ON A NORMAL FORM OF THE ORTHOGONAL TRANSFORMATION II

Hans Zassenhaus

§ 3. Indecomposable matrix pairs II. In this section we continue to study the indecomposable matrix pairs adopting the same notation as in part I of this paper.

LEMMA 2. If the matrix A is regular and if it is symmetric or anti-symmetric such that

$$A^T = \varepsilon A \quad (\varepsilon = \pm 1) \tag{3.1}$$

and if the matrix pair (X,A) is indecomposable then the corresponding representation space either is indecomposable or it is the direct sum of two indecomposable invariant subspaces. These are operator isomorphic if and only if the minimal polynomial m_X of X is equal to $(x-\delta)^{\mu}$ where

$$\delta^{\mu-1} + \varepsilon = 0, \quad \delta = \pm 1; \tag{3.2}$$

at any rate there is even a decomposition of the representation space into the direct sum of two isotropic indecomposable invariant subspaces provided the characteristic of F is not 2.

Proof. From (3.1) it follows that

$$f(u,v) = \varepsilon f(v,u) \quad \text{for } u,v \text{ of } M \tag{3.3}$$

so that f is symmetric if $\varepsilon = 1$ and f is anti-symmetric if $\varepsilon = -1$. Since A is regular, it follows that the linear subspace m' orthogonal to a given r-dimensional subspace m is obtained by solving a system of r independent linear homogeneous equations, hence dim m' = d-r, dim m + dim m' = dim M. If m and m' have only 0 in common then M is the direct sum of m and m'. Since M is orthogonally indecomposable and since m' is invariant if m is invariant, it follows that for each invariant subspace m that is neither 0 nor M, also $m \cap m' \neq 0$.

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There is a decomposition (1.13) of M into the direct sum of non-vanishing indecomposable invariant subspaces $M_1, \ldots,$ M_r. If r=1 then we are finished. Let $r > 1$. Hence $M_i \cap M_i' \neq 0$ for $i = 1, 2, \ldots, r$. The characteristic polynomial of the linear transformation σ_i induced by σ on M_i is equal to its minimal polynomial, namely to $P_i^{\mu_i}$ where P_i is an irreducible polynomial with highest coefficient 1 and with degree d_i. Hence there is precisely one minimal invariant subspace $\neq 0$ of M_i viz. $m_i =$ $P_i(\sigma)^{\mu_i - 1} M_i$ and therefore m_i is contained in M_i'. Let $\mu_1 \geqslant \mu_i$ for $i=1,2,\ldots,r$. Since $\dim m_1' = d - \dim m_1 < d$ it follows that $f(m_1, M_i) \neq 0$ for some $i > 1$. Say $f(m_1, M_2) \neq 0$. Hence by Lemma 1 one has $P_2 = P_1^*$, moreover $f(P_1(\sigma)^{\mu_1 - 1} u, v) \neq 0$ for some u of M_1, v of M_2 and $f(\sigma P_1(\sigma)^{\mu_1-1} u, \sigma v) = f(P_1(\sigma)^{\mu_1-1} u, v)$ $\neq 0$, hence $xP_1(x)^{\mu_1 - 1} \not\equiv 0(P_1^{\mu_1})$, $x \not\equiv 0(P_1)$, hence x and P_1 are mutually prime. Therefore also the polynomials x and $P_1^{\mu_1}$ are mutually prime and hence the congruence $xU \equiv 1(P_1^{\mu_1})$ is solvable by a polynomial U(x) of F[x] so that by the argument used in the proof of Lemma 1 it follows that

$$(3.4) \qquad f(u, R(\sigma)v) = f(R(U(\sigma))u, v)$$

for u of M_1, v of M_2 and for any polynomial R(x) of F[x]. In particular

$$f(u, P_2^{\mu_1-1}(\sigma)v) = f(P_2^{\mu_1-1}(U(\sigma))u, v)$$

$$= f(P_1^{\mu_1-1}(\sigma^{-1})u, v) = f(P_1^{\mu_1-1}(\sigma)u, v) \neq 0$$

for some u of M_1, v of M_2. Hence $P_2^{\mu_1-1}(\sigma)M_2 \neq 0$, $\mu_2 \geqslant \mu_1$ $\mu_1 = \mu_2 = \mu_i$.

