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A CHARACTERISTIC SUBGROUP OF π-STABLE GROUPS

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1. Introduction. All groups in this paper are assumed to be *finite.*

Let G be a group with $o_p(G) \neq 1$ which is p-constrained and p-stable, p odd. If P is an S_p-subgroup of G, then by Glauberman's Theorem, [**3**, 8.2.11],

$$G = O_{p'}(G)N_G(ZJ(P)).$$

In particular, if $O_{p'}(G) = 1$, then $ZJ(P) \lhd G$.

The object of this paper is to generalize the above result by replacing the prime p by a set of odd primes π.

We obtain the following result:

THEOREM A. *Let G be a π-stable D_π^N group, where π is a set of primes. Assume that $F(G)$ is Abelian or $2 \notin \pi$. Let K be an S_π-subgroup of G. If $C_G(O_\pi(G)) \subseteq O_\pi(G)$, then $ZJ(K)$ char G.*

Note. If $|K|$ is odd, then $ZJ(K) \neq 1$ by [**1**, Theorem 1].

Some related results were obtained by Mann in [**7**].

COROLLARY. *Let G be a π-solvable group, where $2, 3 \notin \pi$. Let K be an S_π-subgroup of G and assume that $O_{\pi'}(G) = 1$. Then $ZJ(K)$* char *G.*

The same is true if we replace the assumption that $3 \notin \pi$ by the assumption that G has an Abelian S_2-subgroup, by a result of Glauberman and the author [**1**, Theorem 2(c)].

Our notation is standard and is taken mainly from [**3**]. In particular, let G be a group, then $F(G)$ denotes the *Fitting subgroup* of G and $[A, B, C]$ denotes the *triple commutator* $[[A, B], C]$ of three subgroups A, B, C of G. Moreover, $d(G)$ is the maximum of the orders of the Abelian subgroups of G. Let $A(G)$ be the set of all Abelian subgroups of order $d(G)$ in G. Then, as in [**3**], $J(G)$ is the subgroup of G generated by $A(G)$, that is, the *Thompson subgroup* of G.

Following Wielandt we consider the following statements about a group G.

E_π: G has an S_π-subgroup.

C_π: G has an S_π-subgroup and any two such subgroups are conjugate.

D_π: G satisfies C_π and every π-subgroup of G is contained in an S_π-subgroup.

D_π^N: G and every normal subgroup of G satisfy D_π.

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We say that G is a π-*stable* group if it satisfies the following condition:

Let K be an arbitrary π-subgroup of G. Let A be an arbitrary π-subgroup of $N_G(K)$. Then, if $[K, A, A] = 1$, we have

$$AC_G(K)/C_G(K) \subseteq O_\pi(N_G(K)/C_G(K)).$$

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2. Some properties of $A(G)$. The following two basic results were proved in [**1**]:

THEOREM 2.1 [**1**, Proposition 1]. *Suppose G is a group, $A \in A(G)$, B is a nilpotent subgroup of G, and A normalizes B. Assume that B has an Abelian S_2-subgroup. Assume also that either $|A|$ is odd or B is Abelian. Then AB is nilpotent.*

THEOREM 2.2 [**1**, Theorem 2]. *Suppose π is a set of primes, G is a finite π-solvable group, and K is an S_π-subgroup of G. Assume that G has an Abelian S_2-subgroup and that $O_{\pi'}(G) = 1$. Then:*

(a) $O_2(G) = O_2(ZJ(G)) = O_2(ZJ(K)) = O_2(K)$;

(b) *if* $2 \notin \pi$, *then for every* $p \in \pi - \{3\}$ *and* $A \in A(K)$, $O_p(A) \subseteq O_p(G)$;

(c) *if* $2 \notin \pi$, *then* $ZJ(K) \lhd G$; *and*

(d) *if* $2 \notin \pi$, *then the prime divisors of* $d(K)$, *of* $|ZJ(K)|$, *and of* $|F(G)|$ *coincide.*

