallmatrix}\big)\big\rangle$. Let $S(A)$ be the set of all $\sim$-equivalence classes in $M_\infty(A)_+$. This is an abelian monoid, which is partially ordered with $\langle x\rangle \le \langle y\rangle$ if and only if $x \lesssim y$. By [43, Lemma 2.2], if A is σ-unital, then $S(A)$ has an order-unit, which we shall denote by $u_{S(A)}$. If $x \in A_+$, denote by A_x the hereditary C^*-subalgebra of A generated by x, that is, $A_x = \overline{xAx}$.

Corollary 2.5 *Let A be a σ-unital simple C^*-algebra with real rank zero and stable rank one. Let $n \in \mathbb{N}$ and $x \in M_n(A) \setminus \{0\}$. Then $V\Big(\mathcal{M}\big(M_n(A)_x\big)\Big)$ is isomorphic to a submonoid of $S(A)$.*

Proof Since $V\big(M_n(A)_x\big) \cong V(A)$ we conclude from Theorem 2.4 that the monoid $V\Big(\mathcal{M}\big(M_n(A)_x\big)\Big)$ is isomorphic to a submonoid of $\Lambda_{\sigma,D}\big(V(A)\big)$. Now, by [43, Theorem 2.8], $S(A)$ is order-isomorphic to $\Lambda_{\sigma,D}\big(V(A)\big)$. ■

3 The Monoid Representation

The isomorphism between $V\big(\mathcal{M}(A)\big)$ and $W_\sigma^D\big(V(A)\big)$ given in Theorem 2.4 for σ-unital C^*-algebras with real rank zero and stable rank one allows us to concentrate on monoids of intervals in a given monoid. We therefore work with the monoid $W_\sigma^D(M)$ of countably generated intervals I in M such that there exist n in $\mathbb{N}$ and a countably generated interval J with $I + J = nD$. Here M is a monoid and D is a countably generated interval in M which is generating, that is, D generates M as a monoid. Our goal is to relate $W_\sigma^D(M)$ to a monoid of lower semicontinuous affine functions defined on the state space of M, and translate subsequently the results to the setting of simple C^*-algebras. The resulting representation of $V\big(\mathcal{M}(A)\big)$ is connected with the ones obtained by Elliott in [22] and by Lin in [36, Theorem 7].

Definition 3.1 (*cf.* [**2**]) Let M be a monoid. For x and y in M we write $x \leq^* y$ provided that there exists a non-zero element z in M such that $x + z = y$. An *atom* of M is a non-zero element a in M such that there is no b in M satisfying $0 \leq^* b \leq^* a$. We say that M is an *atomic* monoid if every element of M can be written as a sum of atoms.

If $x, y \in M$, we write as usual $x < y$ if and only if $x \leq y$ and $x \neq y$. Note that, if M is cancellative, then both relations $<$ and $\leq^*$ coincide.

Recall that a *state* on a preordered abelian group G with order-unit u is a group morphism $s\colon G \to \mathbb{R}$ such that $s(G^+) \subseteq \mathbb{R}_+$ and $s(u) = 1$ (see [26]).

Definition 3.2 (*cf.* [**9**]) Let (M, u) be a monoid with order-unit. A *state* on (M, u) is a monoid morphism $s\colon M \to \mathbb{R}_+$ such that $s(u) = 1$. We denote by $\mathrm{St}(M, u)$ the set of states on (M, u).

It is easy to see that, if (M, u) is a monoid with order-unit, then

$$\mathrm{St}(M, u) = \mathrm{St}\big(G(M), u\big)$$

where $G(M)$ is the Grothendieck group of M. A state on (M, u) is said to be *discrete* if $s\big(G(M)\big)$ is a cyclic subgroup of $\mathbb{R}$ ([26]).

If M is an algebraically ordered conical simple monoid with order-unit u, then $\mathrm{St}(M, u)$ is non-empty if and only if M is partially ordered (see, *e.g.* [26, Corollary 4.4]). It is a standard fact that, if M is an atomless simple monoid and u is an order-unit in M, then $\mathrm{St}(M, u)$ has no discrete states.

Definition 3.3 ([**31**], [**30**]) Let M be a monoid. An interval I in M is *soft* if and only if I is non-zero and for each x in I, there exist y in I and n in $\mathbb{N}$ such that $(n+1)x \leq ny$.

As noted in [30, Section 4] (see also [31, Lemma 7.4, Proposition 7.5]), if M is the positive cone of a simple and weakly unperforated Riesz group G, then every non-zero interval in M is either soft or of the form $[0, x]$ for some non-zero element x in M. This fact can be extended to the non-cancellative case, for countably generated intervals.

Lemma 3.4 *Let M be a conical simple refinement monoid. Then every element in $\Lambda_\sigma(M)$ is either soft or of the form $[0, x]$ for some x in M, but not both.*

Proof Suppose first that M is atomless. We show that every interval I generated by a strictly increasing sequence $x_1 < x_2 < \cdots$ is soft. There exists t in M^* such that $x_1 + t = x_2$. Since M has no atoms, is simple and has refinement, there exists y in M^* such that $y \leq^* \{t, x_1\}$. Further, y is an order-unit, so that $x_1 \leq ky$ for some k. Now:

$$(k+1)x_1 = kx_1 + x_1 \leq kx_1 + ky = k(x_1 + y) \leq^* k(x_1 + t) = kx_2.$$

Similarly, for each i, there exists k with $(k+1)x_i \leq kx_{i+1}$. Hence I is soft. Note finally that every countably generated interval is either generated by a strictly increasing sequence or of the form $[0, x]$ for some x in M.

Now suppose that M has an atom and hence is atomic, because it is simple and has refinement. By [2, Lemma 1.6], M is the infinite cyclic monoid and the result is clear in that case. ∎

The following is the appropriate formulation in the monoid-theoretical context of the notion of weak unperforation for partially ordered abelian groups (see [7], [23, 2.1], [30, Section 8]).

Definition 3.5 Let M be a cancellative monoid. We say that M is *strictly unperforated* if, whenever $nx < ny$ for some n in $\mathbb{N}$, it follows that $x < y$.

Let M be a monoid and assume that there exists an order-unit u in M. Write $S_u = \mathrm{St}(M, u)$ and denote by $\phi_u \colon M \to \mathrm{Aff}(S_u)$ the natural representation map (given by evaluation). If $f, g \in \mathrm{Aff}(S_u)$, we write $f \ll g$ for uniform strict inequality, that is, $f(s) < g(s)$ for all s in S_u. In some important cases it is possible to recover the ordering in M from that of $\mathrm{Aff}(S_u)$. We record this in the following (known) lemma, which is a slight generalization of a result due to Effros, Handelman and Shen ([19, Theorem 1.4]).

Lemma 3.6 *Let M be a strictly unperforated cancellative monoid and let u in M be an order-unit. Let $x, y \in M$. Suppose that $\phi_u(x) \ll \phi_u(y)$. Then $x < y$.*

Proof If $\phi_u(x) \ll \phi_u(y)$, then there exists m in $\mathbb{N}$ such that $m([y] - [x])$ is an order-unit in $G(M)$, where $G(M)$ is the Grothendieck group of M (see [26, Theorem 4.12]). In particular, $m([y] - [x]) > 0$ and since M is cancellative, this implies that $mx \leq^* my$ in M, whence $x < y$, because M is strictly unperforated and cancellative. ∎

If $X \in \Lambda(M)$, set $\rho(X) = \sup \phi_u(X)$, where "sup" stands for pointwise supremum. Then $\rho(X) \in \mathrm{LAff}(S_u)^+$. The following is similar to [30, Lemma 5.2].

Lemma 3.7 *Let M be a simple refinement monoid which is non-atomic, strictly unperforated and cancellative. Let $u \in M^*$.*

(1) If X is a non-zero element in $\Lambda(M)$, then $\rho(X) \in \mathrm{LAff}(S_u)^{++}$ and it is bounded away from zero.
(2) $\rho(X + Y) = \rho(X) + \rho(Y)$ for any intervals X and Y in M.
(3) If $f \in \mathrm{LAff}(S_u)^{++}$, then $\rho'(f) = \{x \in M \mid \phi_u(x) \ll f\}$ is a soft interval in M and $\rho\rho'(f) = f$.
(4) If $X \in \Lambda(M)$ is soft, then $\rho'\rho(X) = X$.

Proof If $X \in \Lambda(M)$ and it is non-zero, then X contains a non-zero element x in M, which is an order-unit (because M is simple). Then $\rho(X) \in \mathrm{LAff}(S_u)^{++}$. Since x is an order-unit, there exists a natural number n such that $u \leq nx$, and so $\rho(X) \geq \phi_u(x) \geq 1/n$. Thus (1) is established.

Since $\phi_u(X + Y) = \phi_u(X) + \phi_u(Y)$, we get (2).

Let $f \in \mathrm{LAff}(S_u)^{++}$. Evidently $\rho'(f)$ is non-empty and order-hereditary. To see that it is upward directed, let $x, y \in \rho'(f)$ and $g = \sup\{\phi_u(x), \phi_u(y)\}$. Then g is upper semicontinuous, convex and $g \ll f$. By [26, Theorem 11.12], there exists h in $\mathrm{Aff}(S_u)$ such that $g \ll h \ll f$. Choose $\epsilon > 0$ satisfying $g \ll h-\epsilon \ll h+\epsilon \ll f$. Since M has no discrete states and it is a refinement monoid, we can use [41, Theorem 3.5] to conclude that the image of $G(M)^+$ under the natural representation map ϕ_u is dense in $\mathrm{Aff}(S_u)^+$. (Notice also that $G(M)^+ = M$.) Therefore there exists z in M such that $\|h - \phi_u(z)\| < \epsilon$. It follows then that $\phi_u(x), \phi_u(y) \ll \phi_u(z)$ and thus $x, y \leq z$ by Lemma 3.6. Note also that $z \in \rho'(f)$. This shows that $\rho'(f)$ is an interval. To see that $\rho\rho'(f) = f$, let $s \in S_u$ and take $\alpha < f(s)$. By [26, Theorem 11.8], there exists g in $\mathrm{Aff}(S_u)^{++}$ such that $\alpha < g(s)$ and $g \ll f$. Let $\epsilon > 0$ be such that $\alpha < g(s) - \epsilon$ and $g + \epsilon \ll f$. By [41, Theorem 3.5], there exists w in M satisfying $\|g - \phi_u(w)\| < \epsilon$. Therefore $\phi_u(w)(s) > \alpha$ and $\phi_u(w) \ll f$, proving that $\rho\rho'(f) = f$. Finally, let $x \in \rho'(f)$. Then $f - \phi_u(x)$ is a strictly positive lower semicontinuous affine function. A similar argument to the one used before shows that there exists a non-zero element y in M such that $x + y \in \rho'(f)$. Thus the proof of [31, Lemma 7.4] implies that $\rho'(f)$ is soft and generating. This gives (3).

For (4), see the arguments used in [31, Proposition 7.7]. ■

Let M be a partially ordered monoid, and assume that u is an order-unit in M. Let D be a generating interval in M with a countable cofinal subset and set $d = \sup \phi_u(D) = \rho(D)$. Define:

$$W_\sigma^d(S_u) = \{f \in \mathrm{LAff}_\sigma(S_u)^{++} \mid f + g = nd \text{ for some } g \text{ in } \mathrm{LAff}_\sigma(S_u)^{++} \text{ and } n \text{ in } \mathbb{N}\}.$$

Notice that $W_\sigma^d(S_u)$ can be also described as the elements f in $\mathrm{LAff}_\sigma(S_u)^{++}$ such that $f + g = nd$ for some n and some g in $\mathrm{LAff}_\sigma(S_u)^+$.

Consider now the set $M \sqcup W_\sigma^d(S_u)$, where $\sqcup$ stands for disjoint union of sets. Define a monoid structure on $M \sqcup W_\sigma^d(S_u)$ by extending the given addition operations on M and $W_\sigma^d(S_u)$ and by $x + f = \rho[0, x] + f = \phi_u(x) + f$, if $x \in M$ and $f \in W_\sigma^d(S_u)$. Note that $x + f \in W_\sigma^d(S_u)$ and that d is an order-unit for $M \sqcup W_\sigma^d(S_u)$.

Theorem 3.8 *Let M be a conical simple refinement monoid and let $u \in M^*$. Let D be a soft non-zero interval in M with a countable cofinal subset. Define a map*

$$\varphi\colon \big(W_\sigma^D(M), D\big) \to \big(M \sqcup W_\sigma^d(S_u), d\big)$$

by $\varphi(X) = \rho(X) \in W_\sigma^d(S_u)$ if X is a soft interval, and $\varphi([0, x]) = x$ if $x \in M$. Then φ is a normalized monoid morphism. If, further, M is non-atomic, strictly unperforated and cancellative, then φ is an isomorphism.

Proof First note that, if X and Y are intervals in M, then

$$\rho(X + Y) = \sup\{\phi_u(x + y) \mid x \in X \text{ and } y \in Y\} = \rho(X) + \rho(Y).$$

In particular, if $X \in W_\sigma^D(M)$ and is soft, then $\rho(X) \in W_\sigma^d(S_u)$. Since every countably generated interval in M is either soft or of the form $[0, x]$ for some x in M (by Lemma 3.4), we see that φ is well-defined.

(i) φ is a normalized monoid morphism. We only need to check that $\varphi([0,x] + X) = \varphi([0,x]) + \varphi(X)$, if $x \in M^*$ and X is a soft interval in $W_\sigma^D(M)$. Since X is soft, we have that $[0,x] + X$ is soft (see the proof of [31, Lemma 8.1]). Then $\varphi([0,x]+X) = \rho([0,x]+X) = \rho([0,x]) + \rho(X) = \phi_u(x) + \rho(X) = x + \rho(X) = \varphi([0,x]) + \varphi(X)$. Note also that $\varphi(D) = \rho(D) = d$, because D is soft.

Now assume that M is moreover non-atomic, strictly unperforated and cancellative.

(ii) φ is injective. Suppose that $\varphi(X) = \varphi(Y)$. Note that, by definition of φ and by Lemma 3.4, it is not possible for just one of X or Y to be soft. If both X and Y are soft, then $\varphi(X) = \rho(X) = \rho(Y) = \varphi(Y)$. Thus $X = \rho'\rho(X) = \rho'\rho(Y) = Y$, by Lemma 3.7. If, otherwise, neither X nor Y are soft, then $X = [0,x]$ and $Y = [0,y]$ for some elements x, y in M, and so $\varphi(X) = x = y = \varphi(Y)$. Hence $X = Y$.

(iii) φ is surjective. If $x \in M$, then x comes from $[0,x]$. We need an easy observation here, consisting of the fact that $[0,x] \in W_\sigma^D(M)$ whenever $x \in M$:

There exists n such that $x \in nD$. Then, if $Y = \{y \in M \mid x + y \in nD\}$ we have that $[0,x] + Y = nD$, by [30, Lemma 3.8]. Using that M is cancellative and D is countably generated, it is easy to see that Y is countably generated too.

Let $f \in W_\sigma^d(S_u)$ and take $X = \rho'(f)$, which is soft by Lemma 3.7. Notice also that $\rho(X) = f$. Since there exist a function h in $\mathrm{LAff}_\sigma(S_u)^{++}$ and a natural number n with $f + h = nd$, we have that $X + \rho'(h) = nD$. Finally, we have to check that X and $\rho'(h)$ are countably generated. We know that $f = \sup g_n$ (pointwise), where $\{g_n\}$ is an increasing sequence and $g_i \in \mathrm{Aff}(S_u)^{++}$ for all i. Denote by $G(M)$ the Grothendieck group of M. As in the proof of Lemma 3.7, we can use [41, Theorem 3.5] to conclude that the image of $G(M)^+$ under the natural representation map ϕ_u is dense in $\mathrm{Aff}(S_u)^+$.

If n is large enough, there exist elements x_n in M such that $0 \ll g_n - 1/2^n \ll \phi_u(x_n) \ll g_{n+1} - 1/2^{n+1}$. Since $\phi_u(x_n) \ll \phi_u(x_{n+1})$, we get that $x_n \le x_{n+1}$, by Lemma 3.6. Also, since $f = \sup g_n$, it follows that $f = \sup \phi_u(x_n)$. Therefore $x_n \in X$ for all n. Now, if $x \in X$, then $\phi_u(x) \ll \sup \phi_u(x_n)$. For each s in S_u, there exists a natural number n with $s(x) < s(x_n)$. Let $U_n = \{s \in S_u \mid s(x) < s(x_n)\}$. Then $S_u = \cup_n U_n$ and by compactness it follows that there exist $x_{i_1}, \dots, x_{i_k}$ with $S_u = \cup_l U_{i_l}$. It is clear that there exists k in $\mathbb{N}$ such that $x_{i_l} \le x_k$ for all l, and then $\phi_u(x) \ll \phi_u(x_k)$. A second use of Lemma 3.6 allows us to conclude that $x \le x_k$. Hence X is countably generated. Similarly, $\rho'(h)$ is countably generated. ∎

We now translate our results to the context of C^*-algebras. Recall that we work within the class $\mathcal{N}$ consisting of all σ-unital (non-unital) simple C^*-algebras A that are non-elementary, with real rank zero, stable rank one and such that $V(A)$ is strictly unperforated.

