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ORBIT SIZES, LOCAL SUBGROUPS AND CHAINS OF p-GROUPS

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To Laci Kovács on his 65th birthday

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Abstract

Let G be a finite group that acts on a finite group V, and let p be a prime that does not divide the order of V. Then the p-parts of the orbit sizes are the same in the actions of G on the sets of conjugacy classes and irreducible characters of V. This result is derived as a consequence of some general theory relating orbits and chains of p-subgroups of a group.

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1. Introduction

Whenever a finite group G acts via automorphisms on a finite group V, there are two natural permutation actions of G that have equal degrees: one on the set $\mathrm{cl}(V)$ of conjugacy classes of V and one on the set $\mathrm{Irr}(V)$ of irreducible characters of V. We ask what can be said about the numbers and sizes of the orbits in these two actions.

Consider, for example, the situation where G is the alternating group A_5. Let M be the natural 5-dimensional permutation module for G over the field of order 5 and take V to be the submodule of codimension 1 in M, consisting of all vectors with coordinate sum equal to 0. Then G acts via automorphisms on the group V of order $5^4 = 625$, and it is not too hard to compute that the action of G on the classes (elements) of V has 27 orbits: one of size 1, four of size 5, four of size 10, two of size 12, six of size 20, six of size 30 and four of size 60. Similarly, one finds that the

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action of G on the irreducible (linear) characters of V also has 27 orbits: five of size 1, ten of size 20, ten of size 30 and two of size 60.

It is, of course, no surprise that the numbers of orbits in the two actions are equal; that is always true. (See Theorem 6.32 and Corollary 6.33 of [2] for this result of Brauer.) It should also not be a surprise that the sizes of the orbits in the two actions are quite different. Although it is known that in the case where $|G|$ and $|V|$ are coprime, the actions of G on the classes and irreducible characters of V are permutation isomorphic, this is certainly not true in general. (The existence of the permutation isomorphism in the coprime case is a consequence of the Glauberman-Isaacs character correspondence. See Theorem 13.24 of [2] and the discussion following that result.)

A closer scrutiny of our example shows that the sizes of the orbits in the two actions are not totally unrelated. Of the 27 orbits in each action, for example, exactly 22 of them have size divisible by 2, exactly 12 have size divisible by 4 and exactly 12 have size divisible by 3. (But note that the numbers of orbits with size divisible by 5 are not equal in the two actions and neither are the numbers of orbits with size divisible by 12.) One of the main results of this paper explains these 'coincidences'.

THEOREM A. *Let G act via automorphisms on V, where G and V are arbitrary finite groups, and let p be any prime not dividing $|V|$. Then for each integer $e \geq 0$, there are equal numbers of orbits of size divisible by p^e in the actions of G on* $\mathrm{cl}(V)$ *and on* $\mathrm{Irr}(V)$.

We shall obtain Theorem A as an application of the elementary theory that we develop in this paper. Motivated by an analogy between the orbits of a permutation action of a group G and the irreducible representations of G, we present some connections between the set of orbits and the p-local structure of G, where p is some prime that we hold fixed throughout our discussion. Our orbit results are easy, however, while the corresponding representation theoretic assertions appear to be deep, or are as yet only conjectures.

Suppose that G acts on some set Ω and let X be an orbit of this action. We define the *defect* of X (with respect to our fixed prime p) to be the integer $d \geq 0$ such that p^d is the exact p-part of the integer $|G|/|X|$, and we refer to a Sylow p-subgroup of the stabilizer in G of any point in X as a *defect group* of X. Clearly, then, the defect groups of an orbit X (of the action of G on Ω) form a uniquely determined conjugacy class of p-subgroups of order p^d in G, where d is the defect of X. We can now state a stronger form of Theorem A, which is the version we will prove.

THEOREM B. *Let G act via automorphisms on V, where G and V are arbitrary finite groups, and let p be a prime not dividing $|V|$. Then for each p-subgroup P of G, there are equal numbers of orbits having P as a defect group in the actions of G on* $\mathrm{cl}(V)$ *and on* $\mathrm{Irr}(V)$.

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We need to introduce some notation. Recall that a *section* of a group G is a factor group of the form K/L, where $L \lhd K \subseteq G$. (In particular, since we allow $L = 1$, all subgroups of G are sections.) We will say that a *sectional function* on G is any real-valued function $f(\cdot)$ whose domain is the set of all sections of G. A fairly trivial example of a sectional function is given by the formula $f(K/L) = |K/L|$, but a more interesting example, and one that is vital for our purposes, is the orbit counting function orb$(\cdot)$, which we are about to describe.

