THE RELAXATION METHOD FOR LINEAR INEQUALITIES

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I. Statement of Problem and Main Results

1. The relaxation method. Let A be a closed set of points in the n-dimensional euclidean space E_n. If p and p_1 are points of E_n such that

$$|p - a| > |p_1 - a|, \text{ for every } a \in A, \tag{1.1}$$

then p_1 is said to be *point-wise closer* than p to the set A. If p is such that there is no point p_1 which is point-wise closer than p to A, then p is called a *closest point* to the set A. In 1922 Fejér **(2)** made the interesting observation that the set of closest points to A is identical with the convex hull $K(A)$ of the set A. We have mentioned this remark because it will suggest a way of dealing with our main problem, to which we now turn.

We are given a consistent system of m linear inequalities

$$\sum_{j=1}^{n} a_{ij}x_j + b_i \geqslant 0 \qquad (i = 1, \ldots, m). \tag{1.2}$$

The coefficients a_{ij} and b_i being given numerically, the problem is to devise a numerical procedure which will furnish a solution $(x_1, \ldots, x_n)$ of the system (1.2). In the case of a homogeneous system, i.e. when all $b_i = 0$, we add the obvious requirement that the solution $(x_1, \ldots, x_n)$ obtained be different from the trivial solution $(0, \ldots, 0)$.

A natural approach to this problem will be suggested by Fejér's idea as soon as we place the problem in its customary geometric setting. Each of the inequalities (1.2) defines a closed half-space

$$H_i\colon \sum_{j=1}^{n} a_{ij}x_j + b_i \geqslant 0, \tag{1.3}$$

in terms of which the set of points corresponding to the solutions of (1.2) is identical with the convex polytope

$$A = \bigcap_{i=1}^{m} H_i, \tag{1.4}$$

which is assumed from the outset not to be void. Let $p \notin A$ be given. The following simple construction furnishes a point p_1 which is point-wise closer than p to A: Clearly $p \notin H_j$ for some j. Let p' be symmetric to p with respect to the

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boundary π_j of H_j. If p_1 is on the segment joining p to p' and $p_1 \neq p, p'$, then clearly if $a \in H_j$ then

$$|p - a| > |p_1 - a|$$

holds. As $A \subset H_j$, we see that (1.1) is verified, i.e. p_1 *is point-wise closer than* p *to* A. The numerical "construction" of p_1 is easily done as follows. Let q be the projection of p on the hyperplane π_j, choose a number λ such that $0 < \lambda < 2$ and set

$$p_1 = p + \lambda(q - p).$$

In passing from p to p_1, the point-wise approach to A would seem to be strongest if among the H_j, not containing p, we select the one which is furthest away from p. If $\lambda = 2$ then $p_1 = p'$ and then (1.1) again holds, with the exception that we have the equality sign for the points of A which are on the boundary of H_j, if such points exist.

These remarks suggest the following systematic search for a point of A: Choose a point p at will. If $p \in A$, i.e. its coordinates satisfy (1.2), our quest has ended. If $p \notin A$, let H_j be such that

$$\text{dist}\,(p, H_j) = \max_i \text{dist}\,(p, H_i), \tag{1.5}$$

where dist denotes euclidean distance. Let q be such that

$$q \in H_j, \quad |p - q| = \text{dist}\,(p, H_j). \tag{1.6}$$

If λ is a constant, $0 < \lambda \leqslant 2$, we define

$$p_1 = p + \lambda(q - p). \tag{1.7}$$

For convenience we abbreviate this construction by writing

$$p_1 = F_\lambda(p). \tag{1.8}$$

where $F_\lambda(p)$, defined for $p \notin A$, has been made single-valued by some preassigned rule for choosing j in case that (1.5) should not define j uniquely.[1] If $p_1 \in A$ our process has terminated. If $p_1 \notin A$ we iterate (1.8), obtaining

$$p_2 = F_\lambda(p_1),$$

and we continue in like manner deriving a sequence of points $p = p_0, p_1, p_2, \ldots$, all outside A and connected by the relation

$$p_{\nu+1} = F_\lambda(p_\nu) \qquad (\nu = 0, 1, \ldots). \tag{1.9}$$

There are two alternatives: (1) The process terminates after N steps with a point $p_N \in A$; (2) The process continues indefinitely producing an infinite sequence $\{p_\nu\}$.