Theorem 3.9 *Let A be a C^*-algebra in the class $\mathcal{N}$. Let $u \in V(A)^*$ and set $d = \sup \phi_u\big(D(A)\big)$. Then there is a monoid isomorphism φ from $V\big(\mathcal{M}(A)\big)$ onto $V(A) \sqcup W_\sigma^d(S_u)$ such that $\varphi([1_{\mathcal{M}(A)}]) = d$.*

Proof Let $M = V(A)$ and $D = D(A)$. Then M is a conical monoid. Since A has real rank zero and stable rank one, M is cancellative (by [8, Proposition III.2.4]) and has refinement (see [56, Theorem 1.1]). It follows from the simplicity of A that M is also simple, and non-atomic because A is non-elementary. Also, D is countably generated by the comment in Definition 2.2. By [30, Lemma 11.2], D is soft and non-zero. By Theorem 2.4, there is a normalized monoid isomorphism from $\Big(V\big(\mathcal{M}(A)\big), [1_{\mathcal{M}(A)}]\Big)$ onto $\big(W_\sigma^D(M), D\big)$, and the latter is isomorphic to $\big(M \sqcup W_\sigma^d(S_u), d\big)$ by Theorem 3.8. ∎

Let M be a partially ordered monoid. Let u be an order-unit in M and D a generating interval in M. Set $d = \sup \phi_u(D)$. We have considered the monoid of intervals $W_\sigma^D(M)$, with the algebraic ordering as its natural ordering. We wish to consider now $\Lambda_{\sigma,D}(M)$, endowed with the (partial) ordering given by set inclusion. One reason for this is that, in [43, Theorem 2.8], an ordered monoid isomorphism from $S(A)$ onto $\Lambda_{\sigma,D}\big(V(A)\big)$ was established, where A is a σ-unital C^*-algebra with real rank zero, stable rank one and $D = D(A)$. The ordering given by set inclusion in $\Lambda_{\sigma,D}\big(V(A)\big)$ corresponds then to the natural ordering of $S(A)$. Consider the monoid $M \sqcup \mathrm{LAff}_{\sigma,d}(S_u)^{++}$ with addition defined as before by $x+f = \rho[0,x]+f$, where $x \in M$ and $f \in \mathrm{LAff}_{\sigma,d}(S_u)^{++}$. A similar argument to the one used in Theorem 3.8 shows that $\Lambda_{\sigma,D}(M)$ and $M \sqcup \mathrm{LAff}_{\sigma,d}(S_u)^{++}$ are isomorphic monoids. Thus we now define in this monoid an ordering $\leq_1$ that will correspond to set inclusion and, consequently, another representation of $S(A)$ will be given. The ordering may be expressed in the following terms:

(1) $x \leq_1 y$ for x, y in M if and only if $x \leq y$ in M.
(2) $f \leq_1 g$ for f, g in $\mathrm{LAff}_{\sigma,d}(S_u)^{++}$ if and only if $f(s) \leq g(s)$ for all s in S_u.
(3) $x \leq_1 f$ for x in M and f in $\mathrm{LAff}_{\sigma,d}(S_u)^{++}$ if and only if $\phi_u(x) \ll f$.
(4) $f \leq_1 x$ for x in M and f in $\mathrm{LAff}_{\sigma,d}(S_u)^{++}$ if and only if $f \leq_1 \phi_u(x)$.

It is easy to see that this ordering is partial and translation-invariant. We leave the details to the reader.

Corollary 3.10 *Let M be a conical simple refinement monoid, which is also assumed to be non-atomic, strictly unperforated and cancellative. Let $u \in M$ be an order-unit, D a soft generating interval with a countable cofinal subset and $d = \sup \phi_u(D)$. Then there is a normalized ordered monoid isomorphism*

$$\psi\colon (\Lambda_{\sigma,D}(M), \subseteq, D) \to (M \sqcup \mathrm{LAff}_{\sigma,d}(S_u)^{++}, \leq_1, d),$$

given by $\psi(X) = \rho(X)$ if X is soft and $\psi([0,x]) = x$ if $x \in M$.

Proof As in Theorem 3.8, we see that ψ is a normalized monoid isomorphism. To see that it is in fact an order isomorphism, we proceed as follows.

Let $X, Y \in \Lambda_\sigma^D(M)$. If $X \subseteq Y$, we have four possibilities. If X and Y are not soft, then $X = [0,x]$ and $Y = [0,y]$ for some elements x, y in M. Hence $x \leq y$ in M, and thus $\psi(X) \leq \psi(Y)$. If X is soft and Y is not, then $Y = [0,y]$ for some y in M and $\psi(X) = \rho(X) \leq \phi_u(y)$, whence $\psi(X) \leq \psi(Y)$. If X is not soft (hence of the form $[0,x]$ where $x \in M$) and Y is soft, then there exists by [31, Lemma 7.4] an order-unit

v in M such that $x + v \in Y$. Therefore $\phi_u(x) \ll \phi_u(x + v) \leq \rho(Y)$. The case when both X and Y are soft is trivial.

Conversely, assume that $\psi(X) \leq \psi(Y)$ where $X, Y \in \Lambda_{\sigma,D}(M)$. Again we have to consider four cases. If X and Y are both non-soft, then it clearly follows that $X \subseteq Y$. If X is soft and Y is not, that is, $Y = [0, y]$ for some y in M, then

$$X = \rho'\rho(X) \subseteq \rho'\big(\phi_u(y)\big) = \{z \in M \mid \phi_u(z) \ll \phi_u(y)\} \subseteq [0, y],$$

because M is strictly unperforated. If $X = [0, x]$ and Y is soft, set $f = \psi(Y)$ and then it follows that $\phi_u(x) \ll f$. If Y is generated by an increasing sequence $\{y_n\}$ in M, then $f = \sup \phi_u(y_n)$. Since S_u is compact, there exists n such that $\phi_u(x) \ll \phi_u(y_n)$, whence $x \leq y_n$. Thus $[0, x] \subseteq Y$. Finally, if X and Y are soft it follows from Condition (4) in Lemma 3.7 that $X \subseteq Y$. ■

Theorem 3.11 *Let A be a C^*-algebra in the class $\mathcal{N}$. Let $u \in V(A)^*$ and set $d = \sup \phi_u\big(D(A)\big)$. Then there is an ordered monoid isomorphism from $\big(S(A), \leq\big)$ onto $\big(V(A) \sqcup \mathrm{LAff}_{\sigma,d}(S_u)^{++}, \leq_1\big)$ that maps $u_{S(A)}$ to d. In particular, $S(A)$ is not cancellative.*

Proof Let $D = D(A)$. As in Theorem 3.9, $V(A)$ is a conical refinement monoid, which is also simple, cancellative and non-atomic. By [43, Theorem 2.8] $\big(S(A), u_{S(A)}\big)$ is order-isomorphic to $\Big(\Lambda_{\sigma,D}\big(V(A)\big), D\Big)$, where the ordering in the latter is given by set inclusion. The first part of the result follows then from Corollary 3.10.

To see that $S(A)$ is not cancellative, let $x \in V(A)^*$ and $f \in \mathrm{LAff}_{\sigma,d}(S_u)^{++}$. Then $x + f = \phi_u(x) + f$, while $x \neq \phi_u(x)$. ■

4 Scales in C^*-Algebras

By using the representation of $V\big(\mathcal{M}(A)\big)$ as a disjoint union of $V(A)$ and a monoid of affine lower semicontinuous functions, we start a (rather) systematic study of the ideal structure of multiplier algebras, within the class $\mathcal{N}$ of σ-unital non-elementary simple C^*-algebras with real rank zero, stable rank one and with monoid of equivalence classes of projections being strictly unperforated.

One important ingredient in what follows is the scale of the algebra under consideration. Although the notions of continuous, finite or bounded scales in C^*-algebras have been previously considered (see, *e.g.* [34], [35]), we shall develop in the present section some variations of these concepts that will lead to work in a wider context.

Our approach benefits considerably from the previous monoid-theoretical setting and therefore some of the arguments used relate to monoid techniques. This procedure produces, in addition, simpler proofs.

We begin by proving a result concerning a special ideal that will appear repeatedly in the sequel. The existence of such an ideal was first noticed by Lin in [34, Lemma 2] for AF algebras, and later in [35, Remark 2.9] for separable simple C^*-algebras. In [40], the existence of this ideal was shown for simple C^*-algebras with real rank zero and having a non-zero finite projection. In the next result we compute the monoid of isomorphism classes of projections of this ideal for a C^*-algebra in the class $\mathcal{N}$. Our proof also establishes its existence in this particular context.

Proposition 4.1 *Let A be a C^*-algebra in the class $\mathcal{N}$ and let $u \in V(A)^*$. Let $L(A)$ be the smallest closed ideal of $\mathcal{M}(A)$ that properly contains A. Then*

$$V\big(L(A)\big) \cong V(A) \sqcup \mathrm{Aff}(S_u)^{++}.$$

Proof Set $D = D(A)$ and $d = \sup \phi_u(D)$. Let $L = V(A) \sqcup \mathrm{Aff}(S_u)^{++}$, which is an order-ideal of $V(A) \sqcup W_\sigma^d(S_u)$. We shall prove the following assertion: L is the smallest order-ideal of $V(A) \sqcup W_\sigma^d(S_u)$ that properly contains $V(A)$. Therefore the result will follow from Theorems 2.1 and 3.9.

Note that $V(A)$ is cancellative (since $\mathrm{sr}(A) = 1$) and that $V(A) \neq 0$. Therefore $K_0(A)$ is partially ordered and non-zero (since $K_0(A)^+ = V(A)$), and hence $S_u = \mathrm{St}(K_0(A), u)$ is non-empty. It follows that L properly contains $V(A)$.

Let I be an order-ideal that properly contains $V(A)$. Let $f \in I \setminus V(A)$ and $g \in \mathrm{Aff}(S_u)^{++}$. Since f is lower semicontinuous and g is continuous, there exists a natural number n such that $g \ll nf$. Thus $nf - g$ is affine, strictly positive and lower semicontinuous. Since $f = \sup_m f_m$ for an increasing sequence $\{f_m\}$ in $\mathrm{Aff}(S_u)^{++}$ and $g \ll nf$, there exists by compactness a natural number l such that $g \ll nf_k$ for all $k \geq l$. It follows that $nf - g = \sup_{k \geq l}(nf_k - g)$, and $nf_k - g \in \mathrm{Aff}(S_u)^{++}$ for all $k \geq l$. This implies that $g \leq nf$ in $W_\sigma^d(S_u)$. Since I is an order-ideal of $V(A) \sqcup W_\sigma^d(S_u)$ we get that $g \in I$. ∎

Definition 4.2 Let M be a monoid with order-unit u. Suppose that M has a generating interval D. We say that (M, D) has *continuous scale* if the affine function $d = \sup \phi_u(D)$ is continuous. If A is a simple C^*-algebra with real rank zero and $u \in V(A)^*$, we say that A has *continuous scale* if $\big(V(A), D(A)\big)$ has continuous scale.

It should be noted that the definition of continuous scale does not depend on the particular choice of the order-unit u. If there is another order-unit v in M, then [26, Proposition 6.17] shows that the state spaces S_u and S_v are homeomorphic, whence it follows that $\sup \phi_u(D)$ is continuous if and only if $\sup \phi_v(D)$ is. For simple C^*-algebras, this definition is of course equivalent to the ones given in [34], [35] and [36].

At this point, we can derive the characterization of simple C^*-algebras with continuous scale, which is valid in greater generality ([35, Theorem 2.10]). Our proof, though, is simple enough to include it.

Corollary 4.3 *Let A be a σ-unital non-unital simple C^*-algebra with real rank zero, stable rank one and with $V(A)$ strictly unperforated. Then $\mathcal{M}(A)/A$ is simple if and only if A is elementary or A has continuous scale.*

Proof If A is elementary, then it is well-known that $\mathcal{M}(A)/A$ is simple. So, suppose that A is non-elementary. Let $M = V(A)$ and let $u \in M^*$. Set $d = \sup \phi_u\big(D(A)\big)$. If d is continuous, then all functions in $W_\sigma^d(S_u)$ are continuous whence $W_\sigma^d(S_u) = \mathrm{Aff}(S_u)^{++}$. By Proposition 4.1, $\mathcal{M}(A)/A$ is simple. The converse is proved similarly. ∎

The next instance that will be of interest to us is that concerning the case where the scale is bounded, or finite, and not continuous. We first note the following:

Lemma 4.4 *Let M be a conical simple refinement monoid with order-unit u, and let D be a countably generated soft interval in M. Set $d = \sup \phi_u(D)$. Then d is finite if and only if it is bounded.*

Proof Assume that d is unbounded. Then, for each k there exists s_k in S_u satisfying $d(s_k) > 2^{2k}$. Note that $d = \sup_n \phi_u(x_n)$, for a strictly increasing sequence of elements $\{x_n\}$ in M^*. Therefore we can choose a subsequence $\{x_{n_k}\}$ such that $s_k(x_{n_k}) > 2^{2k}$ for all k. So, without loss of generality, we may assume that $s_k(x_k) > 2^{2k}$ for all k. Define $s = \sum_{k=1}^{\infty} 1/2^k s_k$, an element of S_u. Then $d(s) = \infty$. ■

We can therefore use the previous lemma as a motivation to give the corresponding notions of bounded and finite scale. For a compact convex set K, denote by $\partial_e K$ the set of its extreme points.

Definition 4.5 Let M be a monoid with order-unit u and generating interval D. We say that (M, D) has *finite scale* (resp. *bounded scale*) if the restriction of $d = \sup \phi_u(D)$ to $\partial_e S_u$ is finite (resp. bounded). If A is a simple C^*-algebra with real rank zero and $u \in V(A)^*$, we say that A has *finite scale* (resp. *bounded scale*) if $\big(V(A), D(A)\big)$ has finite scale (resp. bounded scale).

Notice again that this definition is independent of the choice of the order-unit u. The notion of finite scale differs from the one given in [34] in that the condition on d is required only for the extreme boundary of the state space. On the other hand, if f is a lower semicontinuous affine function on a compact convex set K and f is bounded on $\partial_e K$, then f is bounded. Namely, if $f \le c$ on $\partial_e K$ for some constant bound c, then because it is affine, $f \le c$ on the convex hull K' of $\partial_e K$. By lower semicontinuity, $f \le c$ on the closure of K', which is K by the Krein-Mil'man Theorem.

Now it is possible to construct simple C^*-algebras with finite but not bounded scale, as the following example shows. For a compact space X, we denote by $L(X)$ the additive monoid of lower semicontinuous functions on X with values on $\mathbb{R} \cup \{+\infty\}$.

Example 4.6 There exists a simple AF algebra whose scale is finite but not bounded.

Proof The line of attack will be to prove first that there exist a simple conical refinement monoid M, which is cancellative, unperforated and non-atomic, and a soft countably generated interval D such that the scale with respect to D is finite but not bounded. Let $X = [0, 1]$, a compact Hausdorff space. Let $C(X, \mathbb{R})$ be the ring of real-valued continuous functions over X, which is separable since X is metrizable. Let G be a countable dense subgroup of $C(X, \mathbb{R})$ containing the constant function 1 and equip G with the strict ordering; hence $G^+ = \{f \in G \mid f \gg 0\} \cup \{0\}$. Since G is dense in $C(X, \mathbb{R})$, we have that G is an interpolation group (in the sense of [26]), and it follows that $M = G^+$ is a refinement monoid (see [26, Proposition 2.1]). Also M is a simple conical monoid, cancellative, unperforated and non-atomic. The fact that M contains no atoms can be proved directly, but it also follows from the assumption that G is dense in $C(X,\mathbb{R})$, whence it is non-cyclic, and so by [26, Proposition 14.3] G contains no atoms, and so does M. Fix $u = 1$ as order-unit and set $S_1 = \mathrm{St}(M,1) = \mathrm{St}(G,1)$. Since G is dense in $C(X,\mathbb{R})$, the restriction map $\mathrm{St}\big(C(X,\mathbb{R}),1\big) \to \mathrm{St}(G,1)$ is an affine homeomorphism. Now note that $\mathrm{St}\big(C(X,\mathbb{R})\big) = M_1^+(X)$, the set of all probability measures over X, by [26, Proposition 6.8], whence $\partial_e S_1$ is affinely homeomorphic to $\partial_e M_1^+(X)$, and the latter is homeomorphic to X, by [26, Proposition 5.24]. Let $d_0 \in L(X)^{++}$ be defined by the rule $d_0(x) = 1/x$ for $x \neq 0$ and $d_0(0) = 2$. Let $d \in L(\partial_e S_u)^{++}$ be defined by composing d_0 with the homeomorphism from $\partial_e S_u$ onto X. Using [30, Lemma 7.2], we can extend d to a lower semicontinuous affine function defined on S_1, which is also strictly positive, and we will denote this function again by d. Set $D = \{f \in M \mid f \ll d\}$. Then D is a non-zero soft countably generated interval, and $\rho(D) = d$. Then d restricted to $\partial_e S_1$ is finite but not bounded.

There exists a simple AF C^*-algebra A such that $\big(K_0(A), D(A)\big)$ is isomorphic to (G,D) (see [21]). In particular $V(A) \cong M$. Denote by u the order-unit in $D(A)$ corresponding to the element 1 in D. Now we get from the previous argument that A has finite but not bounded scale. ∎

We now come to the main result of this section, in which we analyze the C^*-algebras with finite scale. As it turns out, in special cases they can be characterized by a stable range condition on a quotient of the corona algebra. Recall that, if $J \subset I$ are closed ideals of a C^*-algebra B with real rank zero, then the natural map $V(I) \to V(I/J)$ induces a monoid isomorphism $V(I)/V(J) \cong V(I/J)$, by [3, Proposition 1.4 and Theorem 7.2].

