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A REMARK ON PRODUCTS OF LOCALLY SOLUBLE GROUPS

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It is shown that if a group G is a product of two normal subgroups, each of which is locally soluble-of-finite-rank, then G is locally soluble if and only if it is locally of finite rank.

A well-known example due to P. Hall indicates that the normal product G of two locally soluble subgroups H and K need not be locally soluble. On the other hand, it is equally well-known that if H and K are locally polycyclic, then so is G. In searching for classes of locally soluble groups which contain all locally polycyclic groups and which are closed under normal joins, one is naturally led to consider products of groups which are locally soluble-of-finite-rank. Using some recent results of Kropholler [1], we are at least able to prove the following:

THEOREM. *Suppose the group G is the product of normal subgroups H and K, each of which is locally soluble-of-finite-rank. Then G is locally soluble if and only if G is locally of finite rank.*

An immediate consequence is that if a group G is locally soluble (or locally of finite rank), then it has a normal subgroup which is locally soluble-of-finite-rank and contains all normal subgroups of G which have this property.

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Proof of the theorem. Since H and K are locally soluble-of-finite-rank, so is $G/(H \cap K)$, because it is the direct product of the images in it of H and K. Let $F = \langle X,Y \rangle$, where X,Y are finitely generated subgroups of H,K respectively. Since every finitely generated subgroup of G is contained in such an F, we restrict our attention to such subgroups.

If G is locally of finite rank then $L := F \cap H \cap K$ has finite rank and is locally soluble. So L is hyperabelian and in fact has a characteristic ascending abelian series [2, Lemma 10.39]. Since F/L is soluble, it follows that F is hyperabelian and, being finitely generated of finite rank, F is soluble (see Corollary 1 to the above-mentioned lemma).

Now suppose G is locally soluble.

If F has finite rank, then we know from Proposition 1 of [1] that *F may be written as a finite product* $\langle x_1 \rangle \ldots \langle x_n \rangle$ *of cyclic subgroups.* We show that each x_i may be chosen to lie in $C = X \cup Y \cup [X,Y]$. It is clear from the proof of Kropholler's result that we need only consider the case where F is metabelian-by-finite. Suppose N is a normal abelian subgroup of F such that F/N is polycyclic, and write $M = [X,Y] \cap N$. Clearly $F/[X,Y]$ may be written in the required form, and therefore so also may F/M, since $[X,Y]/M$ is polycyclic. Proceeding as in the proof of [1,Proposition 1] we deduce that $F = \langle x_1 \rangle \ldots \langle x_n \rangle$, where each x_i belongs to C and hence to $H \cup K$.

Next, suppose only that $F/A = \bar{F}$ has finite rank for some normal abelian subgroup A which is either torsion-free or periodic. Then $\bar{F} = \langle \bar{x}_1 \rangle \ldots \langle \bar{x}_n \rangle$, say, where each $x_i \in H \cup K$. Suppose that for every element a of A, $a^{\langle x_i \rangle}$ has finite rank for each $i = 1,\ldots,n$. In the case where A is torsion-free this means that each $a^{\langle x_i \rangle}$ is finite dimensional as a $\mathbb{Q}$-vector space and, as in the proof of [1,Proposition 2], it follows that $a^F = a^{\langle x_1 \rangle \ldots \langle x_n \rangle}$ has finite rank. If A is periodic, a similar argument (with $\mathbb{Q}$ replaced by $\mathbb{F}_p$) gives us that $a^F/(a^F)^p$ is finite, for each prime p, and hence that a^F has finite abelian section rank. Using the terminology introduced in [1], we have

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that A is in either case a constrained $\mathbb{Z}F$-module. By Lemma 1 and Theorems 1 and 2 of [1], F therefore has finite rank.

For general $F = \langle X,Y \rangle$ we certainly know that F/L has finite rank, where $L = F \cap H \cap K$. We can take a characteristic series in L

$$1 = L_m \triangleleft L_{m-1} \triangleleft \ldots \triangleleft L_1 \triangleleft L_0 = L$$

such that each factor L_j/L_{j+1} is abelian and either torsion-free or periodic. Suppose F/L_j has finite rank. Then we may write $\bar{F} = F/L_j = \langle \bar{x}_1 \rangle \ldots \langle \bar{x}_n \rangle$ where each $x_i \in H \cup K$. Let a be an element of $A = L_j/L_{j+1}$. Since H and K are locally of finite rank, $a^{\langle x_i \rangle}$ has finite rank for $i = 1,\ldots,n$. As before, we deduce that F/L_{j+1} has finite rank. The result follows by induction.

References

[1] P.H. Kropholler, "On finitely generated soluble groups with no large wreath product sections", *Proc. London Math. Soc.*, to appear.

[2] D.J.S. Robinson, "Finiteness conditions and generalised soluble groups", (*Springer, Berlin*, 1972).

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