2. Statement of the main theorems. S. Agmon **(1)** has recently shown that if $0 < \lambda < 2$ and the sequence $\{p_\nu\}$ is infinite, then p_ν converges, as $\nu \to \infty$,

[1]For instance the smallest j satisfying (1.5).

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to a point on the boundary of A. We give a new proof of this result (Theorem 1, Case 1, and Theorem 2, Case 1, below). Our main contribution, however, is the investigation of the case when $\lambda = 2$. Throughout this paper we denote by r the dimensionality of the polytope A defined by (1.4). As we assume that A is not void, r may have any value from zero to n. We denote by L_r the r-flat which contains A.

THEOREM 1. *We assume that $r = n$, i.e. A is not contained in any hyperplane of E_n. Let $\{p_\nu\}$ be a sequence of points obtained by the process described in* § 1. *There are two cases:*

CASE 1. *If $0 < \lambda < 2$ then either $\{p_\nu\}$ terminates or else p_ν converges to a point l on the boundary of A.*

CASE 2. *If $\lambda = 2$ then the sequence $\{p_\nu\}$ always terminates.*

The formulation of our results for the case when $r < n$ requires the following remarks concerning spherical surfaces. Let L_r be a given r-flat in E_n, $0 \leqslant r \leqslant n - 1$. We are also given a point p, $p \notin L_r$. *Let X be the locus of points x such that*

$$|x - a| = |p - a|, \qquad \textit{for every } a \in L_r. \tag{1.10}$$

We claim that X is a spherical surface S_{n-r-1} of dimension $n - r - 1$. Thus if $r = 0$, then L_r reduces to a single point a and X is evidently the S_{n-1} with center at a passing through p. In the other extreme case when $r = n - 1$, L_r is a hyperplane and the locus X contains exactly two points: the point p and its symmetric image with respect to L_r. These two points form a S_0 located on the line through p which is normal to L_r. A general proof of our assertion is as follows: Let b be the orthogonal projection of p onto L_r. Erect at b the $(n - r)$-flat L'_{n-r} which is normal to L_r. Evidently $p \in L'_{n-r}$. *Then $x \in X$ if and only if $x \in L'_{n-r}$ and $|x - b| = |p - b|$.* Indeed, assume that $x \in X$. By (1.10), for $a = b$, we obtain that $|x - b| = |p - b|$. This last equality and (1.10) show that for every $a \in L_r$ the two triangles xba and pba are congruent. Since $\angle pba = 90°$ we conclude that $\angle xba = 90°$. Hence the line joining b and x is normal to L_r and $x \in L'_{n-r}$. Conversely, if $x \in L'_{n-r}$ and $|x - b| = |p - b|$, let us show that $x \in X$. This is now clear because the two triangles pba and xba $(a \in L_r)$ are right-angled at b and have equal legs respectively. This implies (1.10), hence $x \in X$. The locus X may accordingly be defined by the two conditions

$$x \in L'_{n-r}, \quad |x - b| = |p - b|,$$

and is therefore seen to be identical with the spherical surface S_{n-r-1} of L'_{n-r} having its center at b and passing through p. We shall refer to S_{n-r-1} as a spherical surface having L_r as its axis, for indeed, by (1.10), L_r is precisely the locus of points a with the property of being equidistant from all points of S_{n-r-1}.

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THEOREM 2. *We assume that* $r < n$, $A \subset L_r$. *Let* $\{p_\nu\}$ *be a sequence of points obtained by the process of* § 1.

CASE 1. *If* $0 < \lambda < 2$, *then* $\{p_\nu\}$ *either terminates or else* p_ν *converges to a point* l *of* A.