Suppose that G acts on some set Ω. (We find it helpful to think of Ω as some large set, containing many orbits of different sizes, but at the opposite extreme, we also want to allow the possibility that Ω is empty, in which case, of course, it contains no orbits at all.) Now let K/L be a section of G. Given the action of G on Ω, there is a uniquely determined induced action of the section K/L on the (possibly empty) set of L-fixed points in Ω, and we write orb(K/L) to denote the number of orbits of this action. Similarly, we can define the sectional function zer$(\cdot)$, which counts the number of orbits of K/L that have p-defect zero. (We assume that p is some particular prime that we are holding fixed.) Thus zer(K/L) is the number of orbits of K/L whose size is divisible by the full p-part of $|K/L|$.

We can now state a fundamental formula, which, as we shall see, is very easy to prove.

THEOREM C. *Let G act on Ω. Then*

$$\mathrm{orb}(G) = \sum_{P} \mathrm{zer}(\mathbf{N}_G(P)/P)\,,$$

where the sum runs over a set of representatives for the conjugacy classes of p-subgroups P of G.

Although Theorem C is nearly trivial, it is interesting to note that it bears a striking formal resemblance to the Alperin Weight Conjecture. Recall that the simplest (block-free) form of the AWC is just the formula of Theorem C, where we replace orb(G) by $l(G)$, which is the number of p-regular conjugacy classes of G, and we replace zer(K/L) by $z(K/L)$, which is the number of p-defect zero irreducible characters of K/L.)

A fundamental result of Knörr and Robinson in [4] shows that the Alperin Weight Conjecture can be restated in an alternative form, in terms of certain chains of p-subgroups of G. (The corresponding restatement of the much more elementary block-free version of the conjecture also appears in the expository paper [3].) A similar restatement of Theorem C is an essential ingredient of our proof of Theorem B.

We consider *normal p-chains* in G, which are, by definition, totally ordered (and possibly empty) collections σ of nonidentity p-subgroups of G such that all of the members of σ are normal in the largest one. We observe that G acts by conjugation on the set of normal p-chains in G and we write G_σ to denote the stabilizer of the chain σ in G. Also, if σ is a nonempty normal p-chain, we write $\mathbf{M}(\sigma)$ to denote the largest member of σ, and if σ is empty, we write $\mathbf{M}(\sigma) = 1$. For all normal p-chains σ, therefore, we see that $\mathbf{M}(\sigma)$ is a normal p-subgroup of G_σ, and $\mathbf{M}(\sigma) > 1$ if and only if σ is nonempty.

The Knörr-Robinson restatement of the block-free form of the AWC is the formula

$$z(G) = \sum_\sigma (-1)^{|\sigma|} l(G_\sigma),$$

where σ runs over a set of representatives for the G-orbits of normal p-chains in G. Our analogous restatement of Theorem C is the following.

THEOREM D. *Let G act on Ω. Then*

$$\mathrm{zer}(G) = \sum_\sigma (-1)^{|\sigma|}\, \mathrm{orb}(G_\sigma/\mathbf{M}(\sigma)),$$

where σ runs over a set of representatives for the G-orbits of normal p-chains in G.

The analogy between the Knörr-Robinson formula and the formula in Theorem D may seem imperfect because the latter requires that we evaluate the function $\mathrm{orb}(\cdot)$ on the *section* $G_\sigma/\mathbf{M}(\sigma)$, whereas in the Knörr-Robinson formula, the function $l(\cdot)$ is evaluated at the *subgroup* G_σ. But this imperfection is only illusory since, in fact, $l(G_\sigma/\mathbf{M}(\sigma)) = l(G_\sigma)$. (In general, it is easy to see that $l(N/P) = l(N)$ whenever P is a normal p-subgroup of a group N.) Our derivation of Theorem D from Theorem C exactly parallels the derivation of the Knörr-Robinson formula from the AWC, and we present this argument in a general form that we call 'Knörr-Robinson inversion'.

We mention that Glauberman and Külshammer have found a completely different proof of our Theorem B, independent of Theorem C and Theorem D. Their argument depends on the observation that if p does not divide $|V|$ and F is a suitable field of characteristic p, then the FG-permutation modules corresponding to the actions of G on $\mathrm{cl}(V)$ and $\mathrm{Irr}(V)$ are isomorphic. We discuss this approach in the final section of this paper.