If $P_1 \neq P_1^*$ then $P_2 \neq P_2^*$ and by Lemma 1 both M_1 and M_2 are isotropic. Moreover m_1, m_2 are the only minimal subspaces of $M_1 + M_2$ and $f(m_1, M_2) \neq 0$, also $f(M_1, m_2) \neq 0$ as shown above, hence $f(m_2, M_1) = f(M_1, m_2) \neq 0$ and therefore $(M_1 + M_2)$ $\cap (M_1 + M_2)' = 0$, $M = M_1 + M_2$.

If $P_1 = P_1^*$ then the polynomials P_2 and P_1 are equal to the same polynomial P of degree n. Every minimal invariant subspace $m \neq 0$ of $M_1 + M_2$ is contained in $m_1 + m_2$. If $m \neq m_1$ then $m + m_1 = m_1 + m_2$ and hence $f(m, M_1) = f(m + m_1, M_1) =$ $f(m_1 + m_2, M_1) = f(m_2, M_1) \neq 0$, $(M_1 + M_2) \cap (M_1 + M_2)' = 0$, $M = M_1 + M_2$.

Any element u of M is contained in an indecomposable

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invariant component G of M and the intersection $G \cap G'$ does not vanish hence we have identically

$$(3.5) \qquad f(u, P(\sigma)^{\mu-1} \sigma^j u) = 0$$

for u of M and any integer j. It follows that for any two elements u, v of M

$$0 = f(u+v, P(\sigma)^{\mu-1}\sigma^j(u+v)) = f(u, P(\sigma)^{\mu-1}\sigma^j v) + f(v, P(\sigma)^{\mu-1}\sigma^j u)$$

and according to (3.4)

$$f(v, P(\sigma)^{\mu-1}\sigma^j u) = f(P(\sigma^{-1})^{\mu-1}\sigma^j v, u).$$

Because of the symmetry of P it follows that

$$P(\sigma^{-1}) = \sigma^{-n} P(\sigma)$$

$$0 = f(u, \sigma^{-(\mu-1)n+j} P(\sigma)^{\mu-1}(\sigma^{2j+n(\mu-1)} + \varepsilon)v),$$

$$0 = \sigma^{(\mu-1)n+j} P(\sigma)^{\mu-1}(\sigma^{2j+n(\mu-1)} + \varepsilon),$$

$P(x)$ divides $x^{2j+n(\mu-1)} + \varepsilon$,

$P(x)$ divides $(x^2-1)x^{n(\mu-1)}$,

$P(x)$ divides $x^2 - 1$,

$P(x) = x - \delta$ and (3.2).

If $\mu = 1$ then Lemma 2 is proved already.

If $\mu > 1$ then $P(\sigma)^{\mu-1} M = M_P = m_1 \dotplus m_2$ and hence dim $M_P = 2$, dim $M_P' =$ dim M - dim $M_P = 2(\mu-1)$. On the other hand it follows from $P(\sigma) u = 0$ according to (3.4) that $f(u, P(\sigma)v) = f(P(\sigma^{-1}) u, v) = f(\sigma^{-n}P(\sigma) u, v) = 0$ so that $M_P \supseteq P(\sigma) M$. Since dim $P(\sigma)M = 2(\mu - 1)$ it follows that $M_P' = P(\sigma)M$.

If $\mu=2$ then there is a basis a_{i1}, a_{i2} of M_i such that $P(\sigma)a_{i1} = a_{i2}$, $f(a_{i1}, a_{i2}) = 0$ $(i=1,2)$, $f(a_{12}, a_{22}) = 0$, hence $f(a_{11}, a_{22}) \neq 0$, $f(a_{21}, a_{12}) \neq 0$. If the characteristic of f is not 2 then set $b_{11} = a_{11} - \frac{1}{2}f(a_{11}, a_{11}) f(a_{11}, a_{22})^{-1} a_{22}$, $b_{12} = a_{12}$, $b_{21} = a_{21} - \frac{1}{2} f(a_{21}, a_{21}) f(a_{21}, a_{12})^{-1} a_{12}$, $b_{22} = a_{22}$, so that M is the

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direct sum of the isotropic indecomposable invariant subspaces $Fb_{i1} + Fb_{i2}$ (i = 1, 2).