CASE 2. *If* $\lambda = 2$, *then* $\{p_\nu\}$ *either terminates or else there is a number* ν_0 *such that the points* p_ν, *for* $\nu \geqslant \nu_0$, *are on a spherical surface* S_{n-r-1} *having* L_r *as its axis.*

3. Remarks. (a) The procedure here described for finding a solution of (1.2) is called the *relaxation method*, especially if $\lambda = 1$, when it may also be called the *projection method*. We speak of *under-relaxation* or *over-relaxation* depending on whether $0 < \lambda < 1$ or $1 < \lambda < 2$. The case when $\lambda = 2$ is an extreme case of over-relaxation which may also be called the *reflexion method*.

(b) Theorem 1, Case 2, describes the main advantage of the reflexion method ($\lambda = 2$). No other value of λ, $0 < \lambda < 2$, has the property of always leading to a terminating sequence if $r = n$. If $0 < \lambda \leqslant 1$, this is easily shown by consideration of a triangle A in E_2. For a λ with $1 < \lambda < 2$, an example of a non-terminating sequence in E_2 is constructed as follows (Fig. 1): Let $\angle pqO = 90°$

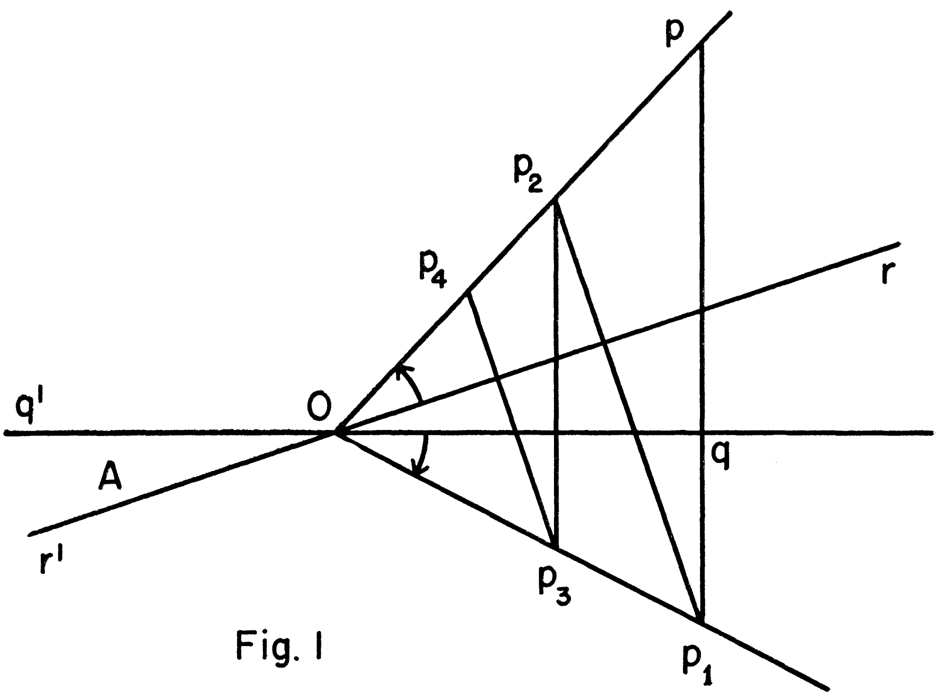

Fig. I

and p, q, p_1 be such that $pp_1/pq = \lambda$, hence $pq > qp_1$. Draw the ray Or such that $\angle pOr = \angle qOp_1$ and produce Or into a full line $r'Or$. Let A be the intersection of the closed half-plane below $q'q$ and the closed half-plane above $r'r$.

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Starting with p and iterating the process $p_1 = F_\lambda(p)$, we obtain an *infinite* sequence of points $\{p_\nu\}$ which oscillate between the rays Op and Op_1, converging to O.

(c) Let $\lambda = 2$, $r < n$ and let us suppose that the sequence $\{p_\nu\}$ is infinite. By Theorem 2, Case 2, for $\nu > \nu_0$, all p_ν are on a S_{n-r-1} having L_r as its axis. Since p_ν and $p_{\nu+1}$ are both on S_{n-r-1}, the hyperplane with respect to which p_ν and $p_{\nu+1}$ are symmetric to each other must contain the axis L_r of S_{n-r-1}. We may state this result as

COROLLARY 1. *Let $r < n$ and let the reflexion process ($\lambda = 2$) lead to an infinite sequence $\{p_\nu\}$. Then there exists an integer ν_0 such that all the hyperplanes*

$$\pi_i\colon \sum_{j=1}^{n} a_{ij}x_j + b_i = 0 \tag{1.11}$$

which are actually used in the reflexion process for $\nu > \nu_0$ contain the entire polytope A and therefore also the r-flat L_r which contains A. Any such hyperplane, or combination of such independent hyperplanes, may therefore be used to reduce the problem to one of a dimension less than n.