2. Theorem C

We are assuming that G acts on a set Ω and that p is some given prime.

LEMMA 2.1. *Let $P \subseteq G$ be a p-subgroup. Then the number of G-orbits in Ω having defect group P is exactly* $\mathrm{zer}(\mathbf{N}_G(P)/P)$.

PROOF. Write $N = \mathbf{N}_G(P)$ and let Λ be the (possibly empty) set of P-fixed points in Ω. To prove the result, we construct a bijection θ from the set of G-orbits in Ω that have defect group P onto the set of defect-zero orbits of the action of N/P on Λ.

Let $A \subseteq \Omega$ be a G-orbit with defect group P. By definition, P is a Sylow p-subgroup of G_α, for some point $\alpha \in A$. Then P fixes α, and so $\alpha \in \Lambda$, and we let Y be the (N/P)-orbit containing α. We want to set $\theta(A) = Y$, and so we must show that Y depends only on A and not on the choice of the particular point α. Suppose then that P is also a Sylow p-subgroup of G_β, where $\beta \in A$. Since A is a G-orbit, we can write $\beta = \alpha \cdot g$ for some element $g \in G$, and thus $P^g \subseteq (G_\alpha)^g = G_\beta$. But $P \in \mathrm{Syl}_p(G_\beta)$, and thus by Sylow's theorem, $(P^g)^x \subseteq P$ for some element $x \in G_\beta$. It follows that $P^{gx} = P$, and thus $gx \in N$ and we have $\alpha \cdot gx \in Y$ since Y is the N-orbit containing α. But $\alpha \cdot gx = (\alpha \cdot g) \cdot x = \beta \cdot x = \beta$, and thus Y is the (N/P)-orbit containing β. We can thus set $\theta(A) = Y$, and θ is a well defined function.

We claim that Y has defect zero as an N/P-orbit. To see why this is so, let Q/P be a Sylow p-subgroup of the point stabilizer $(N/P)_\alpha$. Then Q is a p-subgroup of G that fixes α and we have $P \subseteq Q \subseteq G_\alpha$. But $P \in \mathrm{Syl}_p(G_\alpha)$, and thus $P = Q$ and Q/P is the trivial subgroup of N/P. Since Q/P is a defect group for Y as an (N/P)-orbit, it follows that Y has defect zero, as desired.

To see that θ is injective, suppose also that $\theta(B) = Y$, where $B \subseteq \Omega$ is some G-orbit having defect group P. Let $\beta \in B$ be a point such that $P \in \mathrm{Syl}_p(G_\beta)$. Then $\beta \in Y$ and since Y is an N-orbit, it follows that $\alpha \cdot n = \beta$, for some element $n \in N$. But then α and β lie in the same G-orbit, and thus $A = B$, as required.

Finally, we must prove that θ maps onto the set of defect-zero (N/P)-orbits in Λ, and so we suppose that Z is any such orbit and we let $\gamma \in Z$. Since $\gamma \in \Lambda$, we know that P stabilizes γ, and thus $P \subseteq G_\gamma$. Let $P \subseteq Q \in \mathrm{Syl}_p(G_\gamma)$ and note that if $Q > P$ then $Q \cap N > P$, and thus $(Q \cap N)/P$ is a nontrivial p-subgroup of $(N/P)_\gamma$. This is impossible, however, since we assumed that γ lies in the defect-zero (N/P)-orbit Z. It follows that $Q = P$, and thus P is a defect group for the G-orbit C containing γ. We conclude that $\theta(C) = Z$, and thus θ is surjective, as required. $\square$

PROOF OF THEOREM C. We can count the total number $\mathrm{orb}(G)$ of G-orbits on Ω by summing the numbers of G-orbits having defect group P, where P runs over a set of representatives for the conjugacy classes of p-subgroups of G. The result is now immediate from Lemma 2.1. $\square$

It is somewhat more convenient to use a modified version of Theorem C, where we sum over all p-subgroups of G and not just over a set of representatives for the conjugacy classes of p-subgroups. To correct for the resulting overcount, we must, of course, divide the term corresponding to a p-subgroup P by the number $|G : \mathbf{N}_G(P)|$ of subgroups in the class of P. A bit of algebraic simplification yields the following.