Theorem 4.7 *Let A be a C^*-algebra in the class $\mathcal{N}$. Let $u \in V(A)^*$ and suppose that S_u is metrizable (this is the case if A is separable). Then A has finite scale if and only if the monoid $V\big(\mathcal{M}(A)\big)/V\big(L(A)\big)$ is cancellative. If, further, $\mathcal{M}(A)$ has real rank zero, then A has finite scale if and only if $\mathrm{sr}\big(\mathcal{M}(A)/L(A)\big) = 1$.*

The key that led to the proof of Theorem 4.7 relies on the translation of the problem in terms of monoids. In fact, the monoid approach has become the only way to the solution we have been able to trace. Before proving the theorem we therefore establish the corresponding result in the context of Riesz monoids.

In order to deal with the restrictions of lower semicontinuous affine functions defined on a compact convex set to its extreme boundary we need a simple fact, which is in [1, Lemma II.7.1].

Lemma 4.8 *Let K be a compact convex set, and let $f, g\colon K \to \mathbb{R}$ be two lower semicontinuous affine functions. If $f|_{\partial_e K} = g|_{\partial_e K}$, then $f = g$.* ∎

Proposition 4.9 *Let M be a cancellative simple monoid and let $u \in M^*$. Suppose that M has a generating interval D with a countable cofinal subset and that (M,D) has finite scale. If I is an ideal of $M \sqcup W_\sigma^d(S_u)$ which properly contains M, then $\big(M \sqcup W_\sigma^d(S_u)\big)/I$ is cancellative.*

Proof First note that, since M is non-zero and cancellative, the state space S_u is non-empty. Therefore the proof of Proposition 4.1 shows that $L = M \sqcup \mathrm{Aff}(S_u)^{++}$ is the smallest order-ideal of $M \sqcup W_\sigma^d(S_u)$ that properly contains M.

Suppose that I is an order-ideal of $M \sqcup W_\sigma^d(S_u)$ that properly contains M. Then there exists $\varnothing \neq E \subseteq W_\sigma^d(S_u)$ such that $I = M \sqcup E$. Denote by $\sim$ the congruence modulo I. If $f+g \sim f+h$, where $f, g, h \in W_\sigma^d(S_u)$, then $f+g+x_1 = f+h+x_2$, for some elements x_1, x_2 in $M \sqcup E$. If $x_i \in E$ for $i = 1, 2$, then since the scale is finite $(g+x_1)|_{\partial_e S_u} = (h+x_2)|_{\partial_e S_u}$, and so $g+x_1 = h+x_2$ by Lemma 4.8. Thus $g \sim h$. If $x_1 \in E$ and $x_2 = x \in M$, then by a similar argument $g + x_1 = h + \phi_u(x)$, but by minimality of L we see that $\phi_u(x) \in \mathrm{Aff}(S_u)^{++} \subseteq E$, so that $g \sim h$. The other possibilities are treated in a similar fashion. ∎

When the simplex S_u is metrizable and M is a cancellative simple Riesz monoid, the converse to the above proposition is also true, namely that if $\left(M \sqcup W_\sigma^d(S_u)\right)/L$ is cancellative, then (M, D) has finite scale. In order to prove this, we need a technique consisting of dropping infinite values of some functions. It is also convenient to note the probably standard fact that $\mathrm{LAff}(K) = \mathrm{LAff}_\sigma(K)$ for a compact metrizable Choquet simplex K. Recall that, if K is a simplex and if F is a face of K, then the union of faces of K that are disjoint from F is again a face, called the *complementary face* of F denoted by F' (see [1, Proposition 2, Theorem 1] and [26, Proposition 10.12]).

Proposition 4.10 *Let K be a Choquet simplex. Let $s \in \partial_e K$ and let $f \in \mathrm{Aff}(K)^{++}$ with $f(s) \geq 1$. Denote by $\{s\}'$ the complementary face of $\{s\}$. Then there exists a function g in $\mathrm{LAff}(K)^{++}$ which equals f on $\{s\}'$ and 1 on s.*

Proof If $f(s) = 1$, take $g = f$. Hence we may assume that $f(s) > 1$. By [26, Theorem 11.28], K equals the direct convex sum of $\{s\}$ and $\{s\}'$, so there exists a unique affine function g on K such that $g|_{\{s\}'} = f|_{\{s\}'}$ and $g(s) = 1$. Note that $g \leq f$.

We have to check that g is lower semicontinuous. Notice that $h = f - g$ equals 0 on $\{s\}'$ and equals $a = f(s) - 1 > 0$ on s. If h were upper semicontinuous, then $g = f - h$ would be lower semicontinuous. Let $\lambda \in [0, \infty)$. Pick any element x in K. There exists $0 \leq \alpha \leq 1$ such that $x = \alpha s + (1-\alpha)t$, for some t in $\{s\}'$. Then $h(x) = \alpha a$. Therefore, if $\lambda > a$, we get $h^{-1}[\lambda, \infty) = \varnothing$, which is closed. Now, if $0 \leq \lambda \leq a$, consider the map:

$$\varphi \colon [\lambda, a] \times \overline{\{s\}'} \to h^{-1}[\lambda, \infty),$$

defined by the rule $\varphi(\gamma, \bar{t}) = (\gamma/a)s + (1-\gamma/a)\bar{t}$, so that

$$h\big((\gamma/a)s + (1-\gamma/a)\bar{t}\big) = \gamma + (1-\gamma/a)h(\bar{t}).$$

Now $\bar{t} = \beta s + (1-\beta)t$, where $\beta \in [0,1]$ and $t \in \{s\}'$, so that $h(\bar{t}) = \beta a$. Then $h\big((\gamma/a)s + (1-\gamma/a)\bar{t}\big) = \gamma + (a-\gamma)\beta \geq \lambda$. Therefore φ is well-defined. Also φ is continuous and onto: if $x \in h^{-1}[\lambda, \infty)$, then $x = \alpha s + (1-\alpha)t$ for some α in $[0,1]$, t in $\{s\}'$, and $\alpha a \geq \lambda$. Thus $\varphi(\alpha a, t) = x$. Since $[\lambda, a] \times \overline{\{s\}'}$ is compact, we get that $h^{-1}[\lambda, \infty)$ is closed. ∎

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Corollary 4.11 *Let K be a metrizable Choquet simplex and let $f \in \mathrm{LAff}(K)^{++}$ such that $f(s) = \infty$ for some s in $\partial_e K$. Then there exists a function g in $\mathrm{LAff}(K)^{++}$ such that $g(s) = 1$ and $f + g = 2f$.*

Proof Write $f = \sup_n f_n$, where $\{f_n\}$ is an increasing sequence in $\mathrm{Aff}(K)^{++}$. Since $f(s) = \infty$, we may assume that $f_i(s) > 1$ for all i. Let $\{s\}'$ be the complementary face of $\{s\}$ and use Proposition 4.10 to build functions g_n in $\mathrm{LAff}(K)^{++}$ such that $g_n|_{\{s\}'} = f_n|_{\{s\}'}$ and $g_n(s) = 1$. Note that $g_n \leq g_{n+1}$. Let $g = \sup_n g_n$. It follows that $g \in \mathrm{LAff}(K)^{++}$. Also, $(g + f)|_{\{s\}\cup\{s\}'} = 2f|_{\{s\}\cup\{s\}'}$ and since K is the convex hull of $\{s\} \cup \{s\}'$, we get $g + f = 2f$. ∎

By applying the above several times in a row we can drop the values of f at finitely many extreme points where f is infinite.

Corollary 4.12 *Let K be a metrizable Choquet simplex and let $f \in \mathrm{LAff}(K)^{++}$ such that there exist $s_1, \ldots, s_n$ in $\partial_e K$ with $f(s_i) = \infty$ for all i. Then there exists a function g in $\mathrm{LAff}(K)^{++}$ such that $g(s_i) = 1$ for all i and $g + f = 2f$.*

Proof Without loss of generality, we may assume that $s_1, \ldots, s_n$ are distinct. By dropping $f(s_1)$ to 1 we get g_1 in $\mathrm{LAff}(K)^{++}$ such that $g_1|_{\{s_1\}'} = f|_{\{s_1\}'}$ and $g_1(s_1) = 1$. Since $s_2 \in \{s_1\}'$, we have $g_1(s_2) = \infty$. So we can drop $g_1(s_2)$ to 1 and get a function g_2 in $\mathrm{LAff}(K)^{++}$ such that $g_2|_{\{s_2\}'} = g_1|_{\{s_2\}'}$ and $g_2(s_2) = 1$. Note that, since $s_1 \in \{s_2\}'$, we get that $g_2(s_1) = g_1(s_1) = 1$, and also $\{s_1, s_2\}' \subseteq \{s_1\}', \{s_2\}'$, so that $g_2|_{\{s_1,s_2\}'} = f|_{\{s_1,s_2\}'}$. In particular $g_2(s_3) = \infty$.

Continuing in this way we get functions g_n belonging to $\mathrm{LAff}(K)^{++}$ such that $g_n(s_i) = 1$ for all i and $g_n|_{\{s_1,\ldots,s_n\}'} = f|_{\{s_1,\ldots,s_n\}'}$. Take $g = g_n$. Since K equals the direct convex sum of the convex hull of $\{s_1, \ldots, s_n\}$ and $\{s_1, \ldots, s_n\}'$, and $(g + f)|_{\{s_1,\ldots,s_n\}\cup\{s_1,\ldots,s_n\}'} = 2f|_{\{s_1,\ldots,s_n\}\cup\{s_1,\ldots,s_n\}'}$, we get $g + f = 2f$, as wanted. ∎

The same kind of proof yields a slight extension of the previous result.

Corollary 4.13 *Let K be a metrizable Choquet simplex and let $f \in \mathrm{LAff}(K)^{++}$ such that there exist distinct states $s_1, \ldots, s_n$ in $\partial_e K$ with $f(s_i) = \infty$. Then, for fixed real numbers $a_1, \ldots, a_n > 0$, there exists a function g in $\mathrm{LAff}(K)^{++}$ such that $g(s_i) = a_i$ for all i and $g + f = 2f$.* ∎

The arguments in [29, Theorem 1.2] show that, if M is a Riesz monoid with order-unit u, then $\mathrm{St}(M, u)$ is a Choquet simplex. We will make use of this fact in the following.

Theorem 4.14 *Let M be a cancellative simple Riesz monoid and let $u \in M^*$. Suppose that M has a generating interval D with a countable cofinal subset and that S_u is metrizable. Then (M, D) has finite scale if and only if $\left(M \sqcup W_\sigma^d(S_u)\right)/L$ is cancellative (where $L = M \sqcup \mathrm{Aff}(S_u)^{++}$).*

Proof Necessity is shown in Proposition 4.9. Assume that $\left(M \sqcup W_\sigma^d(S_u)\right)/L$ is cancellative and that $d|_{\partial_e S_u}$ is not finite. Then $d(s) = \infty$ for some s in $\partial_e S_u$. Apply Corollary 4.11 to construct a function d' in $\mathrm{LAff}(S_u)^{++}$ such that $d'(s) = 1$ and $d + d' = 2d$. Then, in particular, $d' \in W_\sigma^d(S_u)$ and in the quotient modulo L we have $[d] + [d'] = 2[d]$. By assumption, $\left(M \sqcup W_\sigma^d(S_u)\right)/L$ is cancellative and consequently $[d] = [d']$. Hence there exist affine and continuous functions f_1 and f_2 on S_u such that $d + f_1 = d' + f_2$, which gives a contradiction after evaluating at s. ■

Proof of Theorem 4.7 The first part of the statement follows from Theorem 3.9 and from Theorem 4.14, taking $M = V(A)$.

If $RR\left(\mathcal{M}(A)\right) = 0$, then

$$V\left(\mathcal{M}(A)\right)/V\left(L(A)\right) \cong V\left(\mathcal{M}(A)/L(A)\right),$$

so A has finite scale if and only if $V\left(\mathcal{M}(A)/L(A)\right)$ is cancellative, and, by [8, Proposition III.2.4], this is equivalent to saying that $\mathrm{sr}\left(\mathcal{M}(A)/L(A)\right) = 1$. ■

To conclude this section we derive two consequences. Recall that a closed ideal I in a C^*-algebra B is *stably cofinite* if the algebra B/I is stably finite. In [30, Section 16], Goodearl asks if the smallest stably cofinite closed ideal of $\mathcal{M}(A)$ that contains A coincides with the smallest closed ideal of $\mathcal{M}(A)$ that properly contains A, where A is a simple non-elementary σ-unital (non-unital) C^*-algebra with bounded scale, real rank zero, stable rank one and with $V(A)$ strictly unperforated, and assuming moreover that $RR\left(\mathcal{M}(A)\right) = 0$. We solve this problem by giving a positive answer in a wider context. It also follows from Proposition 4.9 that we do not need separability for this.

Theorem 4.15 *Let A be a C^*-algebra in the class $\mathcal{N}$. If A has finite scale and $RR\left(\mathcal{M}(A)\right) = 0$, then for every closed ideal I of $\mathcal{M}(A)$ properly containing A we have that $\mathrm{sr}\left(\mathcal{M}(A)/I\right) = 1$. In particular, these ideals are stably cofinite and $L(A)$ is the smallest stably cofinite closed ideal of $\mathcal{M}(A)$ that contains A.*

Proof If I is a closed ideal of $\mathcal{M}(A)$ properly containing A, then $L(A) \subseteq I$ (by Proposition 4.1). Since the real rank of $\mathcal{M}(A)$ is zero and A has finite scale, we have that $\mathrm{sr}\left(\mathcal{M}(A)/L(A)\right) = 1$ by Theorem 4.7. Therefore $\mathcal{M}(A)/I$, being a quotient of $\mathcal{M}(A)/L(A)$, has stable rank one by [45, Theorem 4.3]. ■

Definition 4.16 Let M be a monoid. We say that M is *stably finite* provided that, whenever $x + y = y$ for x, y in M, then $x = 0$. An order-ideal I of M is said to be *stably cofinite* if M/I is a stably finite monoid.

Stably finite monoids have sometimes been called strict (see [6]). Note that a C^*-algebra A is stably finite if and only if the monoid $V(A)$ is stably finite.

A natural question is if the hypotheses of Theorem 4.15 are satisfied except that the scale is not finite, does $\mathcal{M}(A)/L(A)$ fail to be stably finite? The answer is "yes" at least in the case when d is infinite at some state s in $\partial_e S_u$ such that s lies in the closure of $\{s\}'$. By Proposition 4.10 (applied to the constant function 2), there exists a function g in $\mathrm{LAff}(S_u)^{++}$ such that $g(s) = 1$ and $g = 2$ on $\{s\}'$. Then $d + g = d + 2$ (since this holds in $\{s\} \cup \{s\}'$), so in $\big(V(A) \sqcup W_\sigma^d(S_u)\big) / \big(V(A) \sqcup \mathrm{Aff}(S_u)^{++}\big)$ we have $[d] + [g] = [d]$. Further, g is not continuous because $s \in \overline{\{s\}'}$, so $[g] \neq 0$. Using Theorem 3.9, it follows that $V\big(\mathcal{M}(A)/L(A)\big)$ is not stably finite, and therefore $\mathcal{M}(A)/L(A)$ is not stably finite.

In the case where the scale is bounded and not continuous, something more can be said about the structure of $V\big(\mathcal{M}(A)/L(A)\big)$. Recall from [30] (see also [8]) that a (finite) 1-*quasitrace* on a C^*-algebra A is a map $\tau\colon A \to \mathbb{C}$ which is linear on commutative $*$-subalgebras of A, satisfies $\tau(x^*x) = \tau(xx^*) \geq 0$ for all x in A, and $\tau(a + bi) = \tau(a) + i\tau(b)$ if $a, b \in A_{sa}$. A (finite) *quasitrace* on A is a (finite) 1-quasitrace that extends to a (finite) 1-quasitrace on $M_2(A)$, meaning that there exists a (finite) 1-quasitrace τ' on $M_2(A)$ such that $\tau'\left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right) = \tau(x)$. It follows from [8, Propositions II.4.1 and II.4.2] that any quasitrace τ extends uniquely to all matrix algebras $M_n(A)$, all these extensions being determined by τ. The norm of τ is defined as the supremum of its values on the positive unit ball of A and is known to be finite ([8, Corollary II.2.3]). We denote by $\mathrm{QT}(A)$ the set of all normalized quasitraces on A, that is, quasitraces with norm 1.

Theorem 4.17 *Let A be a C^*-algebra in the class $\mathcal{N}$. Suppose that A has bounded but not continuous scale and that the real rank of $\mathcal{M}(A)$ is zero. Let*

$$F' = \big\{ \tau \in \mathrm{QT}\big(\mathcal{M}(A)\big) \mid \tau|_A = 0 \big\}.$$

Then F' is a closed face of $\mathrm{QT}\big(\mathcal{M}(A)\big)$ and $V\big(\mathcal{M}(A)/L(A)\big) \cong (\mathrm{Aff}\, F')^+$.

Proof Note that, by Theorem 4.15, $V\big(\mathcal{M}(A)/L(A)\big)$ is cancellative. Hence

$$V\big(\mathcal{M}(A)/L(A)\big) = K_0\big(\mathcal{M}(A)/L(A)\big)^+.$$

Now [30, Theorem 16.4] says that

$$K_0\big(\mathcal{M}(A)/L(A)\big)^+ \cong (\mathrm{Aff}\, F')^+. \qquad \blacksquare$$

5 Quasitraces

The purpose of this section is to establish the exact relationship between the state space S_u and the semifinite quasitraces on A, for a simple C^*-algebra A with real rank zero. This will be used in the sequel to express conditions related to algebras without finite scale in terms of quasitraces.