(d) When the inequalities (1.2) are all homogeneous, we wish to find a point of A distinct from its vertex o. The relaxation method may well lead to the trivial solution o, if $0 < \lambda < 2$. Thus if $1 < \lambda < 2$ and if A is the "cone" in E_2 of Fig. 1, we have the infinite sequence $\{p_\nu\}$ converging to o. By Theorem 1, Case 2, and Theorem 2, Case 2, this can never happen if $\lambda = 2$.

In II and III we prove the Theorems 1 and 2 respectively. In IV we discuss the behavior of the reflexion process for a special kind of infinite family of half-spaces, namely all half-spaces of support of a bounded and closed convex set in E_n. A study of this problem, suggested by our previous discussion, seems justified by its own geometric interest.

II. A PROOF OF THEOREM 1

4. On Fejér-monotone sequences of points. Let A be defined by (1.4) and let $q_0, q_1, q_2, \ldots$ be an infinite sequence of points outside A with the following properties:

$$q_i \neq q_{i+1}, \tag{2.1}$$

$$|q_i - a| \geqslant |q_{i+1} - a|, \text{ for all } a \in A,\ i = 0, 1, \ldots. \tag{2.2}$$

The sequence $\{q_\nu\}$ is approaching the set A point-wise and we summarize this situation by saying that the sequence $\{q_\nu\}$ is *Fejér-monotone* with respect to A. Concerning such sequences we prove

LEMMA 1. *Let the sequence $\{q_\nu\}$ be Fejér-monotone with respect to the polytope A, assumed to be of dimension r.*

CASE 1. *If $r = n$ then the sequence $\{q_\nu\}$ converges to a point.*

CASE 2. *If $r < n$ then the sequence $\{q_\nu\}$ either converges to a point or else the set of its limit points lies on a spherical surface S_{n-r-1} whose axis is the r-flat L_r spanned by A.*

Proof. CASE 1. Let $r = n$ and consider the spherical surfaces

(2.3) $$S^{\nu}(a) : |x - a| = |q_{\nu} - a| \qquad (a \in A, \nu = 0, 1, \ldots).$$

By (2.2) the surface $S^{\nu}(a)$ is non-expanding as its center a is kept fixed and ν increases. Therefore the following limits exist

(2.4) $$\lim_{\nu \to \infty} |q_{\nu} - a| = R(a) \qquad (a \in A).$$

Define the surface

(2.5) $$S(a) : |x - a| = R(a) \qquad (a \in A),$$

and let us consider the set

(2.6) $$X = \bigcap_{a \in A} S(a).$$

(a) *Every limit point l of the sequence $\{q_{\nu}\}$ is in X.* Indeed by (2.4) we see that $l \in S(a)$, for every $a \in A$, hence $l \in X$, by (2.6). We conclude that X is not void because the bounded sequence $\{q_{\nu}\}$ has at least one limit point.

(b) *The set X contains exactly one point l.* Indeed, if $l \neq l'$, $l \in X$, $l' \in X$, then let π denote the hyperplane of points equidistant from l and l'. If $a \in A$ then l, l' being both in X, are also both in $S(a)$. Hence $a \in \pi$ and we conclude that $A \subset \pi$ in contradiction to our assumption that $r = n$.

(c) *We conclude that $q_{\nu} \to l$.* Indeed, by (a) and (b) we see that $l = X$ is the only limit point of the sequence $\{q_{\nu}\}$.