Apply induction over μ. Let $\mu > 2$. The given indecomposable matrix pair induces on $M_{P^{\mu-1}}/M_P$ a matrix pair to which the inductional assumption can be applied so that $P(\sigma)M$ will be the direct sum of two isotropic indecomposable invariant subspaces L_1, L_2 each of dimension $\mu - 1$. Since $\dim L_i' = \dim M - \dim L_i = \mu + 1$, but $L_i' \cap P(\sigma)M = L_i + P(\sigma)^{\mu-2} L_k$ with $i \neq k$, it follows that there is an element x_i of L_i' which does not belong to $P(\sigma)M$. Since $P(\sigma)M = M_P'$ it follows that there is an element y_i of M_P for which $f(x_i, y_i) \neq 0$. The element $z_i = x_i - \frac{1}{2}f(x_i, y_i)^{-1} f(x_i, x_i)y_i$ belongs to L_i' but not to $P(\sigma)M$ so that $f(z_i, z_i)$ vanishes and therefore the invariant subspace $F[\sigma] z_i = K_i$ generated by z_i is an isotropic indecomposable subspace of dimension μ for which $P(\sigma)K_i$ belongs to $L_i' \cap P(\sigma)M$ and therefore $P(\sigma)^2 K_i$ belongs to $P(\sigma)L_i$ so that K_1 and K_2 intersect in 0 and $M = K_1 \dot{+} K_2$, q.e.d.

For the following we remark that every matrix Y satisfying

$$(3.6) \qquad \chi_Y = m_Y = Q(x) = x^n + \alpha_{n-1} x^{n-1} + \dots + \alpha_0$$

is similar to the matrix

$$Y_Q = \begin{bmatrix} 0 & & & & & -\alpha_0 \\ 1 & 0 & & & & -\alpha_1 \\ & 1 & \cdot & \cdot & & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot\ 0 & -\alpha_{n-2} \\ & & & & 1 & -\alpha_{n-1} \end{bmatrix}$$

Moreover we define the matrix pairs

$$(3.7) \quad (X(Y, \varepsilon), A(Y, \varepsilon)) = \left(\begin{pmatrix} Y & 0 \\ 0 & Y^{-T} \end{pmatrix}, \begin{pmatrix} 0 & I_\mu \\ \varepsilon I_\mu & 0 \end{pmatrix} \right) \qquad (\varepsilon = \pm 1)$$

and we observe that the matrix pairs $(X(Y, \varepsilon), A(Y, \varepsilon))$ and $(X(TYT^{-1}, \varepsilon), A(TYT^{-1}, \varepsilon))$ are equivalent. Now we have

THEOREM 1. If the characteristic of the field of reference is not 2 then

a) any indecomposable matrix pair with decomposable first constituent and regular second constituent that is either symmetric

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or anti-symmetric is equivalent to the matrix pair (3.7) where the matrix Y satisfies (3.6) such that the polynomial $Q(x) = P(x)^{\mu}$ is a power of an irreducible polynomial P(x) which is either asymmetric or of the form $x - \delta$ where δ satisfies (3.2), and conversely; b) any matrix pair with indecomposable first constituent and regular second constituent that is either symmetric or anti-symmetric, is equivalent to the matrix pair

$$(X, A)' = ((X_{ik}), (A_{ik})) = (X, A_{\mu}(C)) \qquad (i, k = 1, 2, \ldots, \mu)$$

where

$$(3.8) \quad X_{ik} = \delta_{ik} Y_P + \delta_{i,k+1} Y_P \qquad (i, k = 1, 2, \ldots, \mu)$$

$$A_{\mu}(C) = (A_{ik}), \; A_{ik} = \gamma_{ik} C \qquad (i, k = 1, 2, \ldots, \mu)$$

and either

$$(3.9) \quad \mu = 2\nu, \; \gamma_{ik} = (-1)^i \binom{\nu - i}{k - \nu - 1} + (-1)^k \binom{\nu - k}{i - \nu - 1}$$

or

$$(3.10) \qquad \mu = 2\nu - 1,$$

$$\gamma_{ik} = (-1)^i \binom{\nu - i}{k - \nu} + \tfrac{1}{2}(-1)^{\nu+1} \binom{\nu - i - 1}{k - \nu} + (-1)^k \binom{\nu - k}{i - \nu} + \tfrac{1}{2}(-1)^{k+1} \binom{\nu - k - 1}{i - \nu}$$

such that C is a regular matrix of degree n satisfying

$$(3.11) \qquad C^T = (-1)^{\mu+1} \varepsilon C,$$

$$(3.12) \qquad Y_P^T C Y = C$$

and the polynomial P(x) is irreducible symmetric.