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COROLLARY 2.2. *Let G act on Ω. Then*

$$|G|\ \mathrm{orb}(G) = \sum_{P} |\mathbf{N}_G(P)|\ \mathrm{zer}(\mathbf{N}_G(P)/P)\,,$$

where P runs over the set of all p-subgroups of G.

Of course, since Corollary 2.2 is valid for every permutation action of every group, it can also be applied to an arbitrary section K/L of G in its induced action on the set of L-fixed points in Ω. In particular, if we limit ourselves to sections where L is a p-group, we obtain the following.

COROLLARY 2.3. *Let G act on Ω and suppose that Q is a p-subgroup of G and that $Q \triangleleft K \subseteq G$. Then*

$$|K|\ \mathrm{orb}(K/Q) = \sum_{P} |\mathbf{N}_K(P)|\ \mathrm{zer}(\mathbf{N}_K(P)/P)\,,$$

where P runs over the set of p-subgroups such that $Q \subseteq P \subseteq K$.

PROOF. In order to apply Corollary 2.2 to the induced action of K/Q on the set Λ of Q-fixed points in Ω, we must consider p-subgroups P/Q of K/Q. We need to evaluate the quantity $\mathrm{zer}(\mathbf{N}_{K/Q}(P/Q)/(P/Q))$, where here, the sectional function $\mathrm{zer}(\cdot)$ refers to the action of the group K/Q on the set Λ and the induced actions of appropriate sections of K/Q. But $\mathbf{N}_{K/Q}(P/Q) = \mathbf{N}_K(P)/Q$, and so the group $\mathbf{N}_{K/Q}(P/Q)/(P/Q)$ is naturally isomorphic to the group $\mathbf{N}_K(P)/P$, which, of course, is a section of G. Moreover, the (P/Q)-fixed points in Λ are exactly the P-fixed points in Ω, and so it follows that $\mathrm{zer}(\mathbf{N}_{K/Q}(P/Q)/(P/Q)) = \mathrm{zer}(\mathbf{N}_K(P)/P)$. Finally, we observe that since we are assuming that Q is a p-group, the p-subgroups of K/Q are exactly the subgroups of the form P/Q, where P runs over all p-subgroups of K that contain Q.

If we now apply Corollary 2.2 to K/Q, we obtain the formula:

$$\frac{|K|}{|Q|}\ \mathrm{orb}(K/Q) = \sum_{P} |\mathbf{N}_K(P)/Q|\ \mathrm{zer}(\mathbf{N}_K(P)/P)\,,$$

where P runs over the set of p-subgroups such that $Q \subseteq P \subseteq K$. Multiplication of both sides by $|Q|$ yields the desired result. □

We mention that if we apply similar reasoning, starting with the formula given by Alperin Weight Conjecture in place of Theorem C, and if we assume that the AWC is valid for all sections K/Q of G, where Q is a p-subgroup, we obtain a formula that is formally identical to that in Corollary 2.3, except that the functions $\mathrm{orb}(\cdot)$ and $\mathrm{zer}(\cdot)$ are replaced by the functions $l(\cdot)$ and $z(\cdot)$, respectively.

Finally, we consider what happens if we try to replace the prime p by some set π of primes. If we assume that G is π-separable, or more generally that G and its subgroups satisfy all of the Sylow-like theorems for π, and if in addition, we assume that a Hall π-subgroup of G is nilpotent, then the π-analogs of Theorem C and the other results of this section remain valid, with proofs that are essentially unchanged.

3. Knörr-Robinson inversion

Let us refer to a section K/Q of G, where Q is a p-subgroup, as a *p-normal* section of G. (Note that in the formula of Corollary 2.3, the section K/Q appearing on the left side and all of the sections $\mathbf{N}_K(P)/P$ on the right side are p-normal.) We can view Corollary 2.3 as a system of linear equations, one for each p-normal section K/Q of G, where the 'unknowns' are the quantities $\operatorname{zer}(X/P)$, as X/P runs over all p-normal sections of G. Since each of the sections X/P appearing on the right side of the equation whose left side is $|K| \operatorname{orb}(K/Q)$ satisfies the inequality $|X/P| \leq |K/Q|$, it follows that the (square) matrix of this linear system is actually upper (or lower) triangular. Also, if we take $P = Q$ on the right side, then since $|\mathbf{N}_K(Q)| = |K| \neq 0$, we see that the diagonal entries of this matrix are nonzero. It is thus easy to solve these equations for the quantities $\operatorname{zer}(X/P)$, and in particular, we can compute $\operatorname{zer}(G)$. We do this in a little greater generality.