CASE 2. Let $r < n$, $A \subset L_r$. If the sequence $\{q_{\nu}\}$ converges to a point then there is nothing to prove. Let us assume that

(2.7) *the sequence $\{q_{\nu}\}$ does not converge to a point.*

In any case the bounded sequence $\{q_{\nu}\}$ has limit points and let p be one of them. Define as before the spheres $S^{\nu}(a)$, $S(a)$ and the set X by (2.3), (2.4), (2.5) and (2.6). By (2.4) we have that $R(a) = |p - a|$ $(a \in A)$. The set X is therefore identical with the set of points x such that

$$|x - a| = |p - a|, \text{ for every } a \in A.$$

Since A spans L_r we may also define X as the locus of points x such that

(2.8) $$|x - a| = |p - a|, \text{ for every } a \in L_r.$$

As shown in § 2, (2.8) defines a S_{n-r-1}, provided that the locus does not reduce to a point of L_r. Since this locus contains all limit points of $\{q_{\nu}\}$, our assumption (2.7) excludes this possibility and (2.8) defined a spherical surface S_{n-r-1}. This completes a proof of Lemma 1.

5. Proof of Theorem 1, Case 1. Here $r = n$; assume the sequence $\{p_{\nu}\}$ to be infinite. As already mentioned in § 1, the sequence $\{p_{\nu}\}$ is Fejér-monotone with respect to A. By Lemma 1, Case 1, the sequence $\{p_{\nu}\}$ converges to a point l:

(2.9) $$\lim p_{\nu} = l.$$

We have to show that $l \in A$. For this purpose we introduce the function

$$d(x) = \max_i \text{dist}\,(x, H_i). \tag{2.10}$$

Observe that $d(x)$ is everywhere continuous and that

$$d(x) \begin{cases} =0, & x \in A, \\ >0, & x \notin A. \end{cases} \tag{2.11}$$

Now (2.9) implies that $|p_{\nu-1} - p_\nu| \to 0$ and therefore also that

$$d(p_\nu) = \frac{1}{\lambda}\,|p_{\nu+1} - p_\nu| \to 0.$$

By the continuity of $d(x)$ we have

$$d(l) = \lim d(p_\nu) = 0$$

and therefore $l \in A$, by (2.11). The point l, being a limit of exterior points p_ν, must be on the boundary of A.

6. Proof of Theorem 1, Case 2. Let $\lambda = 2$, $r = n$, and let us show that the sequence $\{p_\nu\}$ must terminate. Indeed, suppose it were infinite. By Lemma 1, Case 1, we again conclude that

$$\lim p_\nu = l \tag{2.12}$$

and the argument used in the previous paragraph shows that l is on the boundary of A. Let $p_{\nu+1}$ be obtained from p_ν by reflexion in the boundary π_{j_ν} of the half-space H_{j_ν}. By (2.12) it is clear that

$$\lim_{\nu \to \infty} \text{dist}\,(l, \pi_{j_\nu}) = 0. \tag{2.13}$$

The given family of hyperplanes (1.11) being finite, we conclude from (2.13) that

$$\text{dist}\,(l, \pi_{j_\nu}) = 0,$$

provided $\nu \geqslant \nu_0$, hence

$$l \in \pi_{j_\nu}, \qquad \nu \geqslant \nu_0.$$

This conclusion, however, contradicts (2.12). Indeed, every point p_ν ($\nu > \nu_0$) is obtained from the preceding point $p_{\nu-1}$ by reflexion in a hyperplane through l. All these points must therefore lie on the spherical surface

$$|x - l| = |p_{\nu_0} - l| \;(> 0)$$

and can therefore never converge to l, as (2.12) requires.

III. A Proof of Theorem 2

7. Proof of Theorem 2, Case 1. We assume $r < n$, $0 < \lambda < 2$, and that the relaxation process (1.9) furnishes an infinite sequence $\{p_\nu\}$. We are to show that

p_ν converges to a point l of A. If the sequence $\{p_\nu\}$ converges to a point l then the argument of § 5 shows that $l \in A$ and we are through. Let us now assume that

(3.1) *the sequence* $\{p_\nu\}$ *does not converge*

and show that we shall reach a contradiction. By Lemma 1, Case 2, and (3.1), we conclude that all the limit points of $\{p_\nu\}$ are on a spherical surface S_{n-r-1} having L_r as an axis.