Conversely, if P(x) is a symmetric irreducible polynomial and if C is a regular matrix satisfying (3.11), (3.12) then by means of (3.8), (3.9) and (3.10) a matrix pair $((X_{ik}), (A_{ik}))$ is defined with indecomposable first constituent and regular second constituent that is either symmetric or anti-symmetric.

Proof of a). According to the two previous lemmas there is a decomposition of any representation space M into the direct sum of two isotropic indecomposable invariant subspaces M_1, M_2 of equal dimension. Let $a_1, a_2, \ldots, a_m$ be an F-basis of

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M_1. Since dim $M_1' = \dim M - \dim M_1 = \dim M_1$, and M_1 is contained in M_1' it follows that $M_1' = M_1$, $M_1' \cap M_2 = 0$ and therefore the equations $f(a_i, b_k) = \delta_{ik}$ $(i, k = 1, 2, \ldots, m)$ which for fixed value of k can be transformed into a system of linear equations for the coefficients of b_k with respect to a basis of M_2, have precisely one solution in M_2. Moreover the elements b_1, b_2, ..., b_m form an F-basis of M_2 because any linear relation $\sum \lambda_k b_k = 0$ implies that $0 = f(a_i, \sum \lambda_k b_k) = \sum \lambda_k f(a_i, b_k) = \lambda_i$
Choosing the basis $a_1, a_2, \ldots, a_m, b_1, \ldots, b_m$ of M we obtain the matrix pair

$$\left(\begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix}, \begin{pmatrix} 0 & I_m \\ \varepsilon I_m & 0 \end{pmatrix}\right) \quad (\varepsilon = \pm 1)$$

where $Z = Y^{-T}$ on account of (0.1).

The restrictions on P(x) mentioned in Theorem 1 a) follow from Lemma 2. Conversely, for any polynomial $Q(x) = P(x)^{\mu}$ with highest coefficient 1 a matrix pair satisfying (3.6), (3.7) has a second constituent that is regular and either symmetric or anti-symmetric. If $P(x) = x - \delta$ and if (3.2) holds then P(x) divides the polynomial $x^{2j + n(\mu - 1)} + \varepsilon$ for all non-negative integers so that (3.5) is identically satisfied for the elements u of a representation space M. Therefore for every indecomposable invariant component of M the intersection with its orthogonal subspace does not vanish so that M is orthogonally indecomposable. If P(x) is an irreducible asymmetric polynomial with highest coefficient 1 then the matrix pair defined by (3.6), (3.7) induces a decomposition of M into the direct sum of two isotropic indecomposable invariant subspaces that are neither orthogonal to each other nor operator isomorphic. Hence there is no other decomposition of M into the direct sum of indecomposable invariant subspaces and thus M is orthogonally indecomposable.

Proof of b). We define linear operators D_o, $D_1 = D$, D_2, ... of $F[x]$ over F by

$$D_i(\sum \alpha_j x^j) = \sum_j \binom{j}{i} \alpha_j x^{j-i} \tag{3.13}$$

satisfying the rules

$$D_h D_i = \binom{h+i}{h} D_{h+i} \tag{3.14}$$

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(3.15) $$D_h^{\,n} = \binom{nh}{h} \frac{((n-1)h)!}{(h!)^{n-1}} D_{nh}$$

(3.16) $$D_o = \underline{1}$$

$$D_1 = D = \frac{d}{dx}$$

$$n!\, D_n = \frac{d^n}{dx^n}.$$

(3.17) $$D_h(PQ) = \sum_{i=0}^{h} D_i(P)\, D_{h-i}(Q)$$

(3.18) $$D_h(P^n) = \sum_{\substack{\alpha_1+\alpha_2+\cdots+\alpha_n = h \\ 0 \leq \alpha_i}} D_{\alpha_1}(P)\, D_{\alpha_2}(P) \ldots D_{\alpha_n}(P)$$

Let m be a linear space, ν be a linear transformation of m, τ be another linear transformation of m that is permutable with ν, let θ_i be an isomorphic mapping of m onto another linear space θ_i m, let

(3.19) $$M = \dot{\sum}_{i=1}^{\mu} \theta_i m$$

be the direct sum of the linear spaces $\theta_i m$, let σ be the linear transformation of M that is defined by