THEOREM 3.1. *Suppose that $f(\cdot)$ and $g(\cdot)$ are arbitrary functions defined on the set of p-normal sections of a group G. Assume for every p-normal section K/Q of G, that these the two functions are related by the formula*

$$(*) \qquad |K| f(K/Q) = \sum_P |\mathbf{N}_K(P)| \, g(\mathbf{N}_K(P)/P),$$

where P runs over all p-subgroups such that $Q \subseteq P \subseteq K$. Then

$$|G| \, g(G) = \sum_\sigma (-1)^{|\sigma|} |G_\sigma| f(G_\sigma/\mathbf{M}(\sigma)),$$

where σ runs over the set of all normal p-chains in G.

PROOF. It clear that the result holds when G is the trivial group, and so we assume that $G > 1$ and we proceed by induction on $|G|$. If we use equation $(*)$ in the case $Q = 1$ and $K = G$ and we separate the term corresponding to $P = 1$ on the right, we obtain

$$(1) \qquad |G| f(G) = |G| \, g(G) + \sum_{P>1} |\mathbf{N}_G(P)| \, g(\mathbf{N}_G(P)/P).$$

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Let us consider the term of the sum in equation (1) corresponding to the nonidentity p-subgroup $P \subseteq G$. For simplicity of notation, we write $N = \mathbf{N}_G(P)$. Note that the p-normal sections of the group N/P can be identified with appropriate p-normal sections of G, and thus the functions $f(\cdot)$ and $g(\cdot)$ naturally define functions on the set of p-normal sections of N/P. Furthermore, using this identification, we see that equations $(*)$ are valid for the group N/P, and thus we can use the inductive hypothesis to evaluate the quantity $|N/P|g(N/P)$.

We need to consider the normal p-chains τ of N/P, and we observe that these are in natural bijective correspondence with the set of those normal p-chains σ of G for which the smallest member of σ is exactly P. (Under this correspondence, for example, the empty chain τ corresponds to the chain $\sigma = \{P\}$.) If τ is any normal p-chain of N/P and σ is the corresponding normal p-chain of G, we see that $(N/P)_\tau = G_\sigma/P$. (This follows since P is a member of σ, and thus every element of G that stabilizes σ must normalize P, and hence such an element lies in N.) Also in this situation, we see that $\mathbf{M}(\tau) = \mathbf{M}(\sigma)/P$ and that $|\sigma| = 1 + |\tau|$.

By the inductive hypothesis, we can now see that the term corresponding to P in the sum appearing in equation (1) is

$$\begin{aligned}|N|\,g(N/P) = |P||N/P|g(N/P) &= |P| \sum_{\tau}(-1)^{|\tau|}|(N/P)_\tau|\,f\left((N/P)_\tau/\mathbf{M}(\tau)\right) \\ &= |P| \sum_{\sigma} -(-1)^{|\sigma|}|G_\sigma/P|\,f\left(G_\sigma/\mathbf{M}(\sigma)\right) \\ &= -\sum_{\sigma}(-1)^{|\sigma|}|G_\sigma|\,f\left(G_\sigma/\mathbf{M}(\sigma)\right),\end{aligned}$$

where in the first sum, τ runs over all normal p-chains in N/P and in the second and third sums, σ runs over the set of normal p-chains in G for which P is the smallest member.

If P is any nonidentity p-subgroup of G, we write $\mathscr{C}(P)$ to denote the set of normal p-chains of G for which P is the smallest member. Then every nonempty normal p-chain in G lies in exactly one of the sets $\mathscr{C}(P)$. We can now return to equation (1), which becomes the following:

$$|G|f(G) = |G|\,g(G) - \sum_{P>1}\sum_{\sigma\in\mathscr{C}(P)}(-1)^{|\sigma|}|G_\sigma|\,f\left(G_\sigma/\mathbf{M}(\sigma)\right).$$

It follows that

$$\begin{aligned}|G|\,g(G) &= |G|f(G) + \sum_{P>1}\sum_{\sigma\in\mathscr{C}(P)}(-1)^{|\sigma|}|G_\sigma|\,f\left(G_\sigma/\mathbf{M}(\sigma)\right) \\ &= \sum_{\sigma}(-1)^{|\sigma|}|G_\sigma|\,f\left(G_\sigma/\mathbf{M}(\sigma)\right),\end{aligned}$$

where in the final sum, σ runs over the set of all normal p-chains in G. (Note that the term in this sum corresponding to the empty chain is exactly $|G|f(G)$.) This completes the proof. □

COROLLARY 3.2. *Let G act on Ω. Then*

$$|G|\ \mathrm{zer}(G) = \sum_{\sigma} (-1)^{|\sigma|} |G_\sigma|\ \mathrm{orb}(G_\sigma/\mathbf{M}(\sigma)),$$

where σ runs over the set of normal p-chains in G.