(a) *Our assumptions imply that*

$$\inf |p_{\nu+1} - p_\nu| = c > 0. \tag{3.2}$$

Indeed, consider the function $d(x)$, defined by (2.10), for the points x on the surface S_{n-r-1}. Since $A \cap S_{n-r-1} = 0$, we conclude that

$$d(x) > 0, \text{ if } x \in S_{n-r-1}.$$

Since $d(x)$ is continuous and S_{n-r-1} is compact, we conclude that

$$\gamma = \min d(x) > 0, \qquad x \in S_{n-r-1}.$$

Let us select γ_1 fixed such that $0 < \gamma_1 < \gamma$. Let δ be positive and let N_δ denote the set of points x defined by

$$\text{dist}\,(x, S_{n-r-1}) < \delta.$$

Again by the continuity of $d(x)$ we can select δ so small that

$$d(x) > \gamma_1, \qquad x \in N_\delta. \tag{3.3}$$

By Lemma 1, Case 2, we have $p_\nu \in N_\delta$, provided $\nu > \nu_0$. Now (3.3) implies that

$$|p_{\nu+1} - p_\nu| = \lambda\, d(p_\nu) > \lambda\, \gamma_1,$$

provided $\nu > \nu_0$. This proves (3.2) with $c \geqslant \lambda\, \gamma_1 > 0$.

We may now easily show that the assumption (3.1) leads to a contradiction. Indeed, let the point α on S_{n-r-1} be a limit point of $\{p_\nu\}$. For an appropriate subsequence $\{\nu'\}$ of the sequence of all integers $\{\nu\}$ we have

$$p_{\nu'} \to \alpha \in S_{n-r-1}.$$

For a subsequence $\{\nu''\}$ of $\{\nu'\}$ we may also assume that

$$p_{\nu''} \to \alpha, \text{ and } p_{\nu''+1} \to \beta \in S_{n-r-1}.$$

By (3.2) we conclude that $\alpha \neq \beta$; in fact

$$|\alpha - \beta| \geqslant c > 0.$$

Select on the line through α and β a point η such that $\beta - \alpha = \lambda(\eta - \alpha)$, and notice, because of $0 < \lambda < 2$, that η is nearer to β than to α:

$$|\eta - \alpha| > |\eta - \beta|. \tag{3.4}$$

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For the subsequence $\{\nu''\}$, the half-spaces H_j, used in obtaining $p_{\nu''+1}$ from $p_{\nu''}$, must converge to the half-space

$$H : |x - \alpha|^2 - |x - \beta|^2 \geqslant |\eta - \alpha|^2 - |\eta - \beta|^2; \tag{3.5}$$

in fact H_j must already be identical with H, for sufficiently large ν'', because of the finiteness of the number of half-spaces H_i. This, however, leads to a contradiction, for on the one hand $A \subset H_j$ implies that $A \subset H$. Hence $x \in A$ implies $x \in H$ and therefore by (3.5) and (3.4)

$$|x - \alpha|^2 - |x - \beta|^2 \geqslant |\eta - \alpha|^2 - |\eta - \beta|^2 > 0,$$

or

$$|x - \alpha| > |x - \beta|. \tag{3.6}$$

On the other hand, A being on the axis of S_{n-r-1}, we must have in A the equality $|x - \alpha| = |x - \beta|$ in contradiction to (3.6). Thus our assumption (3.1) is untenable and the proof is completed.

8. Proof of Theorem 2, Case 2. We assume $r < n$, $\lambda = 2$, and that the reflexion process produces an infinite sequence $\{p_\nu\}$. This sequence cannot possibly converge, for its limit l would belong to A (§ 5) and would then have to terminate (§ 6). By Lemma 1, Case 2, the only alternative is that the sequence $\{p_\nu\}$ converges to a spherical surface S_{n-r-1} of axis L_r. Then (3.2) or

$$\inf |p_{\nu+1} - p_\nu| > 0 \tag{3.7}$$

again holds. Out of A select $r + 1$ fixed points $a_0, a_1, \ldots, a_r$ spanning L_r and let π_{j_ν} be the reflecting hyperplane used in obtaining $p_{\nu+1}$ as the point symmetric to p_ν. Since all limit points of p_ν are on S_{n-r-1}, it is clear that

$$\lim_{\nu\to\infty} |a_k - p_\nu| = \lim_{\nu\to\infty} |a_k - p_{\nu+1}| = R(a_k) \qquad (k = 0, \ldots, r).$$