The core of the section is concerned with the semifinite version of Blackadar and Handelman's result [8, Theorem III.1.3]. Many of these results are possibly known, but since we could not locate references in the literature, we provide proofs for the convenience of the reader, which show how our case is obtained from the unital case.

We give the definition of a (not necessarily finite) quasitrace, which is the obvious modification of the definition of a trace (see, *e.g.* [42, 5.2.1]), assuming also extendibility to matrices over the algebra.

Definition 5.1 Let A be a C^*-algebra. A 1-*quasitrace* on A is a map $\tau\colon A_+ \to [0,\infty]$ such that $\tau(\alpha x) = \alpha\tau(x)$ if $x \in A_+$ and $\alpha \in \mathbb{R}_+$, such that $\tau(x+y) = \tau(x) + \tau(y)$, whenever x and y are commuting elements in A_+, and such that $\tau(xx^*) = \tau(x^*x)$ for all x in A. A *quasitrace* on A is a 1-quasitrace τ that extends to a 1-quasitrace τ_n on $M_n(A)$ for each n.

Here we use the convention that $0\cdot\infty = 0$, so that $\tau(0) = 0$. Viewing A as the upper left hand corner subalgebra of $M_n(A)$, the extension τ_n in 5.1 of τ means that $\tau(x) = \tau_n(xe_{11})$, where e_{11} is the matrix unit in $M_n(\tilde{A})$.

If τ is a quasitrace, set $F_\tau = \{x \in A_+ \mid \tau(x) < \infty\}$, which is non-empty since $0 \in F_\tau$. We say that τ is *densely defined* if F_τ is dense in A_+, and we denote the set of densely defined quasitraces by $\mathrm{QT}_d(A)$. We also use $\mathrm{LQT}(A)$ to denote the set of lower semicontinuous quasitraces, while $\mathrm{LQT}_d(A)$ stands for the set of lower semicontinuous, densely defined, quasitraces. Note that all the sets considered are convex.

It is natural then to consider quasitraces that satisfy certain finiteness conditions. Some of them are closely related to the property of being order-preserving, which does not seem to follow directly from the definitions.

Lemma 5.2 *Let τ be an order-preserving quasitrace on a C^*-algebra A. Then $F'_\tau = \{x \in A \mid x^*x \in F_\tau\}$ is an ideal of A.*

Proof Since $\tau(0) = 0$, we have that F'_τ is non-empty. Let $x \in F'_\tau$ and $y \in A$. Then $\tau\big((xy)^*(xy)\big) = \tau(xyy^*x^*) \le \|y\|^2\tau(xx^*)$, because τ is order-preserving. Since $\tau(xx^*) < \infty$, we get that $xy \in F'_\tau$. Similarly $yx \in F'_\tau$. Therefore F'_τ is closed under products by elements of A. The same method of proof as in [8, Corollary II.1.11] shows that, if $a, b \in A_+$, then $\tau(a+b) \le 2\big(\tau(a) + \tau(b)\big)$. Now, if $x, y \in F'_\tau$, we have that $(x+y)^*(x+y) \le 2(x^*x + y^*y)$, whence $\tau\big((x+y)^*(x+y)\big) \le 4\big(\tau(x^*x) + \tau(y^*y)\big) < \infty$. ■

For any C^*-algebra A we denote by $K(A)$ the Pedersen ideal of A (see [42, 5.6]).

Corollary 5.3 *Let A be a C^*-algebra and let $\tau \in \mathrm{QT}_d(A)$. If τ is order-preserving, then $\tau|_{K(A)_+}$ is finite. In particular, $\tau(p) < \infty$ for any projection p in A.*

Proof F'_τ is an ideal of A by Lemma 5.2. Let $x \in F_\tau$, and note that $x^2 = x^{1/2}xx^{1/2} \le \|x\|x$. Therefore, using that τ is order-preserving we get $\tau(x^2) \le \|x\|\tau(x) < \infty$. Thus $x \in (F'_\tau)_+$, showing that $F_\tau \subset (F'_\tau)_+$. Since τ is densely defined, we get that F'_τ is dense and therefore it contains $K(A)$. Let $x \in K(A)_+$. Then $x^{1/2} \in K(A)_+$ and thus $\tau(x) = \tau(x^{1/2}x^{1/2}) < \infty$, whence $K(A)_+ \subseteq F_\tau$. Therefore the first part of the result follows. If p is a projection in A, then $p \in K(A)$ by [42, 5.6.3] and so $\tau(p) < \infty$. ■

Observe that, if τ is a quasitrace on a C^*-algebra A and if a, b are commuting elements in A_+ such that $a \le b$, then $\tau(a) \le \tau(b)$. This is clear if $\tau(b) = \infty$, while if $\tau(b) < \infty$ we have that $\tau(b) = \tau(b-a) + \tau(a) \ge \tau(a)$ since $b-a$ and a are commuting elements of A_+.

Adopting the terminology of [17], we say that a quasitrace $\tau\colon A_+ \to [0,\infty]$ on a C^*-algebra A is *semifinite* if every non-zero element in A_+ majorizes a non-zero element at which τ is finite. For $\epsilon > 0$, denote by f_ϵ the continuous function from $\mathbb{R}$ to $\mathbb{R}$ which is 0 on $(-\infty,\epsilon]$, linear on $[\epsilon,2\epsilon]$ and 1 on $[2\epsilon,\infty)$.

Proposition 5.4 *Let A be a C^*-algebra and let $\tau \in \mathrm{LQT}_d(A)$. Each one of the following conditions implies the next one:*

(1) τ is order-preserving;
(2) $\tau|_{K(A)_+}$ is finite;
(3) τ is semifinite.

Further, (1) and (2) are always equivalent; and if A is simple then they are all equivalent.

Proof (1) $\Rightarrow$ (2) follows from Corollary 5.3.

(2) $\Rightarrow$ (1). Suppose that a, b are elements in A such that $0 \le a \le b$. Then there exists a sequence $\{t_n\}$ of elements in the unit ball of A such that $a^{1/2} = \lim_{n\to\infty} t_n b^{1/2}$ (see [32, Lemma A-1]). Then $a = \lim_{n\to\infty} b^{1/2}t_n^*t_nb^{1/2}$. Since τ is lower semicontinuous we have that $\tau(a) \le \liminf_n \tau(b^{1/2}t_n^*t_nb^{1/2})$. Therefore it clearly suffices to show that $\tau(b^{1/2}c^*cb^{1/2}) \le \tau(b)$ whenever $\|c\| \le 1$. Write $b = \lim_n b_n$, where $\{b_n\}$ is an increasing sequence of commuting elements in $K(A)_+$ that also commute with b as in [42, Proof of Theorem 5.6.1]. Then $cbc^* = \lim_n cb_nc^*$, and since $B_n = \overline{b_nAb_n} \subset K(A)$, we have that $\tau|_{(B_n)_+}$ is a finite quasitrace and therefore it is order-preserving, by [8, Corollary II.2.5]. Thus

$$\tau(b^{1/2}c^*cb^{1/2}) = \tau(cbc^*) \le \liminf_n \tau(cb_nc^*) \le \liminf_n \tau(b_n) = \sup_n \tau(b_n) = \tau(b),$$

since $b_n \le b$ for all n and they commute with b and τ is lower semicontinuous.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (2). Now assume that A is simple. We first prove that $\tau\big(f_\epsilon(x)\big) < \infty$ whenever $\epsilon > 0$ and $x \in K(A)_+$.

Let x be a non-zero element in $K(A)_+$, and choose $0 \ne y$ in $K(A)_+$ with $\tau(y) < \infty$. Then $K(A) = K(A)yK(A)$, and therefore $\langle x\rangle \le n\langle y\rangle$ in $S(A)$ for some $n \ge 1$. By [48, Proposition 2.4], there exist $\delta' > 0$ and r in $M_n(A)$ such that

$$f_\epsilon(x) = r\,\mathrm{diag}\big(f_{\delta'}(x),\dots,f_{\delta'}(x)\big)\,r^*.$$

Thus, if we set $v = r\,\mathrm{diag}\big(f_{\delta'}(x),\dots,f_{\delta'}(x)\big)^{1/2}$, we have that $f_\epsilon(x) = vv^*$ and

$$v^*v = \mathrm{diag}\big(f_{\delta'}(y)^{1/2},\dots,f_{\delta'}(y)^{1/2}\big)\,r^*r\,\mathrm{diag}\big(f_{\delta'}(y)^{1/2},\dots,f_{\delta'}(y)^{1/2}\big).$$

Taking $\delta = \delta'/2$ we get $(v^*v)t = t(v^*v) = v^*v$ with $t = \mathrm{diag}\big(f_\delta(y),\dots,f_\delta(y)\big)$. Note that $\|v^*v\| = \|f_\epsilon(x)\| \le 1$ and so $v^*v \le 1$. Thus $v^*v \le t$. Hence

$$\tau\big(f_\epsilon(x)\big) = \tau(vv^*) = \tau(v^*v) \le \tau(t) = n\tau\big(f_\delta(y)\big) \le (n/\delta)\tau(y) < \infty.$$

Now choose $0 \neq x$ in $K(A)_+$ and let $\epsilon > 0$ such that $f_\epsilon(x) \neq 0$. Then $K(A) = K(A)f_\epsilon(x)K(A)$ and so $\langle x\rangle \leq n\langle f_\epsilon(x)\rangle$ in $S(A)$ for some $n \geq 1$. For each m there exists as before $\delta_m > 0$ such that $\tau\big(f_{1/m}(x)\big) \leq n\tau\Big(f_{\delta_m}\big(f_\epsilon(x)\big)\Big)$. Taking into account that $f_{\delta_m}\big(f_\epsilon(x)\big) \leq f_{\epsilon/2}(x)$ we get $\tau\big(f_{1/m}(x)\big) \leq n\tau\big(f_{\epsilon/2}(x)\big)$. Since τ is lower semicontinuous we have that

$$\tau(x) \leq \liminf_m \tau\big(xf_{1/m}(x)\big) \leq n\|x\|\tau\big(f_{\epsilon/2}(x)\big) < \infty,$$

as desired. ■

Corollary 5.5 *Let A be a simple C^*-algebra and let $\tau \in \mathrm{LQT}(A)$. If $\tau|_{K(A)_+} < \infty$ then $\tau_n|_{K(M_n(A))_+} < \infty$ for all n. In particular, the extensions τ_n are uniquely determined by τ.*

Proof It suffices to check that $\tau_2|_{K(M_2(A))_+} < \infty$. Let $a \in K\big(M_2(A)\big)_+ = M_2\big(K(A)\big)_+$, and let B be the hereditary C^*-subalgebra of A generated by the entries of a. By [42, Proposition 5.6.2], $B \subseteq K(A)$ and thus $a \in M_2(B)_+$. By [17, Lemma 5.6], B is algebraically simple, hence so is $M_2(B)$. Therefore, if $x \in B_+$ and $x \neq 0$, we have $M_2(B) = M_2(B)xe_{11}M_2(B)$. Fix $\epsilon > 0$. An argument similar to the one used in the proof of Proposition 5.4 shows that $\tau_2\big(f_\epsilon(a)\big) \leq n\tau\big(f_\delta(x)\big)$ for some $\delta > 0$ and some natural number n, and by hypothesis $\tau\big(f_\delta(x)\big) < \infty$. Now, since $\epsilon f_\epsilon(a) \leq a$, we see that τ_2 is semifinite and so $\tau_2|_{K(M_2(A))_+} < \infty$, by Proposition 5.4.

Suppose now that $\bar\tau$ and $\bar\tau'$ are lower semicontinuous 1-quasitraces on A such that $\bar\tau|_{K(A)_+} = \bar\tau'|_{K(A)_+}$. Let $x \in A_+$. Since by [42, Proof of Theorem 5.6.1] there exists an increasing sequence $\{x_n\}$ of commuting elements in $K(A)_+$ that also commute with x and with limit x, we get that $\bar\tau(x) = \sup \bar\tau(x_n) = \sup \bar\tau'(x_n) = \bar\tau'(x)$.

Finally, if τ_2' is another extension of τ to $M_2(A)_+$, then $\tau_2'|_{M_2(B)_+}$ is a finite 1-quasitrace that extends $\tau|_{B_+}$ for any singly generated hereditary C^*-subalgebra $B \subseteq K(A)$. Using [8, Propositions II.4.1 and II.4.2] we get that $\tau_2'|_{K(M_2(A))_+} = \tau_2|_{K(M_2(A))_+}$, whence $\tau_2' = \tau_2$ and τ_2 is determined by τ. ■

Let A be a C^*-algebra and let $x \in K(A)_+$. Set $Q = \big\{\tau \in \mathrm{LQT}_d(A) \;\big|\; \tau|_{K(A)_+} < \infty\big\}$ and $Q_x = \{\tau \in Q \mid \tau(x) = 1\}$. Note that, if $x \neq 0$ and A is simple, then $\mathbb{R}_+Q_x = Q$. To see this, let $\tau \in Q$. If $\tau = 0$, then clearly $\tau \in \mathbb{R}_+Q_x$. If, on the other hand, $\tau \neq 0$, then since τ is determined by its values on $K(A)_+$ (by the proof of Corollary 5.5) it follows that $\tau|_{K(A)_+} \neq 0$, and hence $\tau(x) > 0$. For, if $\tau(x) = 0$, then $\tau\big(f_\epsilon(x)\big) = 0$ for all $\epsilon > 0$. Observe that $K(A) = K(A)xK(A)$ by simplicity of A. Let $y \in K(A)_+$. Then, as in the proof of Proposition 5.4, there exists n in $\mathbb{N}$ such that for all $\epsilon > 0$, there is $\delta > 0$ and a positive number c_n satisfying $\tau\big(f_\epsilon(y)\big) \leq c_n\tau\big(f_\delta(x)\big)$. Then $\tau\big(f_\epsilon(y)\big) = 0$ for all $\epsilon > 0$. Since τ is lower semicontinuous and $y = \lim_n f_{1/2n}(y)yf_{1/n}(y)$, we finally get $\tau(y) \leq \liminf_n \|y\|\tau\big(f_{1/n}(y)\big) = 0$. Thus $\tau|_{K(A)_+} = 0$, a contradiction to $\tau(x) = 0$. Thus $\tau = \tau(x)\big(\tau/\tau(x)\big) \in \mathbb{R}_+Q_x$, and therefore $\mathbb{R}_+Q_x = Q$.

It follows from the proof of Corollary 5.5 that elements of Q which agree on $K(A)_+$ are equal. Therefore $(Q, +)$ is cancellative. Let $X = Q - Q = \{\tau - \tau' \mid \tau, \tau' \in$

$Q\}$, which is a real linear space. As in [17, Section 7], we endow X with the locally convex topology generated by the semi-norms $p_x\colon X \to \mathbb{R}$, $\varphi \mapsto |\varphi(x)|$, where $x \in K(A)_+$, which is called the *weak topology* on X. Since equality of elements in Q is determined by equality on $K(A)_+$, the semi-norms $\{p_x\}$ form a separating family, so that X becomes a Hausdorff space. If A is simple, then Q_x is weakly compact.

Our goal is to show that, if A is a simple σ-unital C^*-algebra with real rank zero and p is a non-zero projection in A, then the spaces Q_p and S_u, where $u = [p] \in V(A)$, are affinely homeomorphic. This was observed in [36, Section 2] without proof, and for C^*-algebras that, moreover, have stable rank one. For convenience of the reader we give an argument which does not need cancellation and is partly based on that used in [30, Theorem 12.3].

Theorem 5.6 **(Blackadar-Handelman)** *Let A be a simple σ-unital C^*-algebra with real rank zero. Let p be a non-zero projection in A and set $u = [p]$ in $V(A)$. Then there exists an affine homeomorphism*

$$\alpha\colon Q_p \to S_u,$$

such that $\alpha(\tau)([q]) = \tau(q)$ for all τ in Q_p and all projections q in $M_\infty(A)$.

Proof It is clear that the map α is affine and continuous. Since both Q_p and S_u are compact spaces, it only remains to show that α is bijective.

Suppose that $\alpha(\tau) = \alpha(\tau')$ for some τ, τ' in Q_p. Then τ and τ' agree on all projections of A. Let $x \in K(A)_+$. Then $B = \overline{xAx} \subseteq K(A)$, hence $\tau|_{B_+}$ and $\tau'|_{B_+}$ are finite quasitraces on B. Note that B has real rank zero, and since τ and τ' coincide on all projections of B, they must agree on all B_+ by uniform continuity (see [8, Corollary II.2.5]). In particular, $\tau(x) = \tau'(x)$. Therefore $\tau|_{K(A)_+} = \tau'|_{K(A)_+}$. Now, given any element x in A_+, there exists an increasing sequence of commuting elements x_n in $K(A)_+$ (which also commute with x) such that $x = \lim_n x_n$. Then $\tau(x) = \sup_n \tau(x_n) = \sup_n \tau'(x_n) = \tau'(x)$, again by [8, Corollary II.2.5]. This shows that α is injective.

Let $s \in S_u$ and fix an (increasing) approximate unit $\{p_n\}$ of A consisting of projections. Since A has real rank zero and $s \neq 0$, we have that $\gamma_n := s[p_n] > 0$ if n is large enough. Therefore we may assume without loss of generality that $\gamma_n > 0$ for all n in $\mathbb{N}$. Denote by $j_n\colon p_nAp_n \to A$ the inclusion maps, which induce monoid morphisms $V(j_n)\colon V(p_nAp_n) \to V(A)$. Note that $\gamma_n^{-1}sV(j_n) \in \mathrm{St}\big(V(p_nAp_n), [p_n]\big)$. By [8, Theorem III.1.3] (or also [9, Theorem 3.5]), there exists a (finite) quasitrace τ_n in $\mathrm{QT}(p_nAp_n)$ such that $\tau_n(q) = \gamma_n^{-1}s[q]$ for all projections q in $M_\infty(p_nAp_n)$, that is, $\gamma_n\tau_n(q) = s[q]$ for all projections q in $M_\infty(p_nAp_n)$.