Theorem D is now immediate since the right side of the equation in that theorem can be rewritten by summing over all normal p-chains σ and dividing each term by $|G : G_\sigma|$, which is the size of the orbit containing the chain σ.

Note that if we assume the Alperin Weight Conjecture for G and its p-normal sections, and we apply Theorem 3.1 with the functions $f(\cdot) = l(\cdot)$ and $g(\cdot) = z(\cdot)$, we obtain what is essentially the (block-free form of) the result of Knörr and Robinson.

Finally, we observe that as in Section 2, if we replace the prime p by a set π of primes, and we make the appropriate assumptions on the structure of G, then everything we have done goes through without essential change.

4. Actions on classes and characters

Suppose now that G acts on some group V and write $\mathrm{orb}_1(\cdot)$ and $\mathrm{orb}_2(\cdot)$ to denote the orbit-counting sectional functions corresponding to the actions of G on $\mathrm{cl}(V)$ and on $\mathrm{Irr}(V)$, respectively. Similarly, let $\mathrm{zer}_1(\cdot)$ and $\mathrm{zer}_2(\cdot)$ be the sectional functions on G that count defect-zero orbits in these two actions.

THEOREM 4.1. *Let G act on V, and suppose that p does not divide $|V|$. Then* $\mathrm{orb}_1(K/Q) = \mathrm{orb}_2(K/Q)$ *and* $\mathrm{zer}_1(K/Q) = \mathrm{zer}_2(K/Q)$ *for every p-normal section K/Q of G.*

PROOF. To prove that $\mathrm{orb}_1(K/Q) = \mathrm{orb}_2(K/Q)$ it suffices to show that the permutation characters π_1 and π_2 corresponding to the actions of K/Q on the sets of Q-fixed conjugacy classes of V and Q-fixed irreducible characters of V are equal.

Write $C = \mathbf{C}_V(Q)$ and note that the section K/Q acts via automorphisms on the group C. Since Q is a p-group and p does not divide $|V|$, it is well known that the map $X \mapsto X \cap C$ defines a bijection from the set of Q-fixed classes of V onto the set $\mathrm{cl}(C)$ of all classes of C. (See Corollary 13.10 of [2], for example.) This bijection is a permutation isomorphism between actions of K/Q on the set of Q-fixed classes of V and on the set $\mathrm{cl}(C)$, and in particular, it follows that these two actions yield equal

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permutation characters of the group K/Q. The character π_1 of K/Q is thus equal to the permutation character of the action of K/Q on $\mathrm{cl}(C)$.

Similarly, the Glauberman correspondence is a natural bijection from the set of Q-fixed irreducible characters of V onto the set $\mathrm{Irr}(C)$. (See Chapter 13 of [2]. Note that in this case, where Q is a p-group, the Glauberman correspondence is especially easy to describe: The Q-fixed irreducible character ψ of V maps to the unique irreducible constituent of ψ_C that has multiplicity not divisible by p.) This defines a permutation isomorphism between the actions of K/Q on the set of Q-fixed irreducible characters of V and on the set $\mathrm{Irr}(C)$, and thus the corresponding permutation characters are equal. It follows that the character π_2 of K/Q is equal to the permutation character of the action of K/Q on $\mathrm{Irr}(C)$.

By the Brauer permutation lemma applied to the action of the group K/Q on the group C, we know that the permutation characters of K/Q corresponding to its actions on $\mathrm{cl}(C)$ and $\mathrm{Irr}(C)$ are equal, and it follows that $\pi_1 = \pi_2$, as desired, and thus $\mathrm{orb}_1(K/Q) = \mathrm{orb}_2(K/Q)$.