This, together with (3.7), shows that

$$\lim_{\nu\to\infty} \text{dist}\,(a_k, \pi_{j_\nu}) = 0 \qquad (k = 0, \ldots, r).$$

By the finiteness of our supply of reflecting hyperplanes we conclude from the last relations that

$$a_k \in \pi_{j_\nu}, \qquad \nu \geqslant \nu_0;\ k = 0, \ldots, r,$$

or what amounts to the same thing:

$$L_r \subset \pi_{j_\nu}, \qquad \nu \geqslant \nu_0.$$

In other words: there is a number ν_0 such that all reflexions for $\nu \geqslant \nu_0$ are performed with respect to hyperplanes π_{j_ν} which contain the axis L_ν of S_{n-r-1}. This, however, requires that

$$p_\nu \in S_{n-r-1}, \qquad \nu \geqslant \nu_0.$$

Indeed, if $p_{\nu_0} \notin S_{n-r-1}$, then all p_ν $(\nu \geqslant \nu_0)$ would lie on a surface S'_{n-r-1} of axis L_ν, passing through p_{ν_0}, and could then not converge to S_{n-r-1}, as we assumed.

IV. The Reflexion Process with Respect to a Convex Domain

9. Statement of the problem. We have so far discussed the behavior of the reflexion process with respect to a finite family $\{H\}$ of half-spaces in E_n. Do the results obtained extend to infinite classes of half-spaces? We deal here only with the following special case of this problem: Let A be a given closed and bounded convex set in E_n. A closed half-space H belongs to the family F if and only if the boundary of H is a hyperplane of support of A and $A \subset H$. Let $p \notin A$ and let q be the point of A which is nearest to p. Let π_0 be the hyperplane through q which is normal to the segment joining p and q and let H_0 be the closed half-space, bounded by π_0, which does not contain p. Evidently $H_0 \in F$; also

$$\text{dist}\ (p, H_0) = \max_{H \epsilon F} \text{dist}\ (p, H)$$

Indeed, if there were a $H \in F$ such that dist $(p, H) >$ dist $(p, H_0) = |p - q|$, then $q \notin H$, in contradiction to the fact that $q \in A = \bigcap H$. This shows that the reflexion process with respect to the family $F = \{H\}$ amounts to the construction of the point

$$p_1 = p + 2(q - p) = F(p) \qquad (p \notin A). \tag{3.1}$$

Let us call $p_1 = F(p)$ the *image* of p with respect to A.

If $p_1 \notin A$ we may form $p_2 = F(p_1)$ and continue in like manner obtaining a sequence of points $p = p_0, p_1, p_2, \ldots$ connected by the relation

$$p_{\nu+1} = F(p_\nu) \qquad (\nu = 0, 1, \ldots). \tag{3.2}$$

We have again the old alternative: (1) The process terminates after N steps with $p_N \in A$; (2) The process continues indefinitely, producing an infinite sequence $\{p_\nu\}$.

10. The main result. The behavior of the reflexion process with respect to A is described by the following

Theorem 3. *Let A be a closed convex and bounded set of E_n of dimension r and let L_r be the r-flat containing A. Suppose $p_0 \notin A$ and let $\{p_\nu\}$ be the sequence obtained by the reflexion process* (3.1), (3.2).