(3.20) $$\sigma\left(\sum_{i=1}^{\mu} \theta_i u_i\right) = \sum_{i=1}^{\mu} \theta_i \nu u_i + \sum_{i=2}^{\mu} \theta_i \tau u_{i-1} \quad (u_i \in m)$$

and let $\theta_{\mu+1}, \theta_{\mu+2}, \ldots\ldots$ be the linear mapping of m onto the zero element of M. Then

(3.21) $$\sigma^m(\theta_i u) = \sum_{j=0}^{m} \theta_{i+j}\left(\tau^j \binom{m}{j} \nu^{m-j} u\right)$$

(3.22) $$Q(\sigma)(\theta_i u) = \sum_{j \geq 0} \theta_{i+j}(\tau^j D_j Q(\nu) u)$$

If $P(\nu) = 0$ then

(3.23) $$D_j P^{\mu-1}(\nu) = \sum_{\alpha_1+\alpha_2+\cdots\alpha_{\mu-1} = j} D_{\alpha_1} P(v) D_{\alpha_2} P(v) \ldots D_{\alpha_{\mu-1}} P(v)$$

$$= \begin{cases} 0 & \text{if } j < \mu - 1, \\ (DP(\nu))^{\mu-1} & \text{if } j = \mu - 1 \end{cases}$$

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(3.24) $$D_j P^{\mu}(\nu) = 0 \text{ if } j \leq \mu-1$$

(3.25) $$P^{\mu-1}(\sigma)(\theta_i u) = \sum_{j=0}^{\mu-1} \theta_{i+j}(\tau^j D_j P^{\mu-1}(\nu)(u))$$
$$= \theta_{i+\mu-1}(\tau DP(\nu))^{\mu-1}(u)$$
$$= \begin{cases} 0 & \text{if } i > 1, \\ \theta_{\mu}(\tau^{\mu-1}(DP(\nu))^{\mu-1}(u)) & \text{if } i = 1, \end{cases}$$

(3.26) $$P^{\mu}(\sigma) = 0.$$

Hence for the matrix

(3.27) $$X = (\delta_{ik} Y_p + \delta_{i,k+1} T) \quad (i,k=1,2,\ldots,\mu)$$

satisfying

(3.28) $$TY_P = Y_P T$$

we find

(3.29) $$P^{\mu-1}(X) = (\delta_{i\mu}\delta_{k1} T^{\mu-1} \quad DP(Y_P)^{\mu-1})$$

(3.30) $$P^{\mu}(X) = 0.$$

If P is a separable polynomial then there is a polynomial equation $A(x)P(x) + B(x)DP(x) = 1$ from which it follows that $DP(Y_P)$ is a regular matrix, hence $P^{\mu-1}(X) \neq 0$ if T is not nilpotent. It follows that

(3.31) $$m_X = \chi_X = P$$

if P(x) is irreducible with highest coefficient 1 and distinct from x and if $T = Y_P$. Moreover in this case we have

(3.32) $$P^{\mu-1}(X) = (\delta_{iu}\delta_{k1} Y_P^{\mu-1}(DP(Y_P))^{\mu-1}) .$$

Hence in the case b) the given indecomposable matrix pair is equivalent to a matrix pair $((X_{ik}), (A_{ik}))$ satisfying (3.8).

If $\mu = 0, 1$ then the theorem is clear. Apply induction over μ. Let $\mu > 1$. Let M be a representation space of (X, A). Then $\dim P(\sigma)M = n(\mu-1)$, $\dim (P(\sigma)M)' = n\mu - n(\mu-1) = n$, $(P(\sigma)M)' =$

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$P(\sigma)^{\mu-1}M$, hence the given pair induces an indecomposable pair with representation space $P(\sigma)M/P(\sigma)^{\mu-1}M$. By inductional assumption the given matrix pair is equivalent to a matrix pair $(X, B) = ((X_{ik}), (B_{ik}))$ satisfying (3.8), (3.11), (3.12) and the equations $B_{ik} = A_{ik}$ in the event that $1 < i < \mu$, $1 < k < \mu$. Moreover

(3.33) $$B_{ik} = 0 \text{ if } i = \mu,\ 1 < k \leq \mu \text{ or } 1 < i \leq \mu,\ k = \mu$$

and

(3.34) $$B_{ik} = Y_P^T (B_{ik} + B_{i,k+1} + B_{i+1,k} + B_{i+1,k+1}) Y_P$$

if $1 \leq i \leq \mu$, $1 \leq k \leq \mu$ where by definition $B_{\mu+1,\,k} = B_{i,\mu+1} = 0$.