Since the action of K/Q on the set of Q-fixed classes of V is permutation isomorphic to its action on $\mathrm{cl}(C)$, we see that $\mathrm{zer}_1(K/Q)$ is equal to the number of defect-zero orbits of the action of K/Q on the set $\mathrm{cl}(C)$. Similarly, since we know that the actions of K/Q on the set of Q-fixed-irreducible characters of V and on the set $\mathrm{Irr}(C)$ are permutation isomorphic, we see that $\mathrm{zer}_2(K/Q)$ is equal to the number of defect-zero orbits of the action of K/Q on $\mathrm{Irr}(C)$. To complete the proof of the Theorem, therefore, it suffices to show that the numbers of defect-zero orbits of the actions of K/Q on $\mathrm{cl}(C)$ and $\mathrm{Irr}(C)$ are equal. It suffices, therefore, to prove the special case of Theorem B where $P = 1$, and to apply that result with K/Q in place of G and with C in place of V.

To prove Theorem B in the case where $P = 1$, we must show that $\mathrm{zer}_1(G) = \mathrm{zer}_2(G)$. By Corollary 3.2, however, we see that $\mathrm{zer}_1(G)$ can be expressed in terms of the sectional function $\mathrm{orb}_1(\cdot)$ on G, evaluated at p-normal sections of G. Similarly, Corollary 3.2 tells us that $\mathrm{zer}_2(G)$ can be expressed in terms of the sectional function $\mathrm{orb}_2(\cdot)$ on G, evaluated at p-normal sections of G. We have already seen, however, that the functions $\mathrm{orb}_1(\cdot)$ and $\mathrm{orb}_2(\cdot)$ agree on all p-normal sections of G, and it follows that $\mathrm{zer}_1(G) = \mathrm{zer}_2(G)$. The proof is now complete. $\square$

Proof of Theorem B. By Theorem 2.1, the numbers of orbits of G with defect group P on the sets $\mathrm{cl}(V)$ and $\mathrm{Irr}(V)$ are equal to the numbers of defect-zero orbits of N/P on the P-fixed members of these two sets, where $N = \mathbf{N}_G(P)$. By Theorem 4.1, however, we know that $\mathrm{zer}_1(N/P) = \mathrm{zer}_2(N/P)$, and the result follows. $\square$

We remarked earlier that Theorem C and Theorem D and their related results would remain true if we replace the prime p by a set π of primes, provided that we assume that G and its subgroups satisfy all of the Sylow-like theorems for π and also that a Hall π-subgroup of G is nilpotent. The arguments of this section show that Theorem B would also remain valid in this situation. (But note that the Glauberman character correspondence is not so easy to describe when the group that is acting is not a p-group.)

5. Permutation modules

In this section, we present an alternative proof of Theorem B that was suggested by Glauberman and Külshammer. We mention that the referee has pointed out that most of what follows can also be deduced from the theory of Scott modules, as presented in [1].

LEMMA 5.1. *Let G act on V, where $|V|$ is not divisible by the prime p. Then there exists a characteristic p field F such that the FG-permutation modules corresponding to the actions of G on* $\mathrm{cl}(V)$ *and* $\mathrm{Irr}(V)$ *are isomorphic.*

PROOF. Let X be the character table of V and let $\mathscr{P}$ and $\mathscr{Q}$ be the permutation representations of G (over the complex numbers) corresponding to the actions of G on $\mathrm{cl}(V)$ and $\mathrm{Irr}(V)$, respectively. Then as in the proof of Brauer's permutation lemma, ([2, Theorem 6.32]) we have the matrix equation $\mathscr{P}(g)X = X\mathscr{Q}(g)$, for all elements $g \in G$.

Now let R be the ring of integers in some algebraic number field chosen to be large enough so that R contains all of the entries of the character table X. Let P be a maximal ideal of R containing the prime number p, and let F be the field R/P. By the second orthogonality relation for characters, we know that X^*X is the diagonal matrix whose nonzero entries are the orders of the centralizers of elements representing the conjugacy classes of the group V. (Here, of course, X^* is the conjugate transpose of the matrix X.) Since p does not divide $|V|$, we see that p does not divide the rational integer $\det(X^*X)$, and it follows that $\det(X^*X) \notin P$, and thus also $\det X \notin P$. We conclude that the image of X modulo P is invertible in the field F and therefore, when $\mathscr{P}$ and $\mathscr{Q}$ are viewed as F-representations of G, they are similar. □

Now let M be any right FG-module, where F is a field, and let $K \subseteq G$ be a subgroup. Let U denote the F-subspace of M consisting of the K-fixed points of M and define the F-linear map $T_K : U \to M$ by the formula $T_K(u) = \sum_r ur$, where r runs over a set R of representatives for the right cosets of K in G. It should be clear that T_K, which is called the *relative trace* map, carries U into the subspace consisting of the G-fixed points in M. Also, T_K is independent of the choice of the set R of coset representatives, and thus the nonnegative integer $\dim_F(T_K(U))$ is an invariant, depending only on the FG-module M and the subgroup K. We write $t_K(M)$ to denote this integer and we note that if M is the direct sum of FG-submodules X and Y, then $t_K(M) = t_K(X) + t_K(Y)$.