Case 1. *If $r = n$, then the process always terminates.*

Case 2. *Let $r < n$. If $p_0 \in L_r$ then the process terminates. If $p_0 \notin L_r$ then the process produces an infinite sequence $\{p_\nu\}$ with the following property. There is a number ν_0 such that for all $\nu > \nu_0$ the points p_ν oscillate between two points which are symmetric with respect to L_r.*

Proof. CASE 1. Let $r = n$ and let us assume, to obtain a contradiction, that the sequence $\{p_\nu\}$ is infinite. We know that $\{p_\nu\}$ is a Fejér-monotone sequence and hence that it converges to a point a by Lemma 1, Case 1. Clearly $a \in A$, and hence a is on the boundary of A, by the argument of § 5. Let P be the projection cone of A at the point a, i.e. the intersection of all H whose boundary hyperplanes pass through a. The cone P is convex and of dimension n, since $P \supset A$. There is therefore a half-space $H \in F$, whose boundary π supports P (and also A) at the point a and whose interior normal ai, at a, is wholly interior to P, except for the point a. Let us think of π as horizontal and its normal ai as pointing vertically downward. The point i being interior to P also a certain small spherical neighborhood S of i is in P. Let us call C the slim circular cone, of vertex a and axis ai, which is circumscribed to S. This convex cone C is wholly in P.

Let us denote by Q the convex cone of vertex a which is generated by the interior normals of all H supporting P (or A) at the point a. These H support also C at a. It follows that the closed convex cone Q (called the polar cone of A at a) has only the point a in common with π, Q being below π. Q was defined as the locus of all interior normals of A at a. Let us denote by N a small neighborhood of a on the boundary of A. Let Q' be a given closed and convex cone satisfying the following conditions: (i) Every ray of Q is interior to Q', (ii) Q' has only the point a in common with π. Clearly, a neighborhood N of a exists such that the interior normals to A at the points of N, if transferred parallel to themselves so as to start at a, will all lie in Q'. In fact otherwise we could find normals at points converging to A whose limit (which obviously being a normal at A) would be outside or on the boundary of Q'.

We now return to the sequence of points $\{p_\nu\}$ which converges to a. Let q_ν be the midpoint of the segment $p_\nu p_{\nu+1}$. It is clear by our construction that $q_\nu \in$ boundary of A and that the vectors $q_\nu p_{\nu+1}$ are interior normals to A. Also $p_\nu \to a$ implies that $q_\nu \to a$, hence $q_\nu \in N$, provided $\nu \geqslant \nu_0$. This leads to a contradiction. Indeed, consider the sequence of points

$$p_{\nu_0}, p_{\nu_0+1}, \ldots .$$

We know from what was said above that the vectors $q_\nu p_{\nu+1}$, if transferred to a, lie in Q'. That means that the vectors

$$p_\nu p_{\nu+1} \qquad (\nu \geqslant \nu_0)$$

have a positive component in the direction of the downward vertical vector ai. Since $p_\nu \to a$, we conclude that all points $p_\nu (\nu \geqslant \nu_0)$ are *above* the horizontal plane π. But this implies that also q_ν are above π. This, however, is absurd since $q_\nu \in A$ and A is below π.

CASE 2. If $r < n$ and $p_0 \in L_r$ then again the process must terminate by the previous case because we have no occasion to leave L_r in the course of our process.

We now assume that $r < n$ while $p_0 \notin L_r$. Let L_{r+1} be the $(r + 1)$-flat containing L_r and p_0. Note that we never leave L_{r+1} which amounts to assuming at the start that $r = n - 1$. Let, therefore, A be $(n - 1)$-dimensional, $A \subset L_{n-1}$ and $p_0 \notin L_{n-1}$. Let p'_0 and p'_1 be the projections of p_0 and p_1, respectively, on L_{n-1}. It should be clear that a point q_0 of A is nearest to p_0 if and only if it is nearest to p'_0. It follows that $p_1 = F(p_0)$ implies that $p'_1 = F(p'_0)$. Consider the sequence of reflexions $\{p_\nu\}$. It is clear that the distance from p_ν to L_{n-1} has the same positive value dist (p_0, L_{n-1}), the points p_ν passing from one side of the plane to the other alternately. However, the sequence $\{p'_\nu\}$ of their projections on L_{n-1} has the property $p'_{\nu+1} = F(p'_\nu)$. By the previous case this sequence "terminates" with a first $p'_N \in A$. From that moment onwards the sequence p_ν must oscillate between two points on the normal to L_{n-1} at the point $p'_N = p'_{N+1} = \ldots$.

References

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