Since $p_nAp_n \subseteq p_{n+1}Ap_{n+1}$ for all n and have real rank zero, we see that $\gamma_n\tau_n$ and $\gamma_{n+1}\tau_{n+1}$ agree on p_nAp_n. Therefore the maps $\gamma_n\tau_n$ induce a map $\bar\tau\colon \cup_n p_nAp_n \to [0,\infty)$, which is a 1-quasitrace. It remains to extend $\bar\tau$ to A_+. This cannot be done by uniform continuity since the sequence $\{\gamma_n\}$ need not be bounded. Set $\tau(x) = \sup_n \bar\tau(p_nxp_n)$, which defines a map $\tau\colon A_+ \to [0,\infty]$. We now show that $\tau(xx^*) = \tau(x^*x)$ whenever $x \in A$. Fix $n \leq m$. We compute that

$$\bar\tau(p_nx^*p_mxp_n) = \bar\tau(p_mxp_nx^*p_m) \leq \bar\tau(p_mxx^*p_m) \leq \tau(xx^*).$$

Since $\bar{\tau}|_{p_nAp_n} = \gamma_n\tau_n$ is continuous we get $\bar{\tau}(p_nx^*xp_n) = \lim_{m\to\infty} \bar{\tau}(p_nx^*p_mxp_n) \le \tau(xx^*)$, whence it follows that $\tau(x^*x) \le \tau(xx^*)$, and by symmetry $\tau(xx^*) = \tau(x^*x)$.

It is clear that, if $\alpha > 0$ and $x \in A_+$, then $\tau(\alpha x) = \alpha\tau(x)$. Suppose that x, y are commuting elements in A_+. Let $\delta > 0$. Then there exists n_0 in $\mathbb{N}$ such that, if $n \ge n_0$,

$$\|p_nxp_nyp_n - p_nyp_nxp_n\| \le \|xp_ny - yp_nx\| < \delta.$$

Let $k \in \mathbb{N}$. By [8, Corollary II.2.6], there exists $\delta = \delta(k)$ such that, if $\|xp_ny - yp_nx\| < \delta$, then $\left|\bar{\tau}\big(p_n(x+y)p_n\big) - \bar{\tau}(p_nxp_n) - \bar{\tau}(p_nyp_n)\right| < (1/2^k)$. Therefore there exists n_0 such that $\left|\bar{\tau}\big(p_n(x+y)p_n\big) - \bar{\tau}(p_nxp_n) - \bar{\tau}(p_nyp_n)\right| < (1/2^k)$ whenever $n \ge n_0$. We conclude that $\tau(x+y) \le (1/2^k) + \tau(x) + \tau(y)$, whence $\tau(x+y) \le \tau(x) + \tau(y)$. The converse inequality is proved similarly. Thus $\tau(x+y) = \tau(x) + \tau(y)$.

We now check the lower semicontinuity of τ. Let $\{x_k\}$ be a sequence in A_+ converging to an element x in A_+. For each n in $\mathbb{N}$ we have $\bar{\tau}(p_nxp_n) = \lim_k \bar{\tau}(p_nx_kp_n) \le \liminf_k \tau(x_k)$, since $\bar{\tau}(p_nx_kp_n) \le \tau(x_k)$. Thus $\tau(x) \le \liminf_k \tau(x_k)$, whence τ is lower semicontinuous.

Let $x \in A_+$. If $n \in \mathbb{N}$, we have $x^{1/2}p_nx^{1/2} \le x$, and so $\tau(x^{1/2}p_nx^{1/2}) = \tau(p_nxp_n) \le \|x\|\tau(p_n) = \|x\|\gamma_n < \infty$. Thus τ is semifinite. Also, since $p \sim p'$ for some projection p' in p_nAp_n and some n in $\mathbb{N}$, we have that $\tau(p) = \tau(p') = \gamma_n\tau_n(p') = s[p'] = s[p] = 1$.

Note now that $\cup_n p_nAp_n \subseteq F_\tau$. Hence τ is densely defined. The extensions of $\gamma_n\tau_n$ to $M_k(p_nAp_n)_+$ induce a 1-quasitrace on $M_k(A)_+$ for each k, which is an extension of τ. Therefore $\tau \in Q_p$. Finally, if q is a projection in $M_\infty(A)$, then q is equivalent to a projection q' in $M_\infty(p_nAp_n)$ (for some n). Thus $\tau(q) = \tau(q') = \gamma_n\tau_n(q') = s[q'] = s[q]$, and so α is surjective. ∎

As an application we analyze the structure of the simple C^*-algebras with bounded scale. Blackadar showed in [4, Theorem 4.8] that a simple separable AF algebra has bounded scale if and only if it is algebraically simple. Lin proved later that, if A is a simple σ-unital C^*-algebra with continuous scale then it is algebraically simple ([35, Theorem 3.3]). In the following we extend Blackadar's result to the class of σ-unital simple C^*-algebras with real rank zero and stable rank one.

Theorem 5.7 *Let A be a simple σ-unital C^*-algebra with real rank zero and stable rank one. Assume that $V(A)$ is strictly unperforated. Then the following are equivalent:*

(1) A is algebraically simple;
(2) Every semifinite quasitrace in $\mathrm{LQT}_d(A)$ is finite;
(3) A has bounded scale.

Proof Let p be a non-zero projection in A, set $u = [p]$ in $V(A)$ and $d = \sup \phi_u\big(D(A)\big)$. If $\{e_n\}$ is an approximate unit consisting of projections, then $d = \sup \phi_u([e_n])$.

(1) $\Rightarrow$ (2). Let $\tau \in \mathrm{LQT}_d(A)$ and assume that it is semifinite. Since A is algebraically simple, we have that $K(A) = A$ and by Proposition 5.4, $\tau|_{K(A)_+} < \infty$ whence τ is finite.

(2) $\Rightarrow$ (3). By Lemma 4.4, if d is not bounded then there exists a state s in S_u such that $d(s) = +\infty$, that is, $\sup_n s[e_n] = +\infty$. Since $\mathbb{R}_+ Q_p = Q$ and Q equals the set of semifinite quasitraces in $\mathrm{LQT}_d(A)$ by hypothesis, we have that Q_p consists of the finite quasitraces τ on A such that $\tau(p) = 1$. Let τ be the quasitrace in Q_p that corresponds to s via the homeomorphism α of Theorem 5.6. By [8, Corollary II.2.3], the norm of τ must be bounded, but this contradicts the fact that $\sup \tau(e_n) = \sup s[e_n] = +\infty$.

(3) $\Rightarrow$ (1). Assume that $d \ll k$ for some constant bound k. For every s in S_u we have $d(s) = \sup s[e_n] < k = ks[p]$, whence $s[e_n] < ks[p]$ for all $n \geq 1$. Since $V(A)$ is strictly unperforated and cancellative, we have that $[e_n] \leq k[p]$ in $V(A)$ for all $n \geq 1$ (see Lemma 3.6). Let $x = \sum_{n=1}^{\infty}(1/2^n)e_n$ and note that $x \in A$. It follows from [43, Proposition 2.3] that $\langle x\rangle \leq k\langle p\rangle$ in $S(A)$. Let $B \subseteq M_k(A)$ be the hereditary C^*-subalgebra generated by $\mathrm{diag}(p, \dots, p) \in M_k(A)$. Note that B is simple, unital, with real rank zero and stable rank one, and in particular B is algebraically simple.

Observe now that, if $n \in \mathbb{N}$ then there exists a projection e'_n in $M_k(A)$ such that $e_n \sim e'_n \leq \mathrm{diag}(p, \dots, p)$. We therefore obtain a sequence of projections $\{e'_n\}$ in B such that $e'_n \lesssim e'_{n+1}$ for all $n \geq 1$. Using cancellation of projections it is possible to construct an increasing sequence of projections $\{q_n\}$ in B with $q_n \sim e'_n \sim e_n$ for all $n \geq 1$ (see, *e.g.* [43, Proof of Proposition 2.7]). Let $y = \sum_{n=1}^{\infty}(1/2^n)q_n$, an element of B. Then $\langle x\rangle = \langle y\rangle$ in $S(A)$ by [43, Proposition 2.3] and therefore $x \sim_s y$ ([43, Corollary 2.4]), that is, there exists a in $M_k(A)$ such that $\overline{yBy} = \overline{a^*M_k(A)a}$, while $\overline{aM_k(A)a^*} = \overline{xM_k(A)x} \cong \overline{xAx}$, and the latter equals A since x is a strictly positive element. By [17, Lemma 5.6] $\overline{yBy} \subseteq B$ is algebraically simple, and since $\overline{a^*M_k(A)a}$ and $\overline{aM_k(A)a^*}$ are isomorphic (*e.g.* extending by continuity the isomorphism given in [14, 1.4]), we conclude that A is algebraically simple. ∎

6 Ideals in the Corona Algebra

Lin has shown that, if A is a non-unital and non-elementary separable AF C^*-algebra with a finite number of extremal semifinite traces of which n are infinite, then $\mathcal{M}(A)/A$ has exactly 2^n closed ideals ([34, Theorem 2]). An extension of this result was established by Rørdam in [47, Theorem 4.4] for C^*-algebras of the form $A \otimes K$, where A is a simple unital infinite dimensional C^*-algebra with a certain comparison property and K stands for the C^*-algebra of compact operators over a separable infinite dimensional Hilbert space.

We show in the current section that a similar pattern can be adopted in our setting, thus including Lin's result. In particular, we remove the assumption of the algebra having finitely many extremal semifinite traces. We continue to work with the class $\mathcal{N}$ consisting of all σ-unital (non-unital) simple C^*-algebras A that are non-elementary, with real rank zero, stable rank one and such that $V(A)$ is strictly unperforated.

Let A be a C^*-algebra. In [46, Propositions 4.1–4.3] it is shown that A has a maximal closed ideal $I_{\mathrm{sr}\,1}(A)$ of stable rank one, that can be determined by

$$I_{\mathrm{sr}\,1}(A) = \{a \in A \mid a + (\tilde{A}^{-1})^- = (\tilde{A}^{-1})^-\},$$

where $\tilde{A}^{-1}$ denotes the set of invertible elements in $\tilde{A}$. We give below a description of this ideal for the algebra $\mathcal{M}(A)/L(A)$ under certain additional hypothesis on A, where $L(A)$ stands for the smallest closed ideal of $\mathcal{M}(A)$ that properly contains A.

Proposition 6.1 *Let A be a separable C^*-algebra in the class $\mathcal{N}$. There exists a unique closed ideal $I_{\mathrm{fin}}(A)$ of $\mathcal{M}(A)$ among the ideals properly containing A which is maximal with respect to the property that $V\big(I_{\mathrm{fin}}(A)\big)/V\big(L(A)\big)$ is cancellative. If $RR\big(\mathcal{M}(A)\big) = 0$, then $I_{\mathrm{fin}}(A)/L(A) = I_{\mathrm{sr}\,1}\big(\mathcal{M}(A)/L(A)\big)$.*

Proof Fix u in $V(A)^*$. Let $D = D(A)$ and $d = \sup \phi_u(D)$. Let

$$I_{\mathrm{fin}} := V(A) \sqcup E_{\mathrm{fin}}, \text{ where } E_{\mathrm{fin}} = \{ f \in W^d_\sigma(S_u) \mid f|_{\partial_e S_u} \text{ is finite } \}.$$

Then I_{fin} is an order-ideal of $V(A) \sqcup W^d_\sigma(S_u)$. We shall establish the following statement: I_{fin} is the largest order-ideal of $V(A) \sqcup W^d_\sigma(S_u)$ with respect to the property that I_{fin}/L is cancellative, where $L = V(A) \sqcup \mathrm{Aff}(S_u)^{++}$. Then the first part of the result will follow from Theorems 2.1 and 3.9, taking $I_{\mathrm{fin}}(A)$ as the closed ideal such that $V\big(I_{\mathrm{fin}}(A)\big) \cong I_{\mathrm{fin}}$. First we prove that I_{fin}/L is cancellative. Suppose that $[f] + [g] = [f] + [h]$ in I_{fin}/L, for f, g, h in I_{fin}. Then there exist l_1, l_2 in L such that $f + g + l_1 = f + h + l_2$. We may assume that l_1 and l_2 are affine and continuous functions. Restricting to the extreme boundary we get $(g + l_1)|_{\partial_e S_u} = (h + l_2)|_{\partial_e S_u}$. Then $g + l_1 = h + l_2$ by Lemma 4.8 and consequently $[g] = [h]$ in I_{fin}/L.

Now suppose that $V(A) \sqcup E$ is another order-ideal of $V(A) \sqcup W^d_\sigma(S_u)$ such that $(V(A) \sqcup E)/L$ is cancellative. We claim that $E \subseteq E_{\mathrm{fin}}$.

If not, there exists f in E such that $f(s) = \infty$ for some s in $\partial_e S_u$. By Corollary 4.11 we can construct a function g in $\mathrm{LAff}(S_u)^{++}$ such that $g(s) = 1$ and $g + f = 2f$. Since $f \in E$ and $g \leq 2f$ algebraically, we conclude that $g \in E$. Therefore we have $[g] + [f] = [f] + [f]$ in $\big(V(A) \sqcup E\big)/L$. By cancellation $[g] = [f]$, and so there exist affine continuous functions l_1 and l_2 satisfying $g + l_1 = f + l_2$, which gives a contradiction after evaluating at s.

If $RR\big(\mathcal{M}(A)\big) = 0$, then

$$V\big(I_{\mathrm{fin}}(A)/L(A)\big) \cong V\big(I_{\mathrm{fin}}(A)\big)/V\big(L(A)\big) \cong I_{\mathrm{fin}}/L.$$

Hence $I_{\mathrm{fin}}(A)$ is the largest closed ideal of $\mathcal{M}(A)$ with respect to the property that $V\big(I_{\mathrm{fin}}(A)/L(A)\big)$ is cancellative, which is equivalent to $\mathrm{sr}\big(I_{\mathrm{fin}}(A)/L(A)\big) = 1$ by [8, Proposition III.2.4]. Thus $I_{\mathrm{fin}}(A)/L(A) = I_{\mathrm{sr}\,1}\big(\mathcal{M}(A)/L(A)\big)$. ∎

We shall call $I_{\mathrm{fin}}(A)$ the *finite ideal* of $\mathcal{M}(A)$. Observe that A has finite scale precisely when $I_{\mathrm{fin}}(A) = \mathcal{M}(A)$.

Definition 6.2 Let A be a C^*-algebra. We say that a lower semicontinuous and order-preserving quasitrace τ is *infinite* if $\sup_\lambda \tau(u_\lambda) = +\infty$ for some approximate unit $\{u_\lambda\}_{\lambda\in\Lambda}$ of A.

Note that this definition does not depend on the particular approximate unit. Note also that, if A has real rank zero and τ is infinite, then $\sup \tau(p) = \infty$, where the supremum is taken over the projections p of A.

In the following and subsequent results, we shall identify without further comment the spaces S_u and Q_p, where p is a projection in a simple, σ-unital C^*-algebra A with real rank zero and $u = [p] \in V(A)$, using Theorem 5.6.

Theorem 6.3 *Let A be a separable C^*-algebra in $\mathcal{N}$ and let p be a non-zero projection in A. Assume that A has exactly n infinite extremal quasitraces in Q_p. Then there exist precisely 2^n closed ideals between $I_{\mathrm{fin}}(A)$ and $\mathcal{M}(A)$.*

Proof Set $u = [p]$ in $V(A)$ and $d = \sup \phi_u\big(D(A)\big)$. Notice first that A has exactly n infinite extremal quasitraces in Q_p if and only if the cardinality of the set

$$\Gamma_d := \{s \in \partial_e S_u \mid d(s) = +\infty\}$$

is n, by virtue of Theorem 5.6.

For each subset $\alpha \subseteq \Gamma_d$, define

$$E_\alpha = \{f \in W_\sigma^d(S_u) \mid f|_{\partial_e S_u} \text{ is finite outside } \alpha\}$$

and let I_α be the closed ideal of $\mathcal{M}(A)$ such that $\varphi\big(V(I_\alpha)\big) = V(A) \sqcup E_\alpha$, where φ is the isomorphism constructed in Theorem 3.9. By Corollary 4.12 the ideals $V(I_\alpha)$ form a set of 2^n different order-ideals of $V(A) \sqcup W_\sigma^d(S_u)$. Notice that $\varphi\big(V(I_\varnothing)\big) = I_{\mathrm{fin}}$ and $\varphi\big(V(I_{\Gamma_d})\big) = V(A) \sqcup W_\sigma^d(S_u)$. Assume that $V(A) \sqcup E$ is an order-ideal in $V(A) \sqcup W_\sigma^d(S_u)$ for some $E \subseteq W_\sigma^d(S_u)$ that contains E_{fin}. Let $\alpha \subseteq \Gamma_d$ be the largest subset of Γ_d such that $f|_{\partial_e S_u}$ is finite outside α for every function f in E. Such an α does exist due to the finiteness of Γ_d. By construction, $E \subseteq E_\alpha$, and we claim that $E = E_\alpha$.

