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DECOMPOSITION OF A VON NEUMANN ALGEBRA RELATIVE TO A^*-AUTOMORPHISM

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Let X be any real or complex Banach space. If T is a bounded linear operator on X then denote by $N(T)$ the null space of T and by $R(T)$ the range space of T.

Now if X is finite dimensional and $N(T)=N(T^2)$ then also $R(T)=R(T^2)$. Therefore X admits a direct sum decomposition

$$X = N(T) \oplus R(T).$$

Indeed it is easy to see that $N(T)=N(T^2)$ implies that $N(T)\cap R(T)=\{0\}$ and, using dimension theory of finite dimensional spaces, that $N(T)$ and $R(T)$ span the whole space (see, for example, (**2**, pp. 271–73)).

Now this result is no longer true when X is infinite dimensional. In fact, one cannot even expect a weaker result that $N(T)+R(T)$ is dense in X. For instance, one can find an injective operator whose range is not dense.

However, in the case of a *-automorphism on a von Neumann algebra we are able to show:

Proposition 1. *Let M be a von Neumann algebra and α a *-automorphism of M. Then $(N(\alpha-1)+R(\alpha-1)$ is σ-weakly dense in M.*

Proof. Suppose that $N(\alpha-1)+R(\alpha-1)$ is not σ-weakly dense. Then there is a non-zero σ-weakly continuous linear functional ϕ on M vanishing on $N(\alpha-1)$ and $R(\alpha-1)$. From the fact that ϕ vanishes on $R(\alpha-1)$ we have $\phi(\alpha(x)-x)=0$ or $\phi(\alpha(x))=\phi(x)$ for all $x\in M$. So ϕ is α-invariant. Now let $\phi=|\phi|U$ be the polar decomposition of ϕ (see, e.g. (**1**, p. 62)). Then by the uniqueness of the polar decomposition, we must have that $|\phi|$ is also α-invariant and $\alpha(U)=U$. Then $\alpha(U^*)=U^*$ and as ϕ also vanishes on $N(\alpha-1)$, we get

$$\phi(U^*)=|\phi|(UU^*)=0.$$

Now UU^* is the support projection of $|\phi|$ and therefore $|\phi|=0$ and hence $\phi=0$. This contradiction proves the result.

Remark. Here also $N(\alpha-1)\cap R(\alpha-1)=\{0\}$. Indeed, let $y=\alpha(x)-x$ for some $x\in M$ and $(\alpha-1)(y)=0$ so that $\alpha(y)=y$, then

$$\alpha(x)=y+x$$
$$\alpha^2(x)=\alpha(y)+\alpha(x)=y+y+x=2y+x$$

and by induction $\alpha^n(x) = ny + x$ for all integers $n \geq 1$. But then

$$n\|y\| = \|ny\| = \|\alpha^n(x) - x\| \leq \|\alpha^n(x)\| + \|x\| \leq 2\|x\|$$

so that $n\|y\| \leq 2\|x\|$ for all positive integers. This implies that $\|y\| = 0$ and hence $y = 0$.

Note that we only used here that $\|\alpha\| \leq 1$, so this result appears to be true for any contraction on a Banach space. We now come to the following.

Theorem 2. *The smallest weakly closed subalgebra* M_1 containing $R(\alpha - 1)$ *is a two-sided ideal, invariant under* α. *If* e *is the central projection in* M *such that* $M_1 = Me$, *and if* $f = 1 - e$, *then* f *is the largest projection such that* $\alpha(fx) = fx$ *for all* $x \in M$.

Proof. We first remark that xy and $yx \in R(\alpha - 1)$ for all $x \in N(\alpha - 1)$ and $y \in R(\alpha - 1)$. Now any element in the algebra generated by $R(\alpha - 1)$ is a linear combination of products of elements in $R(\alpha - 1)$ so that still xy and yx belong to the algebra generated by $R(\alpha - 1)$ for all $x \in N(\alpha - 1)$ and y in the algebra generated by $R(\alpha - 1)$. By continuity also xy, $yx \in M_1$ for all $x \in N(\alpha - 1)$ and $y \in M_1$. Obviously this is true for all $x \in R(\alpha - 1)$ and hence for all $x \in N(\alpha - 1) + R(\alpha - 1)$. Then again by continuity this is true for all $x \in M$.

As $R(\alpha - 1)$ is invariant under α so is M_1. Therefore $\alpha(e) = e$ and $\alpha(f) = f$. Moreover as $R(\alpha - 1) \subseteq M_1 = Me$, we will have $f(\alpha(x) - x) = 0$ for all $x \in M$, and as $\alpha(f) = f$, we get

$$\alpha(fx) = fx \text{ for } x \in M.$$

On the other hand let f_1 be a projection such that $\alpha(f_1x) = f_1x$ for all $x \in M$. Then this is true for $x = 1$, so that $\alpha(f_1) = f_1$ and $f_1(\alpha(x)) = f_1x$ or $f_1(\alpha(x) - x) = 0$ for all $x \in M$. Then $f_1y = 0$ for all $y \in M_1$, in particular $f_1e = 0$ and hence $f_1 \leq f$. This completes the proof of the theorem.

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