KRENGEL, U. *Ergodic theorems* (de Gruyter Studies in Mathematics 6, de Gruyter, Berlin-New York, 1985), viii + 357 pp. £39.40.

Two of the best known ergodic theorems, von Neumann's mean ergodic theorem and Birkhoff's pointwise ergodic theorem, deal with convergence of averages. Professor Krengel's research monograph deals entirely with theorems of this kind, giving a comprehensive and up to date account of progress since 1931 in this important part of ergodic theory. Most of the book covers work published since 1960, but earlier work is given due attention.

What does the book contain? I will try to give indications rather than a list of contents. We start with three long chapters, beginning in Chapter 1 with a brief explanation of the setting: a measure space $(\Omega, \mathscr{A}, \mu)$ with classes L_p $(1 \leqq p \leqq \infty)$ of measurable functions f and a linear transformation $T: f \rightarrow Tf$ which may (but need not) be derived from a measurable transformation $\tau: \omega \rightarrow \tau\omega$ by setting $Tf = f \circ \tau$. When τ is measure-preserving, T is an isometry on L_p, and there is a simple proof that the averages $A_n f = n^{-1} S_n f$, where $S_n f = \sum_0^{n-1} T^i f$, converge to a limit assuming only that T is a contraction on L_2. This includes von Neumann's theorem; next comes Birkhoff's theorem that for measure-preserving τ and f in L_1, $A_n f$ converges almost everywhere. We then go on to recurrence and to applications of Birkhoff's theorem to stationary stochastic processes, before we reach an important recent advance: Kingman's subadditive ergodic theorem (1968), with indications of its wide range of applications. The chapter continues with more results related to Birkhoff's theorem.

Chapter 2 (Mean ergodic theorems) begins with the famous results of Yosida and Kakutani (1941), bringing out the importance of compactness conditions on T, and their extension by Eberlein (1949) to general semigroups $\{T_g\}$ of operators. We go on to weak mixing and its connection with the spectrum for a contraction in Hilbert space, catch a glimpse of Furstenberg's theory of multiple recurrence, and finish with splitting theorems (Jacobs-de Leeuw-Glicksberg) which are interesting but too technical to describe here. Chapter 3 (Positive contractions in L_1) starts by describing how this topic was started by E. Hopf (1954), and goes on to the Chacon–Ornstein (1960) theorem about a.e. convergence of ratios $S_n f / S_n g$ for $f \in L_1$, $g \in L_1^+$. The notes (of which more later) explain how this unifies many older results, and the main text discusses how it is connected with convergence results for $A_n f = n^{-1} S_n f$. Next, a long section (Section 3.4) on the existence of finite invariant measures equivalent to μ; a long story with many authors, brought to a happy ending after Hajian and Kakutani (1964) introduced new characters ("weakly wandering sets"). Among many other good things in this chapter there is a drastic example (Chacon 1964) where the averages $A_n f$ diverge for all f in L_1^+ and consequently there can be no σ-finite invariant measure equivalent to μ (the first such example, settling a long-standing problem, was published by Ornstein in 1960).

The remaining five chapters, and a Supplement by A. Brunel (on processes recurrent in the sense of Harris) develop extensions in various directions. They are packed with results discovered in the last dozen years; as a sample to indicate the level, take Akcoglu's theorem (1975) that $A_n f$ converges a.e. when T is a positive contraction in L_p $(1 < p < \infty)$ (the reader may recall, or discover by a backward search, what can be said about $p = 1$).

What is the book like? As the title indicates, it does not cover the whole of ergodic *theory*, so that one should not be surprised to find very little about entropy (only a short account needed for the Shannon–Macmillan–Breiman theorem of information theory in Chapter 9) and nothing about Ornstein theory for Bernoulli transformations. But the restriction to ergodic *theorems* has the advantage that complete and up to date coverage of a major branch of ergodic theory has been achieved. The book is crammed with interesting mathematics, presented at a brisk pace. The reader should have some familiarity with measure theory and with functional analysis, and can then find all that is needed in standard texts such as Dunford and Schwartz, *Linear operators* I and II, (Interscience 1958, 1963). Each section of the book is followed by extensive notes (comments and detailed descriptions of related results with references to a bibliography extending up to 1984–85). The later chapters are intended to be independently readable and it seems to the reviewer that it is possible, as the author claims, to look at a particular section without needing large amounts of earlier material.

The book as a whole will be invaluable for use in advanced courses and seminars and for study by prospective researchers. It is a desirable addition to any library.

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https://doi.org/10.1017/S0013091500017697 Published online by Cambridge University Press