We may assume that $E_{\mathrm{fin}} \subsetneqq E$. Let d' be the function in $\mathrm{LAff}(S_u)^{++}$, constructed using Corollary 4.12, that satisfies $d'|_{\Gamma_d'} = d|_{\Gamma_d'}$ and $d'(s) = 1$ for all s in Γ_d, where Γ_d' is the complementary face of the closed face (of S_u) generated by Γ_d (the latter is the convex hull of Γ_d since Γ_d is finite). Note that $d' + d = 2d$. Thus $d' \in V(A) \sqcup W_\sigma^d(S_u)$ and in fact $d' \in I_{\mathrm{fin}} \subset V(A) \sqcup E$. Also, by construction of α, there exists a function f in E whose restriction to $\partial_e S_u$ is infinite precisely on α. Let d_α be the "α-dropping" of d. That is, if $(\Gamma_d \setminus \alpha)'$ denotes the complementary face of the (closed) face generated by $\Gamma_d \setminus \alpha$, then d_α is constructed in such a way that $d_\alpha|_{(\Gamma_d\setminus\alpha)'} = d|_{(\Gamma_d\setminus\alpha)'}$ and $d_\alpha(s) = 1$ for all s in $\Gamma_d \setminus \alpha$.

Since $\Gamma_d' \subseteq (\Gamma_d \setminus \alpha)'$, we get $d_\alpha|_{\Gamma_d'} = d|_{\Gamma_d'}$. Notice also that $d_\alpha(s) = \infty$ for all s in α. Therefore $(d_\alpha + f)_{\Gamma_d \cup \Gamma_d'} = (d' + f)_{\Gamma_d \cup \Gamma_d'}$, and since S_u equals the direct convex sum of the (closed) face generated by Γ_d and its complementary face Γ_d', we get $d_\alpha + f = d' + f$. Hence $d_\alpha + f \in E$, and consequently $d_\alpha \in E$.

The only thing that we need to check in order to finish the proof is that $E_\alpha = F_\alpha$, where

$$F_\alpha := \{f \in W_\sigma^d(S_u) \mid f + g = md_\alpha \text{ for some } m \text{ in } \mathbb{N} \text{ and } g \text{ in } W_\sigma^d(S_u)\}.$$

Since $d_\alpha \in E$ and $E \subseteq E_\alpha$ we get that $d_\alpha \in E_\alpha$. Therefore the order-ideal generated by d_α is contained in $V(A) \sqcup E_\alpha$, and thus $F_\alpha \subseteq E_\alpha$. Conversely, let $g \in E_\alpha$. In particular, since $E_\alpha \subseteq W_\sigma^d(S_u)$, there exist h in $W_\sigma^d(S_u)$ and a natural number m such that $g + h = md$. We know that $g|_{\partial_e S_u}$ is infinite at most on α. By adding copies of d if necessary, we may assume that $g(t) < m$ for all t belonging to $\Gamma_d \cap \{s \in \partial_e S_u \mid g(s) < \infty\}$. Now we use Corollary 4.13 to get h' in $W_\sigma^d(S_u)$ such that $h'(t) = m - g(t)$ if $t \in \Gamma_d \setminus \alpha$

and $h'|_{(\Gamma_d\setminus\alpha)'} = h|_{(\Gamma_d\setminus\alpha)'}$. We also have $h + h' = 2h$. Now $(g + h')|_{(\Gamma_d\setminus\alpha)\cup(\Gamma_d\setminus\alpha)'} = md_\alpha|_{(\Gamma_d\setminus\alpha)\cup(\Gamma_d\setminus\alpha)'}$, whence $g + h' = md_\alpha$, as desired. ■

We now investigate the quotients $\mathcal{M}(A)/J$ for any closed ideal J of $\mathcal{M}(A)$ that contains $I_{\text{fin}}(A)$, showing that they have a particular structure.

Let M be a monoid. We say that M is *purely infinite* provided that, for any non-zero element x in M, there is $0 \neq y$ in M such that $x + y = x$. Observe that, in case A is a C^*-algebra with real rank zero, then A is purely infinite if and only if $V(A)$ is a purely infinite monoid. Following [16] (see also [54], [39]), a C^*-algebra A is purely infinite provided that every non-zero hereditary C^*-subalgebra of A contains an infinite projection. The following observation is noted in [54, Theorems 1.1 and 1.3]. For our purposes, we find convenient to state it separately as a lemma.

Lemma 6.4 *Let A be a σ-unital C^*-algebra with real rank zero. Let $I \subsetneqq J$ be closed ideals of $\mathcal{M}(A)$. Then every hereditary C^*-subalgebra of J/I is the closed linear span of images of projections via the natural map $\pi\colon J \to J/I$. In particular, any non-zero projection of J/I contains a non-zero subprojection which is the image of a projection in J.*

Proof If B is a hereditary C^*-subalgebra of J/I, then $\pi^{-1}(B)$ is hereditary in $J \subseteq \mathcal{M}(A)$. Then, by [54, Theorem 1.1], we have that $\pi^{-1}(B)$ is the closed linear span of its projections and therefore the first assertion follows.

If p is a non-zero projection in J/I, then apply the first part to the hereditary C^*-subalgebra $B = p(J/I)p \subseteq J/I$. ■

Proposition 6.5 *Let A be a separable C^*-algebra in the class $\mathcal{N}$ and let p be a non-zero projection of A. Assume that A has exactly n infinite extremal quasitraces in Q_p. Then $\mathcal{M}(A)/J$ is a purely infinite C^*-algebra for any closed ideal J of $\mathcal{M}(A)$ that contains $I_{\text{fin}}(A)$. Moreover, if $RR\big(\mathcal{M}(A)\big) = 0$ then $V\big(\mathcal{M}(A)/I_{\text{fin}}(A)\big)$ is isomorphic to $\{2^n, \cup\}$, where 2^n is the Boolean algebra of subsets of an n-element set.*

Proof By Lemma 6.4, in order to show that $\mathcal{M}(A)/J$ is purely infinite, it is enough to show that every projection $\pi(q)$ of $\mathcal{M}(A)/J$ is infinite, where $\pi\colon \mathcal{M}(A) \to \mathcal{M}(A)/J$ is the natural map and q is a non-zero projection of $\mathcal{M}(A)$. For this, it suffices to prove that $V\big(\mathcal{M}(A)\big)/V(J)$ is a purely infinite monoid.

As in Theorem 6.3, set $u = [p]$ in $V(A)$ and $d = \sup \phi_u\big(D(A)\big)$. Then the set Γ_d has cardinality n. Recall that, if φ is the isomorphism constructed in Theorem 3.9, then $\varphi\Big(V\big(I_{\text{fin}}(A)\big)\Big) = V(A) \sqcup E_{\text{fin}}$, where

$$E_{\text{fin}} = \big\{\, f \in W_\sigma^d(S_u) \;\big|\; f|_{\partial_e S_u} < +\infty \big\},$$

as in Proposition 6.1. Write $\varphi\big(V(J)\big) = V(A) \sqcup E$, where $E_{\text{fin}} \subseteq E \subseteq W_\sigma^d(S_u)$. Let $f \in W_\sigma^d(S_u) \setminus E$. Then there exists s in Γ_d such that $f(s) = \infty$. Since $f \in W_\sigma^d(S_u)$ and Γ_d is a finite set, we have that $f|_{\partial_e S_u}$ is infinite at most at n points. By Corollary 4.12, there exists a function g in $\mathrm{LAff}(S_u)^{++}$ such that $g(s) = 1$ whenever $f(s) = \infty$ (for s

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in $\partial_e S_u$) and $g + f = 2f$. Note that $g \in V(A) \sqcup E_{\text{fin}} \subseteq V(A) \sqcup E$, and in the quotient modulo $V(A) \sqcup E$ we have $[f] = 2[f]$. Therefore $V\big(\mathcal{M}(A)\big)/V(J)$ is a purely infinite monoid, as desired.

Now assume that $RR\big(\mathcal{M}(A)\big) = 0$. To prove the second part of the statement, let Σ be the monoid of all the subsets of Γ_d with set union as addition. For each α in Σ, construct d_α in $W_\sigma^d(S_u)$ as in the proof of Theorem 6.3 (that is, $d_\alpha(s) = 1$ for all s in $\Gamma_d \setminus \alpha$ and $d_\alpha|_{(\Gamma_d\setminus\alpha)'} = d|_{(\Gamma_d\setminus\alpha)'}$), and let $d' = d_\varnothing$. Notice that $d_\alpha + d' = 2d_\alpha$ and in general by induction $md_\alpha = d_\alpha + (m-1)d'$ for all m. Therefore we have a map

$$\Sigma \to \big(V(A) \sqcup W_\sigma^d(S_u)\big)/I_{\text{fin}},$$

defined by $\alpha \mapsto [d_\alpha]$. It is easy to check that $(d_\alpha + d_\beta)|_{\partial_e S_u} = (d_{\alpha\cup\beta} + d')|_{\partial_e S_u}$ for any α, β in Σ. It follows then that $d_\alpha + d_\beta = d_{\alpha\cup\beta} + d'$, and hence in $\big(V(A) \sqcup W_\sigma^d(S_u)\big)/I_{\text{fin}}$ we have that $[d_\alpha] + [d_\beta] = [d_{\alpha\cup\beta}]$. Thus the map just defined is a monoid morphism, which is clearly injective. To see that it is surjective, and hence a monoid isomorphism, let $[f] \in \big(V(A) \sqcup W_\sigma^d(S_u)\big)/I_{\text{fin}}$. Then there exists $\alpha \subseteq \Gamma_d$ such that $f \in E_\alpha$. Actually α is unique if we take it to be the smallest possible (that is, $f|_{\partial_e S_u}$ is infinite precisely on α).

There exist a natural number m and a function g in $W_\sigma^d(S_u)$ such that $f+g = md_\alpha$. Let g' be the element in $W_\sigma^d(S_u)$ obtained by dropping to 1 the values at which $g|_{\partial_e S_u}$ is infinite. Since $g|_{\partial_e S_u}$ is infinite at most on α, we easily get $f + g = f + g'$. Thus $f+g' = f+g = md_\alpha = d_\alpha+(m-1)d'$, whence $[f] = [d_\alpha]$ in $\big(V(A)\sqcup W_\sigma^d(S_u)\big)/I_{\text{fin}}$. ∎

Note that, under the assumptions of Proposition 6.5 and in the particular case that A has exactly *one* infinite extremal quasitrace, we have that $V\big(\mathcal{M}(A)/I_{\text{fin}}(A)\big)$ is isomorphic to $\{0, \infty\}$, and this implies that every two non-zero projections in $\mathcal{M}(A)/I_{\text{fin}}(A)$ are equivalent.

We turn our attention now to the case where the C^*-algebra can have infinitely many extremal quasitraces which are infinite. The first observation to be made is that some of the arguments used before can be easily adapted to our present situation. The following generalizes [34, Theorem 3] and [33, Proposition 4.17].

Theorem 6.6 *Let A be a separable C^*-algebra in the class $\mathcal{N}$ and let p be a non-zero projection in A. Let $\mathfrak{c}$ be the cardinal of infinite extremal quasitraces in Q_p and assume that $\mathfrak{c}$ is infinite. Then $\mathcal{M}(A)$ has at least $\mathfrak{c}$ maximal ideals that properly contain $I_{\text{fin}}(A)$, and the quotient of $\mathcal{M}(A)$ by any of these ideals is a purely infinite simple C^*-algebra. Moreover, $\mathcal{M}(A)$ contains an infinite strictly decreasing sequence of closed ideals that contain $I_{\text{fin}}(A)$.*

Proof Let $u = [p] \in V(A)$ and $d = \sup \phi_u\big(D(A)\big)$. Note that, by Theorem 5.6, the cardinality of the set Γ_d defined in the proof of Theorem 6.3 is exactly $\mathfrak{c}$. For any s in Γ_d, let I_s be the closed ideal of $\mathcal{M}(A)$ such that $\varphi\big(V(I_s)\big) = V(A) \sqcup E_s$, where $E_s := \{f \in W_\sigma^d(S_u) \mid f(s) < \infty\}$ and φ is the isomorphism constructed in Theorem 3.9. Note that $I_{\text{fin}} \subsetneqq V(I_s)$, since it is possible to construct a function

d_s in $W^d_\sigma(S_u)$ such that $d_s(s) = 1$ and $d_s|_{\{s\}'} = d|_{\{s\}'}$, using Corollary 4.11. Then $d_s \in V(A) \sqcup E_s$ and $d_s \notin I_{\text{fin}}$.

Let $t, s \in \Gamma_d$ and suppose that $t \neq s$. By construction $d_s(t) = \infty$ and therefore $I_s \neq I_t$, whence all the ideals under consideration are different.

Suppose that there is an order-ideal $V(A) \sqcup E$ of $V(A) \sqcup W^d_\sigma(S_u)$ such that $V(A) \sqcup E_s$ is properly contained in $V(A) \sqcup E$. Then there exists f in $(V(A) \sqcup E) \setminus (V(A) \sqcup E_s)$, so $f(s) = \infty$. Since $f \in W^d_\sigma(S_u)$, there exist n in $\mathbb{N}$ and g in $W^d_\sigma(S_u)$ such that $f + g = nd$. If $g(s) < \infty$ then both f and g belong to E, so $nd \in E$ and hence $d \in E$. If, on the other hand, $g(s) = \infty$, let g' in $W^d_\sigma(S_u)$ be defined by $g'(s) = 1$ and $g'|_{\{s\}'} = g|_{\{s\}'}$, by Corollary 4.11. Then $g' \in E$ and $(f + g')|_{\{s\} \cup \{s\}'} = (f + g)|_{\{s\} \cup \{s\}'}$, so $f + g' = f + g = nd$, whence it follows again that $d \in E$. This implies that $E = W^d_\sigma(S_u)$. Thus I_s is a maximal ideal.

To construct an infinite strictly descending sequence of closed ideals that contain $I_{\text{fin}}(A)$, let $\{s_n\}$ be a sequence of different elements in Γ_d and let I_n be the closed ideal of $\mathcal{M}(A)$ such that

$$\varphi\big(V(I_n)\big) = V(A) \sqcup \{ f \in W^d_\sigma(S_u) \mid f(s_i) < \infty \text{ for all } i \leq n \}.$$

Then $V(I_n)$ form a sequence of order-ideals in $V\big(\mathcal{M}(A)\big)$ that contain $V\big(I_{\text{fin}}(A)\big)$, and $V(I_1) \supsetneqq V(I_2) \supsetneqq V(I_3) \supsetneqq \cdots$. Consequently $I_1 \supsetneqq I_2 \supsetneqq I_3 \supsetneqq \cdots$, and $I_n \supseteq I_{\text{fin}}(A)$ for all n.

To see that $\mathcal{M}(A)/I_s$ is purely infinite simple it is enough to check, as in Proposition 6.5, that the monoid $\big(V(A) \sqcup W^d_\sigma(S_u)\big) / \big(V(A) \sqcup E_s\big)$ is purely infinite. Let $f \in \big(V(A) \sqcup W^d_\sigma(S_u)\big) \setminus \big(V(A) \sqcup E_s\big)$. Then $f(s) = \infty$ whence there exists f' in $W^d_\sigma(S_u)$ such that $f'(s) = 1$ and $f + f' = 2f$. Therefore $[f] = 2[f]$ in $\big(V(A) \sqcup W^d_\sigma(S_u)\big) / \big(V(A) \sqcup E_s\big)$, since $f' \in V(A) \sqcup E_s$. ∎

We consider now the case when $\partial_e Q_p$ is a compact Hausdorff space. In this case the information relative to the elements in $\mathrm{LAff}(Q_p)$ can be translated faithfully to the extreme boundary by using well-known results, being consequently stored in the continuity rather than in the affinity of the functions involved. This also allows for some new constructions. Recall that, for a compact space X, we denote by $L(X)$ the monoid of lower semicontinuous functions on X with values on $\mathbb{R} \cup \{+\infty\}$. We denote by $L_\sigma(X)$ the submonoid of $L(X)$ whose elements are those functions which are pointwise suprema of increasing sequences of continuous functions over X.

Let M be a partially ordered monoid with order-unit u. Let D be a generating interval in M with a countable cofinal subset and set $d = \sup \phi_u(D)$. Assume that $\partial_e S_u$ is a compact Hausdorff space. By [30, Lemma 7.2], there is a monoid isomorphism $\mathrm{LAff}(S_u)^{++} \cong L(\partial_e S_u)^{++}$ given by restriction. Let d_0 be the restriction of d. Define:

$$W^d_0(S_u) = \{ f \in L_\sigma(\partial_e S_u)^{++} \mid f + g = nd_0 \text{ for some } n \text{ in } \mathbb{N} \text{ and } g \text{ in } L_\sigma(\partial_e S_u)^{++} \}.$$

Now the set $M \sqcup W^d_0(S_u)$ is a monoid with addition given by $x + f = r\big(\phi_u(x)\big) + f$, where $x \in M$, $f \in W^d_0(S_u)$ and r denotes the restriction map from $\mathrm{Aff}(S_u)^{++}$ to $C(\partial_e S_u)^{++}$, which is a monoid isomorphism (see [26, Corollary 11.20]). Then $M \sqcup W^d_\sigma(S_u)$ and $M \sqcup W^d_0(S_u)$ are isomorphic monoids.