Hence

(3.35) $$B_{1\mu} = Y_P^T B_{1\mu} Y_P = -A_{2,\mu-1} = A_{1\mu},$$

similarly $B_{\mu 1} = A_{\mu 1}$.

The equation (3.34) suggests considering relations of the form

(3.36) $$U = Y_P^T U Y_P + V$$

where $Y_P^T V Y_P = V$.

It follows that $Y_P^{-T} V = V Y_P$, $Y_P^{-T} U = Y_P + V Y_P$ and by induction over i it follows that $(Y_P^{-i})^T U = U Y_P^i + V(i Y_P^{i-1})$, hence for any polynomial Q

(3.37) $$Q(Y_P^{-1})^T U = UQ(Y_P) + V(DQ(Y_P)),$$

$$0 = P(Y_P) = P(Y_P^{-1}) = UP(Y_P) + V(DP(Y_P)) = V(DP(Y_P)).$$

Since P is separable it follows from $P(Y_P) = 0$ that $DP(Y_P)$ is a regular matrix and hence $V = 0$,

(3.38) $$U = Y_P^T U Y_P$$

(3.39) $$Q(Y_P^{-1})^T U = UQ(Y_P)$$

so that (3.34) splits into

(3.40) $$B_{ik} = Y_P^T \; B_{ik} Y_P$$

and

(3.41) $$0 = B_{i,k+1} + B_{i+1,k} + B_{i+1,k+1} \qquad (1 \leq i \leq \mu, \; 1 \leq k \leq \mu).$$

Thus we obtain for all pairs i,k that are different from 1,1, the relation $A_{ik} = B_{ik}$. In order to make B_{11} vanish, thus reaching full coincidence, we replace the matrix pair $(X, (B_{ik}))$ by the equivalent pair $(X, T^T B T)$ where $T = (\delta_{ik} I_n + \delta_{\mu i} \; \delta_{1k} Q(Y_P^{-1}))$ $(i,k = 1, 2, \ldots, u)$ and the polynomial $Q(x)$ satisfies the equation

(3.42) $$B_{11} + (-1)^{\mu} Q(Y_P^{-1})^T \; C - CQ(Y_P^{-1}) = 0.$$

From (3.11) and (3.39) we infer

(3.43) $$C^{-1} Q(Y_P^{-1}) = Q(Y_P) \; C.$$

A special case of (3.40) is

(3.44) $$B_{11} = Y_P^T \; B_{11} \; Y_P.$$

From (3.12) and (3.44) it follows that

(3.45) $$Y_P^{-1} C^{-1} B_{11} \; Y_P = C^{-1} \; B_{11} = G$$

so that the matrix equation (3.42) is turned into

(3.46) $$G = (-1)^{\mu+1} \; Q(Y_P) + Q(Y_P^{-1}).$$

On the other hand it follows from the equation $B^T = B$

that

(3.47) $$B_{11}^T = B_{11} = CG = G^T \; C^T = (-1)^{\mu+1} \varepsilon \; G^T \; C.$$

The only matrices permutable with Y_P are the polynomials in Y_P or, what amounts to the same, the polynomials in Y_P^{-1} so that $G = R(Y_P^{-1})$ with $R(x)$ being a certain polynomial. Because of (3.47) it satisfies the equation

$$(-1)^{\mu+1} \; R(Y_P^{-1})^T \; C = C \; R(Y_P^{-1}).$$

But from (3.39) it follows that

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$$R(Y_P^{-1})^T C = C R(Y_P)$$

so that

$$(-1)^{\mu+1} R(Y_P) - R(Y_P^{-1}) = 0$$

and (3.46) is solved by setting

$$Q(x) = \tfrac{1}{2}(-1)^{\mu+1} R(x), \qquad \text{q.e.d.} \tag{3.48}$$

Applications of Theorem 1 will be made in the last part of this paper.

(to be continued)

McGill University

CORRECTION TO PART I

Page 32, line 16, For "α_{ik}" read "ξ_{ik}".

Page 34, line 13, For "any x in M" read "any x in M_σ".

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