Following the proof of [**3**, Lemma 8.2.2], we obtain:

LEMMA 2.3. *Let G be a D_π-group and let K be an S_π-subgroup of G. Then we have:*

(i) *If R is a subgroup of K which contains an element of $A(K)$, then $A(R) \subseteq A(K)$ and $J(R) \subseteq J(K)$.*

(ii) *If Q is an S_π-subgroup of G containing $J(K)$, then $J(Q) = J(K)$.*

(iii) *If $Q = K^x$, $x \in G$, then $J(Q) = J(K)^x$.*

(iv) *$J(K)$ is characteristic in any π-subgroup of G in which it lies.*

LEMMA 2.4. *If A is an Abelian subgroup of G, and $[x, A]$ is Abelian for $x \in G$, then $[x, a, b] = [x, b, a]$ for every $a, b \in A$.*

Proof. In general $[xy, z] = [x, z][x, z, y][y, z]$. Since A is Abelian $[xb, a] = [x, a][x, a, b]$ and $[xa, b] = [x, b][x, b, a]$. Thus,

$$[x, b, a]^{-1}[x, a, b] = [xa, b]^{-1}[x, b][x, a]^{-1}[xb, a] = [b, xa][a, x][x, b][xb, a]$$

as $[x, A]$ is Abelian. Therefore,

$$[x, b, a]^{-1}[x, a, b] = b^{-1}a^{-1}x^{-1}bxaa^{-1}x^{-1}axx^{-1}b^{-1}a^{-1}xba = 1,$$

as A is Abelian.

Remark. Following the proofs of [**3**, Lemma 8.2.3 and Theorem 8.2.4] and using Lemma 2.4 instead of [**3**, Lemma 2.2.5(i)] in the proof of [**3**, Theorem 8.2.4], we generalize these results by replacing the p-group P by an arbitrary group G.

Now, using Theorem 2.1 we can generalize the Thompson Replacement [**3**, Theorem 8.2.5]:

THEOREM 2.5. *Let $A \in A(G)$ and let B be an Abelian subgroup of G. Assume A normalizes B, but B does not normalize A. Then there exists an element A^* in $A(G)$ with the following properties:*
(i) $A \cap B \subset A^* \cap B$;
(ii) A^* *normalizes* A.

Proof. Set $N = N_B(A)$; then $B \lhd AB, N \lhd B$ and $N \subset B$ by our hypothesis. Since by Theorem 2.1 $B/N \cap Z(AB/N) \neq 1$, we can choose $x \in B - N$ so that its image lies in $Z(AB/N)$. Then $[x, A] \subseteq N$. Setting $M = [x, A]$, we have that M is Abelian as $N \subset B$. Therefore $A^* = MC_A(M) \in A(G)$, by the generalized Theorem 8.2.4 of [**3**] (see our remark). Now $M \subseteq N \subseteq N_G(A)$ and $C_A(M) \subseteq N_G(A)$, hence $A^* \subseteq N_G(A)$. Furthermore, $A \cap B \subseteq C_G(x) \cap C_G(A)$, so $A \cap B \subseteq A^*$. On the other hand, $M = [x, A] \nsubseteq A$ as $x \notin N$, so $A \cap B \subset M(A \cap B) \subseteq A^* \cap B$ as $M \subseteq A^* \cap B$, completing the proof.

As a corollary, we have

LEMMA 2.6. *Let B be an Abelian normal subgroup of G. Then there exists an element $A \in A(G)$ such that $B \subseteq N_G(A)$.*

Let G be a group and let A and B be subgroups of G. We define inductively:

$$[B, A, 0] = B \text{ and } [B, A, i] = [[B, A, i - 1], A]$$

for $i > 0$.