The following fact about traces and permutation modules is surely known, but for completeness, we present the elementary proof.

THEOREM 5.2. *Let G act on Ω and suppose that M is the corresponding FG-permutation module, where F is some field of characteristic p. Let $P \subseteq G$ be a p-subgroup. Then $t_P(M)$ is equal to the number of G-orbits on Ω that have defect groups conjugate to subgroups of P.*

PROOF. Since M is the direct sum of the FG-permutation modules corresponding to the individual orbits of the action of G on Ω, it follows from the additivity of the invariant $t_P(\cdot)$ that it is no loss to assume that Ω is a single G-orbit. We need to show that $t_P(M) = 1$ if a defect group of Ω is G-conjugate to a subgroup of P and that $t_P(M) = 0$, otherwise.

If X is any subset of Ω, we will write $\hat{X} \in M$ to denote the sum of the members of X. Since G is transitive on Ω, it follows that $\mathbf{C}_M(G) = F \cdot \hat{\Omega}$, and since this fixed point space has dimension 1, we see that in any case, $0 \le t_P(M) \le 1$. We need to show, therefore, that $t_P(M)$ is nonzero precisely when P contains a defect group for Ω.

The space $U = \mathbf{C}_M(P)$ is spanned by the vectors $\hat{X}$, where X runs over the set of P-orbits in Ω. We know that $T_P(\hat{X}) = \sum_x \sum_r xr$, where x runs over X and r runs over a transversal R for the right cosets of P in G. There are exactly $|X||G : P|$ terms in this sum, and we know that each element of Ω occurs equally often (say m times) among these terms. It follows that $m|\Omega| = |X||G : P|$, and we have $T_P(\hat{X}) = \overline{m}\hat{\Omega}$, where $\overline{m}$ is the image of the natural number m in the field F. Thus $t_P(M) \ne 0$ if and only if there is some P-orbit X in Ω such that the integer $m = |X||G : P|/|\Omega|$ is not divisible by p.

To compute the number m, which depends on the P-orbit X, let $x \in X$ and note that $|X| = |P : P \cap G_x|$ and $|\Omega| = |G : G_x|$. Thus

$$m = \frac{|X||G : P|}{|\Omega|} = \frac{|P : P \cap G_x||G : P|}{|G : G_x|} = |G_x : P \cap G_x|,$$

and we see that m is not divisible by p if and only if P contains a full Sylow p-subgroup of G_x. Thus $t_P(M) \ne 0$ if and only if P contains a full Sylow p-subgroup of the stabilizer of some point of Ω or equivalently, if and only if P contains a defect group for Ω. $\square$

COROLLARY 5.3. *Let G act on Ω and let P be a p-subgroup of G. Then the number of orbits of G on Ω that have defect group P is determined by G and the isomorphism type of the FG-permutation module M corresponding to Ω.*

PROOF. Working by induction on $|P|$, we can assume that for each proper subgroup Q of P, the number n_Q of orbits with defect group Q is determined. By Theorem 5.2, we know that $t_P(M) = n_P + \sum_Q n_Q$, where the sum runs over a set of representatives for those G-classes of p-subgroups of G that contain proper subgroups of P. Of course, $t_P(M)$ is determined by the isomorphism type of M. By the inductive hypothesis, all of the quantities n_Q are also determined, and it follows n_P is determined, as desired. $\square$

We see that Theorem B is an immediate consequence of Lemma 5.1 and Corollary 5.3. It is interesting to note that unlike our proof in Section 4, this Glauberman-Külshammer proof of Theorem B does not rely on the Glauberman character correspondence.

We close with the remark that it is not clear whether or not some version of this module-theoretic proof of Theorem B could be made to work in the case where the prime p is replaced by a set π of primes. As we mentioned earlier, however, Theorem B is true in this greater generality provided that we impose suitable conditions on the group G.

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