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Proposition 6.7 *Let A be a separable C^*-algebra in the class $\mathcal{N}$ and let p be a non-zero projection in A. Suppose that $\partial_e Q_p$ is a (metrizable) compact Hausdorff space. Let $\mathfrak{c}$ be the cardinal of infinite quasitraces in $\partial_e Q_p$ and assume that $\mathfrak{c}$ is infinite. Then $\mathcal{M}(A)/I_{\mathrm{fin}}(A)$ is purely infinite and has exactly $\mathfrak{c}$ minimal (non-zero) closed ideals.*

Proof Let $u = [p] \in V(A)$ and $d = \sup \phi_u\big(D(A)\big)$. By Theorem 5.6, we have that $\partial_e S_u$ is compact Hausdorff and by Theorems 2.1 and 3.9 we get $V\big(\mathcal{M}(A)\big) \cong V(A) \sqcup W_\sigma^d(S_u)$. The remark preceding this proposition assures that we have an isomorphism ψ from $V\big(\mathcal{M}(A)\big)$ onto $V(A) \sqcup W_0^d(S_u)$. Note that the set Γ_d defined in the proof of Theorem 6.3 is equal to $\{s \in \partial_e S_u \mid d_0(s) = \infty\}$, where $d_0 = d|_{\partial_e S_u}$, and that the cardinality of Γ_d is exactly $\mathfrak{c}$. In the following we shall write d instead of d_0 for convenience.

First we show that, for every s in Γ_d, there exists a function h in $W_0^d(S_u)$ such that $h(x) = \infty$ if and only if $x = s$.

Since $\partial_e S_u$ is metrizable, the topology on $\partial_e S_u$ is induced by some metric δ. Let $s \in \Gamma_d$. If $t \in \partial_e S_u$, set $g(t) = 1/\delta(t,s)$, and note that g is continuous on $Y := \partial_e S_u \setminus \{s\}$ and lower semicontinuous on $\partial_e S_u$. Consider $h = \inf\{g, d\}$. Then h is lower semicontinuous, positive and $h \leq g$. If $t \in Y$, let $f(t) = d(t) - h(t) = \sup\{d(t) - g(t), 0\}$ and set $f(s) = 0$. In order to prove that $h \in W_0^d(S_u)$, we have to check that f is lower semicontinuous. Let $k \in \mathbb{R}_+$ and define $U_k = \{x \in \partial_e S_u \mid f(x) \leq k\}$. Let t_n be a sequence in U_k that converges to some point t in $\partial_e S_u$. First suppose that $t = s$. Then $f(s) = 0 \leq k$, so $s \in U_k$. Second, if $t \in \Gamma_d \setminus \{s\}$ then there exists n_0 such that $t_n \neq s$ if $n \geq n_0$. Since g is continuous on $\partial_e S_u \setminus \{s\}$ and $d(t) = \infty$, we have that there exists $m_0 \geq n_0$ such that $(d-g)(t_n) \geq 0$ if $n \geq m_0$, and $\lim_{n\to\infty}(d-g)(t_n) = +\infty$. Since $f(t_n)$ is bounded, we obtain a contradiction. Hence, if $t \neq s$, necessarily $t \notin \Gamma_d$. Let therefore $t \notin \Gamma_d$. If $d(t) \leq g(t)$, then $f(t) = 0$ and hence $t \in U_k$. Suppose that $g(t) < d(t)$. As before, there exists n_0 such that $t_n \neq s$ if $n \geq n_0$. Since g is continuous on $\partial_e S_u \setminus \{s\}$ we may also assume that $g(t_n) < g(t) + \epsilon_n$, where ϵ_n is some real sequence that converges to zero. Therefore $h(t_n) < g(t) + \epsilon_n$, and hence $d(t_n) = h(t_n) + f(t_n) < k + g(t) + \epsilon_n$. Since d is lower semicontinuous, this implies that $d(t) \leq \liminf_n d(t_n) \leq k + g(t)$. Thus $f(t) = d(t) - g(t) \leq k$. Therefore $h \in W_0^d(S_u)$ and $h(x) = \infty$ precisely when $x = s$.

In order to construct $\mathfrak{c}$ minimal non-zero closed ideals in $\mathcal{M}(A)/I_{\mathrm{fin}}(A)$, we proceed as follows. Let $s \in \Gamma_d$. Let I^s be the closed ideal of $\mathcal{M}(A)$ such that $\psi\big(V(I^s)\big) = V(A) \sqcup E^s$, where

$$E^s := \{f \in W_0^d(S_u) \mid f|_{\Gamma_d \setminus \{s\}} < \infty\}.$$

Since as we have shown, there exists a function in $W_0^d(S_u)$ which is infinite precisely at one fixed point of Γ_d, we get that $I^s = I^t$ if and only if $s = t$, and that $I_{\mathrm{fin}}(A) \subsetneqq I^s$.

To see that I^s is minimal containing $I_{\mathrm{fin}}(A)$, suppose that there exists a closed ideal $J \subseteq \mathcal{M}(A)$ such that $I_{\mathrm{fin}}(A) \subseteq J \subsetneqq I^s$, and note that $\psi\big(V(J)\big) = V(A) \sqcup E$, for some $E \subseteq W_0^d(S_u)$. As $V(J) \subsetneqq V(I^s)$, there exists a function f in $(V(A) \sqcup E^s) \setminus (V(A) \sqcup E)$, and thus $f(s) = \infty$. Define $f' \colon \partial_e S_u \to (0,\infty]$ by $f'(s) = 1$ and $f'|_{\partial_e S_u \setminus \{s\}} = f|_{\partial_e S_u \setminus \{s\}}$. Then $f' \in L(\partial_e S_u)^{++}$ and $f' + f = 2f$, so that $f' \in W_0^d(S_u)$. In fact, $f' \in I_{\mathrm{fin}}$. Let $g \in E$. If $g(s) = \infty$, then $f + g = f' + g$ whence $f \in V(A) \sqcup E$, a contradiction. Hence $g(s) < \infty$ and consequently $g \in I_{\mathrm{fin}}$. Thus $J = I_{\mathrm{fin}}(A)$, proving that I^s is minimal containing $I_{\text{fin}}(A)$. Hence, $\{I^s/I_{\text{fin}}(A)\}_{s\in\Gamma_d}$ forms a family of $\mathfrak{c}$ minimal non-zero ideals of $\mathcal{M}(A)/I_{\text{fin}}(A)$.

Let I be an order-ideal of $V(A)\sqcup W_0^d(S_u)$ which is minimal containing I_{fin}. To see that $I = I^s$ for some s in Γ_d, it is enough to prove that, if $f \in I \setminus I_{\text{fin}}$, then f is infinite at only one extreme point. Suppose that there exist elements s, s' in Γ_d such that $s \neq s'$ and $f(s) = f(s') = \infty$. Define functions f', f'' in $W_0^d(S_u)$ by $f'(s') = 1$ and $f'|_{\partial_e S_u\setminus\{s'\}} = f$, and by $f''(s) = 1$ and $f''|_{\partial_e S_u\setminus\{s\}} = f$. Then $f' + f'' = 2f$ whence $f', f'' \in I$. Let $I(f)$, $I(f')$ and $I(f'')$ be the order-ideals of $V(A)\sqcup W_0^d(S_u)$ generated by f, f' and f'' respectively. Then we have that

$$I_{\text{fin}} \subsetneqq I_{\text{fin}} + I(f') \subseteq I_{\text{fin}} + I(f) \subseteq I.$$

Thus $I = I_{\text{fin}} + I(f')$. Similarly $I = I_{\text{fin}} + I(f'')$. Hence $f' = g + h$, for some g in I_{fin} and h in $I(f'')$. If $h \in V(A)$, then writing $x = h$ we have that $f' = g + \phi_u(x)$, which gives a contradiction when we evaluate at s, since $f'(s) = f(s) = \infty$. Thus $h \notin V(A)$, so that there exist h_1 in $W_0^d(S_u)$ and a natural number n such that $h + h_1 = nf''$. Thus

$$f' + h_1 = g + h + h_1 = g + nf''.$$

Evaluating again at s we get $f''(s) = \infty$, which is impossible.

We now check that the monoid $V\big(\mathcal{M}(A)\big)/V\big(I_{\text{fin}}(A)\big)$ is purely infinite. Identify $V\big(\mathcal{M}(A)\big)$ with $V(A)\sqcup W_0^d(S_u)$. If $f \in \big(V(A)\sqcup W_0^d(S_u)\big) \setminus I_{\text{fin}}$, then there exists s in $\partial_e S_u$ such that $f(s) = \infty$. Construct a function h in $W_0^d(S_u)$ such that $h(t) = \infty$ precisely when $t = s$. If we let $h'(t) = h(t)$ if $t \neq s$ and $h'(s) = 1$, then $h' + h = 2h$ and so $h' \in W_0^d(S_u)$, and in fact $h' \in I_{\text{fin}}$. Notice that $f + h = f + h'$. Thus $[f] + [h] = [f]$ in $\big(V(A)\sqcup W_0^d(S_u)\big)/I_{\text{fin}}$ with $[h] \neq 0$. As in Proposition 6.5, this implies that $\mathcal{M}(A)/I_{\text{fin}}(A)$ is purely infinite. ∎

Observe that the conclusion achieved in Proposition 6.5 is a stronger form of pure infiniteness for $\mathcal{M}(A)/I_{\text{fin}}(A)$ when $RR\big(\mathcal{M}(A)\big) = 0$, in that we prove $p \sim p \oplus p$ for all projections p in $\mathcal{M}(A)/I_{\text{fin}}(A)$. We note that this stronger conclusion is not available in the setting of Proposition 6.7. For, let A be a C^*-algebra that satisfies the hypotheses of Proposition 6.7 with $\partial_e Q_p = [0,1]$ and $RR\big(\mathcal{M}(A)\big) = 0$. Then it is enough to construct a lower semicontinuous function $f\colon [0,1] \to [0,\infty]$ (where the value ∞ is attained) such that there are no lower semicontinuous functions $g, h\colon [0,1] \to [0,\infty)$ satisfying $f + h = g$ on the set $\{x \in [0,1] \mid f(x) < \infty\}$. Once this function is constructed, let q_1 be the projection in $\mathcal{M}(A)$ corresponding to f via the isomorphism $\psi\colon V\big(\mathcal{M}(A)\big) \cong V(A)\sqcup W_0^d(S_u)$. If $\pi\colon \mathcal{M}(A) \to \mathcal{M}(A)/I_{\text{fin}}(A)$ is the natural quotient map and if $q = \pi(q_1)$, then it follows that $[q] + [q] \neq [q]$ in $V\big(\mathcal{M}(A)/I_{\text{fin}}(A)\big)$. To construct such a function f, set $f(x) = \infty$ if $x = 0$ or $x \notin \mathbb{Q}$, and set $f(x) = q$ if $x \neq 0$ and $x = p/q$ with $\gcd(p,q) = 1$. Then $f^{-1}[0,\alpha]$ is finite for all $\alpha < \infty$, whence f is lower semicontinuous. Suppose that there exist lower semicontinuous functions $g, h\colon [0,1] \to [0,\infty)$ such that $g(x) = f(x) + h(x)$ if $f(x)$ is finite. Note that $g^{-1}[0,k]$ contains a non-empty open set for some k (using Baire's Theorem). Let $x \notin \mathbb{Q}$ such that $(x-\epsilon, x+\epsilon) \subset g^{-1}[0,k]$ for some $\epsilon > 0$. Taking a sequence $\{x_n\}$ in $(x-\epsilon, x+\epsilon)\cap\mathbb{Q}$ that converges to x, we get

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that $f(x) + h(x) \leq k$, since $f + h$ is lower semicontinuous and $f(x_n) + h(x_n) = g(x_n)$ for all n, a contradiction to $f(x) = \infty$.

To conclude this section, we investigate the presence of uncountably many closed ideals in $\mathcal{M}(A)$. Other instances of this phenomenon have been observed in [30, Corollary 16.7] for algebras with bounded scale.

Theorem 6.8 *Let A be a separable C^*-algebra in the class $\mathcal{N}$ and let p be a non-zero projection in A. Suppose that $\partial_e Q_p$ is a (metrizable) compact Hausdorff space that contains a non-isolated infinite quasitrace. Then there exist uncountably many (proper) closed ideals between $L(A)$ and $\mathcal{M}(A)$ that form a chain with respect to inclusion. The same assertion holds if all the infinite extremal quasitraces in Q_p are isolated, but there are infinitely many of them.*

Proof Let $u = [p] \in V(A)$ and $d = \sup \phi_u\big(D(A)\big)$. As in (the proof of) Proposition 6.7 we have an isomorphism $V\big(\mathcal{M}(A)\big) \cong V(A) \sqcup W_0^d(S_u)$, and $\Gamma_d = \{s \in \partial_e S_u \mid d_0(s) = \infty\}$, where $d_0 = d|_{\partial_e S_u}$. For the rest of the proof we shall write d instead of d_0. The hypothesis that $\partial_e Q_p$ contains a non-isolated infinite quasitrace means that there is a non-isolated point s in Γ_d. If $t \in \partial_e S_u$, set $g(t) = \inf\{1/\delta(t,s), d(t)\}$, where δ is the metric on $\partial_e S_u$. As in the proof of Proposition 6.7, we have that $g \in W_0^d(S_u)$. Since g is lower semicontinuous and $\partial_e S_u$ is compact, there exists an $\epsilon > 0$ such that $g \geq \epsilon$. Define $h\colon \partial_e S_u \to (0,\infty]$ by the rule $h(s) = \infty$ and $h(t) = \epsilon$ if $t \neq s$. Now h is upper semicontinuous and $h \leq g \leq d$. By the Sandwich Theorem (*e.g.* [25, 1.7.15]), there exists a continuous function $f\colon \partial_e S_u \to [0,\infty]$ such that $h \leq f \leq g$. Notice that f is infinite precisely at s. Take $Y = \{t \in \partial_e S_u \mid f(t) \leq 1\}$. Let $0 < \alpha < 1$ and define $g_\alpha(t) = f(t)^\alpha$ if $t \in \partial_e S_u \setminus Y$, and $g_\alpha(t) = f(t)$ if $t \in Y$. We use the convention here that $\infty^\alpha = \infty$ so that $g_\alpha(s) = \infty$. Notice that $g_\alpha \leq f$ and, if $\alpha < \beta$, then $g_\alpha \leq g_\beta$.

If $t \in \partial_e S_u \setminus \{s\}$, let $w(t) = d(t) - f(t)$ and set $w(s) = 0$. Then w is lower semicontinuous. To see this, let $k \in \mathbb{R}_+$ and $U_k = \{t \in \partial_e S_u \mid w(t) \leq k\}$. If t_n is a sequence in U_k converging to some t in $\partial_e S_u$, then we have to check that $w(t) \leq k$. If $t = s$ then $t \in U_k$ since $w(s) = 0$. It follows from the definition of w and in a similar way to Proposition 6.7 that, if $t \neq s$, then $t \notin \Gamma_d$. Since f is continuous on $\partial_e S_u$, there is a real sequence ϵ_n converging to zero such that $f(t_n) < f(t) + \epsilon_n$ if n is large enough. Then $d(t_n) = w(t_n) + f(t_n) < k + f(t) + \epsilon_n$, and since d is lower semicontinuous we get $d(t) \leq k + f(t)$, so $w(t) \leq k$. Therefore w is lower semicontinuous and so $f \in W_0^d(S_u)$. If $0 < \alpha < 1$, a similar argument shows that there exist a function g'_α in $L(\partial_e S_u)^{++}$ and n in $\mathbb{N}$ such that $g_\alpha + g'_\alpha = nf$, whence $g_\alpha \in W_0^d(S_u)$ for all α.

Notice also that, if $0 < \alpha < \beta < 1$ the function g_α can be completed to g_β, that is, there exists $h_{\alpha\beta}$ in $L(\partial_e S_u)^+$ such that $g_\alpha + h_{\alpha\beta} = g_\beta$. To prove this we proceed as before, setting $h_{\alpha\beta} := g_\alpha - g_\beta$ on $\partial_e S_u \setminus \{s\}$ and $h_{\alpha\beta}(s) = 0$. Then, if $k \in \mathbb{R}_+$ and t_n is a sequence in $\partial_e S_u$ that converges to some point t in $\partial_e S_u$ different from s and such that $h_{\alpha\beta}(t_n) \leq k$ for all n, we check that $h_{\alpha\beta}(t) \leq k$. If $t \in Y$, then $h_{\alpha\beta}(t) = 0 \leq k$. If $t \notin Y$ then $h_{\alpha\beta}(t) = f(t)^\beta - f(t)^\alpha$. Since f is continuous on $\partial_e S_u$, there exists n_0 such that $t_n \notin Y$ and $t_n \neq s$ if $n \geq n_0$. Thus $h_{\alpha\beta}(t_n) = f(t_n)^\beta - f(t_n)^\alpha$ converges to $h_{\alpha\beta}(t)$ and since $h_{\alpha\beta}(t_n) \leq k$ for all n, we get that $h_{\alpha\beta}(t) \leq k$.

If $\alpha \in (0,1)$, let

$$E_\alpha := \{h \in W_0^d(S_u) \mid h + h' = ng_\alpha \text{ for some } h' \text{ in } L(\partial_e S_u)^{++} \text{ and } n \text{ in } \mathbb{N}\},$$

and let I_α be the closed ideal of $\mathcal{M}(A)$ such that $\psi\big(V(I_\alpha)\big) = V(A) \sqcup E_\alpha$, where ψ is the isomorphism in the proof of Proposition 6.7. We have shown that $I_\alpha \subseteq I_\beta$ whenever $\alpha < \beta$. We claim that the inclusion is proper. Let $\alpha < \beta$ and let $n \in \mathbb{N}$. Consider the difference $(ng_\alpha - g_\beta)|_{\partial_e S_u \setminus Y}$. Let t_k be a sequence in $\partial_e S_u$ converging to s with $t_k \neq s$ for all k. Since Y is closed, there exists k_0 such that $t_k \notin Y$ if $k \geq k_0$. Thus $(ng_\alpha - g_\beta)(t_k) = f(t_k)^\alpha\big(n - f(t_k)^{\beta-\alpha}\big)$ if $k \geq k_0$ and therefore $\lim_k (ng_\alpha - g_\beta)(t_k) = -\infty$. This shows that $g_\beta \notin V(A) \sqcup E_\alpha$ and establishes the claim. Hence $\{I_\alpha\}_{\alpha\in(0,1)}$ is a family of different closed ideals between $L(A)$ and $\mathcal{M}(A)$ that form a chain with respect to inclusion.