Following the proof of [**3**, Theorem 8.2.7], with small changes, and using all the above results we obtain:

THEOREM 2.7. *Let G be a group with $B \lhd G$, $[B, B, B] = 1$ and $B' \subseteq ZJ(G)$; assume also that there exists an integer n and $A \in A(G)$ such that $[B, A, n]$ is Abelian, and $[A, B]'$ is of odd order. Suppose that $B \nsubseteq N_G(A)$. Then there exists an element $A^* \in A(G)$ with the following properties:*
(i) $A \cap B \subset A^* \cap B$;
(ii) $[A^*, B]'$ *has odd order;*
(iii) $A^* \subseteq N_G(A)$;
(iv) $[B, A^*, n]$ *is Abelian.*

As a corollary we have:

COROLLARY 2.8. *Let G be a group with $B \lhd G$, $[B, B, B] = 1$ and $B' \subseteq ZJ(G)$; and assume that there exists an integer n and $A \in A(G)$ such that $[B, A, n]$ is*

Abelian and $[A, B]'$ *is of odd order. Assume also that* $B \not\subseteq N_G(A)$. *Then there exists an element* $A^* \in A(G)$ *so that* $B \subseteq N_G(A^*)$.

3. The main results. *It is well-known that if* G *is a* D_π^N *group with* H *a normal subgroup of* G *and* K *an* S_π*-subgroup of* G, *then*

(i) H *is a* D_π *group with* $H \cap K$ *an* S_π*-subgroup of* H. [**6**, 7.2 Hilfssatz, p. 444]

(ii) G/H *is a* D_π *group with* KH/H *an* S_π*-subgroup of* G/H. [**6**, 7.2 Hilfssatz, p. 444]

(iii) *If* R *is an* S_π*-subgroup of* H, *then* $G = N_G(R)H$. (*Similar to the proof of* [**3**, Theorem 1.3.7].)

By using all the results of Chapter 2, the above result, and following the proof of [**3**, Theorem 8.2.9], we obtain:

THEOREM 3.1. *Let* G *be a* π*-stable* D_π^N *group. Let* K *be an* S_π*-subgroup of* G, B *a nilpotent normal* π*-subgroup of* G, *and assume that there exists an integer* n *such that* $[B, A, n]$ *is Abelian for all* $A \in A(K)$, *and that* $[A, B]'$ *is of odd order, for all* $A \in A(K)$. *Then* $B \cap ZJ(K)$ *is a normal subgroup of* G.

The next simple Lemma yields Theorem A:

LEMMA 3.2. *Let* G *be a* π*-stable* D_π *group. Let* K *be an* S_π*-subgroup of* G *and* A *a normal Abelian subgroup of* K. *If* $C_G(O_\pi(G)) \subseteq O_\pi(G)$, *then* $A \subseteq F(G)$.

Proof. Since G is a D_π group, $Q = K \cap O_\pi(G) = O_\pi(G)$. By assumption $[Q, A, A] = 1$ and $AC_G(Q)/C_G(Q) \subseteq O_\pi(G/C_G(Q))$. Therefore

$$AZ(O_\pi(G))/Z(O_\pi(G)) \subseteq O_\pi(G/Z(O_\pi(G))) = O_\pi(G)/Z(O_\pi(G)).$$

Thus $A \subseteq O_\pi(G)$. Since G is a D_π group and A is a normal abelian subgroup of K, we have $A \subseteq F(O_\pi(G)) \subseteq F(G)$.

We now obtain at once.

THEOREM 3.3. *Let* G *be a* π*-stable* D_π^N *group. Let* K *be an* S_π*-subgroup of* G. *Assume that there exists an integer* n *such that* $[F(G), A, N]$ *is Abelian for all* $A \in A(K)$ *and that* $[A, F(G)]'$ *is of odd order for all* $A \in A(K)$. *If* $C_G(O_\pi(G)) \subseteq O_\pi(G)$, *then* $ZJ(K) \lhd G$.

Proof. Lemma 3.2 implies that $ZJ(K) \subseteq F(G)$. Taking $F(G)$ as B in Theorem 3.1, it follows that $ZJ(K) = ZJ(K) \cap F(G)$ is a normal subgroup of G, as required.

We now obtain the

Proof of Theorem A. If $F(G)$ is Abelian or $2 \notin \pi$ then Theorem 2.1 implies that there exists an integer n such that $[F(G), A, n]$ is Abelian for every $A \in A(K)$. Clearly $[A, F(G)]'$ is of odd order, for all $A \in A(K)$. Therefore Theorem 3.3 implies that $ZJ(K) \lhd$ G. Let α be an automorphism of G, and take $g \in G$ such that $K^\alpha = K^g$. Then $(ZJ(K))^\alpha = ZJ(K^\alpha) = ZJ(K^g) = (ZJ(K))^g = ZJ(K)$. Therefore $ZJ(K)$ char G.

LEMMA 3.4. *Let π be a set of odd primes. Let G be a strongly p-solvable group for every $p \in \pi$. Then G is π-stable.*

Proof. Let K be an arbitrary π-subgroup of G, and let A be a π-subgroup of $N_G(K)$ with the property $[K, A, A] = 1$. Clearly G is a π-solvable group. Hence K is a π-solvable subgroup of G. Therefore K is solvable. Let $K = K_1 \supset \ldots \supset K_{n+1} = 1$ be an $N_G(K)$-invariant normal series of K such that each $\bar{K}_i = K_i/K_{i+1}$, $1 \leqq i \leqq n$, is p_i-elementary Abelian for $p_i \in \pi$, and such that $N_G(K)$ acts irreducibly on $\bar{K}_i$. Let H_i be the kernel of the representation of $N_G(K)$ on $\bar{K}_i$. Since $\bar{N}_i = N_G(K)/H_i$ acts faithfully and irreducibly on $\bar{K}_i$ as a vector space over Z_{p_i}, we have $O_{p_i}(\bar{N}_i) = \bar{1}$, by [**3**, Theorem 3.1.3]. On the other hand, as $[K, A, A] = 1$, certainly $[\bar{K}_i, \bar{A}_i, \bar{A}_i] = \bar{1}$, where $\bar{A}_i$ denotes the image of A in $\bar{N}_i$.

But now if $\bar{x} \in \bar{A}_i$ is p_i'-element then [**3**, Theorem 5.3.6], implies that $\bar{x} = \bar{1}$. If $\bar{x} \in \bar{A}_i$ is p_i-element it follows from [**3**, Theorem 2.6.6], as $O_{p_i}(\bar{N}_i) = \bar{1}$, that $\bar{x} = \bar{1}$, whence

$$A \subseteq \bigcap_{i=1}^{n} H_i \subseteq N_G(K_i) \quad \text{and} \quad [K_i, \bigcap_{i=1}^{n} H_i] \subseteq K_{i+1}$$

for all i, $1 \leqq i \leqq n$. [**3**, Corollary 5.3.3], now yields that

$$\bigcap_{i=1}^{r} H_i/C_G(K) \lhd N_G(K)/C_G(K)$$

is a π-group, so $AC_G(K)/C_G(K)) \subseteq O_\pi(N_G(K)/C_G(K))$ and G is π-stable.

As an immediate corollary we obtain:

COROLLARY. *Let π be a set of odd primes and let G be a π-solvable group. Assume that G has an Abelian S_2-subgroup or $3 \notin \pi$. Assume also that $O_{\pi'}(G) = 1$. Then* $1 \subset ZJ(K)$ char G.

Proof. It is well-known that G is a strongly p-solvable group for every $p \in \pi$. Hence G is π-stable by Lemma 3.4. Lemma 1.2.3 of Hall-Higman implies that $C_G(O_\pi(G)) \subseteq O_\pi(G)$. It is well-known that G is a D_π^N group. Therefore Theorem A implies that $ZJ(K)$ char G. [**1**, Theorem 1] implies that $ZJ(K) \neq 1$.

Note. As mentioned in the introduction, the first part of the Corollary is known from [**1**, Theorem 2(c)].

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