Suppose now that all the infinite quasitraces in $\partial_e Q_p$ are isolated, but there are infinitely many of them. Then Γ_d consists of infinitely many isolated points. Since $\partial_e S_u$ is compact and all the points in Γ_d are isolated, there exists a sequence t_n in Γ_d whose limit is a point s in $\partial_e S_u \setminus \Gamma_d$. Since $d \in L(\partial_e S_u)^{++}$ and $\partial_e S_u$ is compact, there exists $0 < \epsilon < 1$ such that $\epsilon < d(t)$ for all t in $\partial_e S_u$. For $0 < \alpha < 1$, define functions $g_\alpha \colon \partial_e S_u \to (0,\infty)$ by the rule $g_\alpha(t_n) = n^\alpha$ and $g_\alpha(t) = \epsilon$ if $t \neq t_n$. Let $\lambda \in \mathbb{R}_+$, and suppose that $\lambda \geq \epsilon$. Note that, if n is large enough, then $g_\alpha(t_n) > \lambda$. Therefore ${g_\alpha}^{-1}(-\infty,\lambda]$ equals the complementary set (in $\partial_e S_u$) of a tail of the sequence $\{t_n\}$. Since all the points in $\{t_n\}$ are isolated, we conclude that $g_\alpha^{-1}(-\infty,\lambda]$ is closed. If, on the other hand, $\lambda < \epsilon$, then $g_\alpha^{-1}(-\infty,\lambda] = \varnothing$. So $g_\alpha \in L(\partial_e S_u)^{++}$. Notice that $(d - g_\alpha)(t_n) = +\infty$ and $(d - g_\alpha)(t) = d(t) - \epsilon$ if $t \neq t_n$, whence $d - g_\alpha \in L(\partial_e S_u)^{++}$. Thus $g_\alpha \in W_0^d(S_u)$ for all α. Now $g_\alpha \leq g_\beta$ whenever $0 < \alpha < \beta < 1$. Note also that $(g_\beta - g_\alpha)(t_n) = n^\alpha(n^{\beta-\alpha} - 1)$ and $(g_\beta - g_\alpha)(t) = 0$ if $t \neq t_n$, and that $\lim_n (g_\beta - g_\alpha)(t_n) = +\infty$. Therefore $g_\beta - g_\alpha \in L(\partial_e S_u)^+$. Define

$$E_\alpha = \{h \in W_0^d(S_u) \mid h + h' = ng_\alpha \text{ for some } h' \text{ in } L(\partial_e S_u)^{++} \text{ and } n \text{ in } \mathbb{N}\}$$

and let I_α be the closed ideal of $\mathcal{M}(A)$ such that $\psi\big(V(I_\alpha)\big) = V(A) \sqcup E_\alpha$, as before. We have shown that $I_\alpha \subseteq I_\beta$ if $\alpha < \beta$.

Finally, if $k \in \mathbb{N}$ and $0 < \alpha < \beta < 1$, we see that $\lim_n (kg_\alpha - g_\beta)(t_n) = -\infty$, so that the inclusion $I_\alpha \subseteq I_\beta$ is proper and we get an uncountable chain $\{I_\alpha \mid 0 < \alpha < 1\}$. ■

7 Separativity and Stable Rank

Rieffel has shown in [45, Proposition 6.5] that, if a C^*-algebra B with unit contains two isometries with orthogonal ranges, then $\mathrm{sr}(B) = \infty$. This is the case, for example, for $B(\mathcal{H})$ where $\mathcal{H}$ is an infinite dimensional separable Hilbert space. This fact would seem to preclude any stable rank finiteness condition on the multiplier algebra $\mathcal{M}(A)$ of a simple C^*-algebra A. In the same paper, Rieffel asks for finiteness conditions on $\mathcal{M}(A)$ (or on A itself) to ensure that $\mathrm{sr}\big(\mathcal{M}(A)\big) = \mathrm{sr}(A)$ ([45, Question 4.16]).

In the present section we provide a direct way to compute the stable rank of the multiplier algebra $\mathcal{M}(A)$ in terms of the scale of A. We still assume that A belongs to

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the class $\mathcal{N}$ consisting of simple, σ-unital (non-unital), non-elementary C^*-algebras with real rank zero, stable rank one and strict unperforation on $V(A)$. We will also assume that $RR\big(\mathcal{M}(A)\big) = 0$. We therefore obtain that the only possible values for $\operatorname{sr}\big(\mathcal{M}(A)\big)$ are either 2 or ∞, which do not coincide in any case with $\operatorname{sr}(A)$, even when $\mathcal{M}(A)$ is stably finite.

Definition 7.1 (**[13]**, **[3]**) Let M be an abelian monoid. We say that M is *separative* if $a + a = a + b = b + b$ implies $a = b$, for all elements a, b in M. A C^*-algebra A is *separative* if the associated monoid $V(A)$ is separative.

As is seen in [3], separativity is a key to a number of problems concerning cancellation of finitely generated projective modules over regular rings and C^*-algebras of real rank zero. The importance of this concept in our situation is recorded in the following lemma.

Lemma 7.2 *Let A be a simple σ-unital non-unital C^*-algebra with real rank zero, stable rank one, and with $V(A)$ strictly unperforated. Then $\mathcal{M}(A)$ is separative.*

Proof We only need to show that the monoid $V\big(\mathcal{M}(A)\big)$ is separative. If $A \cong K(\mathcal{H})$ for some infinite dimensional separable Hilbert space $\mathcal{H}$, then $\mathcal{M}(A) \cong B(\mathcal{H})$. Since $V\big(B(\mathcal{H})\big) \cong \mathbb{Z}^+ \cup \{\infty\}$ with $\infty + x = x + \infty = \infty$ for all x, it is clear that $\mathcal{M}(A)$ is separative. Thus we may assume that A is non-elementary. Choose a non-zero element u in $V(A)$ and set $d = \sup \phi_u\big(D(A)\big)$. Then $V\big(\mathcal{M}(A)\big) \cong V(A) \sqcup W_\sigma^d(S_u)$, by Theorem 3.9. Let $a, b \in V(A) \sqcup W_\sigma^d(S_u)$, and assume that $a + a = a + b = b + b$. First note that $a \in V(A)$ if and only if $b \in V(A)$. Hence, if $a \in V(A)$, then $a = b$ since $V(A)$ is cancellative.

Secondly, suppose that neither a, nor b, belong to $V(A)$. Set $a = f$ and $b = g$ for some functions f, g in $W_\sigma^d(S_u)$. The equality $2f = 2g$ implies that f is infinite at the same points as g. In particular we get $f = g$, and thus $a = b$. ■

It has been shown in [3] that an appropriate notion of stable rank for elements in the monoid $V(R)$ of isomorphism classes of finitely generated projective right modules over a ring R provides information about the stable rank for a large class of rings.

Definition 7.3 (**[3]**) Let M be a monoid, let $a \in M$ and $n \in \mathbb{N}$. Say that a satisfies the *n-stable rank condition* if the following holds: Whenever $na + h = a + y$ for some elements h, y in M, then there exists an element e in M such that $y = h + e$ and $na = a + e$. The *stable rank* of a, which is denoted by $\operatorname{sr}(a)$ is the least positive integer n such that a satisfies the n-stable rank condition (if such an n exists), or ∞ (if no such n exists).

The following proposition is known. A ring-theoretic version of it may be found in [3, Section 3].

Proposition 7.4 *Let M be a monoid and let $a \in M$.*

(1) Suppose that M is conical. Then $\operatorname{sr}(a) = 1$ if and only if a cancels from sums in M.

(2) *Suppose that M is separative. Then* $\operatorname{sr}(a)$ *is either* 1, 2 *or* ∞.

Proof (1). Suppose that M is conical. If a cancels from sums in M, then clearly $\operatorname{sr}(a) = 1$. Conversely, assume that $\operatorname{sr}(a) = 1$ and that $a + y = a + h$ for some y, h in M. Then there exists e in M such that $y = h + e$ and $a = a + e$. Applying again that $\operatorname{sr}(a) = 1$ to the equality $a + e = a + 0$ we get an element e' in M such that $a = a + e'$ and $0 = e + e'$. Since M is conical, this implies that $e = 0$ and thus $y = h$, as desired.

(2). Now suppose that M is separative and that $\operatorname{sr}(a) \leq n$ for some natural number n. If $2a + h = a + y$ for some h, y in M, then by adding $(n-1)y$ to this equality we get $na + (a + nh) = a + ny$. Since $\operatorname{sr}(a) \leq n$, there exists e in M such that $na = a + e$ and $ny = (a + nh) + e$, whence $a \leq ny$. Applying [3, Lemma 2.1(iv)] to the equation $a + (a + h) = a + y$ we get that $a + h = y$. It follows that $\operatorname{sr}(a) \leq 2$. ∎

Theorem 7.5 *Let A be a separable C^*-algebra in the class $\mathcal{N}$ such that $\mathcal{M}(A)$ has real rank zero. Then $\operatorname{sr}\big(\mathcal{M}(A)\big) = 2$ if and only if A has finite but not continuous scale; and $\operatorname{sr}\big(\mathcal{M}(A)\big) = \infty$ if and only if A has either continuous scale or if the scale is not finite.*

Proof By Lemma 7.2, $\mathcal{M}(A)$ is separative. Therefore the only possible values for the stable rank of the elements of $V\big(\mathcal{M}(A)\big)$ are 1, 2 or ∞, according to Condition (2) in Proposition 7.4. Notice that $\mathcal{M}(A)$ is an exchange ring (in the sense of [51]) because it has real rank zero (see [3, Theorem 7.2]). Therefore the stable rank of $\mathcal{M}(A)$ equals the stable rank of the element $[1_{\mathcal{M}(A)}]$ in $V\big(\mathcal{M}(A)\big)$, by [3, Theorem 3.2]. Let $u \in V(A)^*$ and set $d = \sup \phi_u\big(D(A)\big)$. By Theorem 3.9, there exists a monoid isomorphism φ from $V\big(\mathcal{M}(A)\big)$ onto $V(A) \sqcup W_\sigma^d(S_u)$ such that $\varphi([1_{\mathcal{M}(A)}]) = d$. Hence $\operatorname{sr}[1_{\mathcal{M}(A)}] = \operatorname{sr}(d)$.

Note that $\operatorname{sr}(d) \geq 2$. To see this we characterize the elements in $V(A) \sqcup W_\sigma^d(S_u)$ with stable rank one as the elements in $V(A)$. Since $V(A) \sqcup W_\sigma^d(S_u)$ is a conical monoid, an element has stable rank one if and only if it cancels from sums in $V(A) \sqcup W_\sigma^d(S_u)$, by Condition (1) in Proposition 7.4. Let $x \in V(A)$ and assume that $x + f = x + g$ for some functions f, g in $W_\sigma^d(S_u)$. Then f is infinite precisely when g is, and if $f(s) < \infty$ for some s in S_u, then $s(x) + f(s) = s(y) + g(s)$, whence $f(s) = g(s)$. Thus $f = g$. Assume conversely that $\operatorname{sr}(x) = 1$ and that $x \notin V(A)$. Then $x \in W_\sigma^d(S_u)$. Take any z in $V(A)^*$ and observe that $z + x = \phi_u(z) + x$, which contradicts the fact that x cancels from sums in $V(A) \sqcup W_\sigma^d(S_u)$. Since $d \notin V(A)$, we conclude that $\operatorname{sr}(d) \geq 2$.

Suppose now that A has finite but not continuous scale. Then $d|_{\partial_e S_u}$ is finite and not continuous. In order to see that $\operatorname{sr}(d) = 2$, it suffices to check that d satisfies the 2-stable rank condition. Suppose that, for some elements y, h in $V(A) \sqcup W_\sigma^d(S_u)$, we have that $2d + h = d + y$. If $y \in V(A)$, then the equality $2d + h = d + \phi_u(y)$ says that $d + h = \phi_u(y)$, and this implies that d is continuous, a contradiction. Therefore $y \notin V(A)$. In this case $(d + h)|_{\partial_e S_u} = y|_{\partial_e S_u}$, whence $d + h = y$ and we are done if we choose $e = d$. Consequently $\operatorname{sr}(d) \leq 2$ and since as we have shown, $\operatorname{sr}(d) \geq 2$, we get $\operatorname{sr}(d) = 2$. Therefore $\operatorname{sr}\big(\mathcal{M}(A)\big) = 2$.

Conversely, assume that $\operatorname{sr}\big(\mathcal{M}(A)\big) = 2$. Then $\operatorname{sr}(d) = 2$. If d is a continuous affine function, then d is bounded, so $d \ll (m+1)\phi_u(u)$ for some m in $\mathbb{N}$. Let $x = mu$ and note that $0 \ll d \ll \phi_u(x) + 1 = \phi_u(x + u)$. Let $y = x + u$ and choose h

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in $\mathrm{Aff}(S_u)^{++}$ such that $d + h = \phi_u(y)$. Then $2d + h = d + \phi_u(y) = d + y$, and since $h \notin V(A)$ there is no e in $V(A) \sqcup W_\sigma^d(S_u)$ satisfying $y = h + e$. Hence d does not satisfy the 2-stable rank condition, a contradiction. If, on the other hand, there exists s in $\partial_e S_u$ such that $d(s) = \infty$, we distinguish two possibilities: first, if $d|_{\partial_e S_u}$ is identically infinite, then it is easy to see that $\mathrm{sr}(d) = \infty$ (since in this case $W_\sigma^d(S_u) = \mathrm{LAff}(S_u)^{++}$) and so d does not satisfy the 2-stable rank condition, contradicting our hypothesis; second, if $d|_{\partial_e S_u}$ is finite at some point but $d(s) = \infty$, then by using the techniques of Corollary 4.13 there exists y in $W_\sigma^d(S_u)$ with $y(s) = 1/2$ and $y|_{\{s\}'} = (d+1)|_{\{s\}'}$, where $\{s\}'$ denotes the complementary face of $\{s\}$ in S_u. Thus $2d + h = d + y$, with $h = 1$. If there exists an element e in $V(A) \sqcup W_\sigma^d(S_u)$ such that $y = h + e$, then $1 \le y$, which is impossible since $y(s) = 1/2$. Therefore d does not satisfy the 2-stable rank condition, and this contradicts our assumption that $\mathrm{sr}(d) = 2$.

It is clear that the complementary conditions characterize the multiplier algebras with infinite stable rank. ■

It has been proved in [54, Corollary 1.6] that, in case A is a simple σ-unital C^*-algebra with real rank zero and continuous scale (that is, the quotient $\mathcal{M}(A)/A$ is simple), then $\mathcal{M}(A)/A$ contains two isometries with orthogonal ranges, so in particular $\mathrm{sr}\big(\mathcal{M}(A)/A\big) = \infty$ by [45, Proposition 6.5]. The following corollary deals with the case when $\mathcal{M}(A)/A$ is not simple.

Corollary 7.6 *Let A be a separable C^*-algebra in the class $\mathcal{N}$ and assume that $\mathcal{M}(A)$ has real rank zero. Then $\mathrm{sr}\big(\mathcal{M}(A)/A\big) = 2$ if and only if $\mathrm{sr}\big(\mathcal{M}(A)\big) = 2$.*

Proof Suppose that $\mathrm{sr}\big(\mathcal{M}(A)\big) = 2$. Then $\mathrm{sr}\big(\mathcal{M}(A)/A\big) \le \mathrm{sr}\big(\mathcal{M}(A)\big) = 2$, by [45, Theorem 4.3]. Since every projection in $\mathcal{M}(A)/A$ is infinite ([54, Theorem 1.3(a)]) we have that $V\big(\mathcal{M}(A)/A\big)$ is not cancellative, whence $\mathrm{sr}\big(\mathcal{M}(A)/A\big) = 2$.

Conversely, assume that $\mathrm{sr}\big(\mathcal{M}(A)/A\big) = 2$. Since $\mathrm{sr}(A) = 1$ we get that

$$\mathrm{sr}\big(\mathcal{M}(A)\big) \le \max\big\{\mathrm{sr}(A), \mathrm{sr}\big(\mathcal{M}(A)/A\big) + 1\big\} = 3,$$

using [45, Corollary 4.12]. Since the stable rank of $\mathcal{M}(A)$ is finite, we conclude from Theorem 7.5 that $\mathrm{sr}\big(\mathcal{M}(A)\big) = 2$. ■

Finally, we note that $\mathcal{M}(A)$ is stably finite whenever the scale is not identically infinite (that is, when A is not stable, by [38, Proposition 1.14]). This can be derived again from the corresponding notion stated for monoids. Observe, though, that if d is the scale corresponding to A, then nd is the scale corresponding to $M_n(A)$ and thus A is stable if and only if $M_n(A)$ is stable for some (hence all) natural numbers n. It follows then from [49, Theorem 3.5] that if A is not stable, then $\mathcal{M}(A)$ is stably finite.

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Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona
Spain
e-mail: perera@mat.uab.es