FINITE AMPLITUDE EFFECTS OF TORSIONAL ALFVÉN WAVES IN A FLUX TUBE

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Abstract: Consider a cylindrical, homogeneous magnetic flux tube embedded in a field-free compressible plasma. We employ the method of successive approximations to solve the magnetohydrodynamic equations analytically for an axisymmetrical (m=0) torsional Alfvén mode. The m=0 mode inside the tube may excite surface-, stationary- or progressive waves in the field-free plasma. We conclude that, in contrast to the linear theory, the flux tube may lose energy by wave radiation to the surroundings.

1. INTRODUCTION

Our understanding of the mechanical energy propagation and the heating process in stellar atmospheres is still rather limited. As it appears that structured magnetic fields play a fundamental role, studies of wave propagation in magnetic flux tubes embedded in a compressible plasma are of considerable interest. Recent reviews of small-amplitude wave propagation in flux tubes are given by Roberts (1981), Wilson (1981), Spruit (1982) and Edwin and Roberts (1982).

Investigations into the nonlinear nature of the wave propagation are valuable in understanding the damping mechanisms and the heating processes. Hitherto few attempts in this direction have been done. The interesting possibility of solitons in solar magnetic flux tubes has been pointed out by Roberts and Maugeney (1982). A discussion of the nonlinear aspects of propagation of Alfvén waves has recently been given by Hollweg et al. (1982). They consider the Alfvén waves as time-dependent axisymmetrical twists and solve the equations numerically. One of their simplifying assumptions seems vital, i.e. the suppression of the particle motion in the direction perpendicular to both the average field and to the azimuthal direction. In this paper an analytical study of torsional Alfvén waves in magnetic flux tubes embedded in a field-free compressible plasma is presented. In a thorough discussion different Alfvén wave modes should be considered. We have concentrated our efforts to the axisymmetrical torsional (m=o) mode in

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order to point out differences between linear and non-linear theory. Allowing motions in three dimensions we find that the tube boundary is perturbed in the nonlinear theory. This result contrasts the linear theory of the torsional Alfvén mode where the tube boundary is unperturbed(e.g. Spruit, 1982).

2. BASIC EQUATIONS

Neglecting the effect of gravity the following set of equations will describe an adiabatic, perfectly conducting gas in magneto-hydrodynamics:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{v}) = 0 \qquad \rho\frac{d\underline{v}}{dt} = -\nabla P + \underline{j}\times\underline{B}$$

$$\mu\underline{j} = \nabla\times\underline{B} \qquad \frac{\partial \underline{B}}{\partial t} = \nabla\times(\underline{v}\times\underline{B}) \qquad (1)$$

$$\nabla\cdot\underline{B} = 0 \qquad \frac{d}{dt}(P\rho^{-\gamma}) = 0$$

Here $\underline{v}$ denotes the velocity, ρ the density, P the pressure, $\underline{B}$ the magnetic induction, $\underline{j}$ the current density, μ the magnetic permeability and γ the ratio of specific heats.

3. SECOND ORDER SOLUTION

We shall employ the method of successive approximations to solve equations in (1) up to second order in the field variables. Consider a cylindrical coordinate system(r, θ, z) with the unperturbed magnetic field in the z-direction. A first order symmetrical torsional Alfvén mode (m=0) is assumed and the field variables are written as follows:

$$\underline{B} = \underline{B}_z^{(0)} + \underline{b}_\theta^{(1)} + (\underline{b}_r^{(2)} + \underline{b}_z^{(2)}) = \underline{B}_z + \underline{b}_\theta + (\underline{b}_r + \underline{b}_z)$$

$$\underline{v} = 0 + \underline{v}_\theta^{(1)} + (\underline{v}_r^{(2)} + \underline{v}_z^{(2)}) = \underline{v}_\theta + (\underline{v}_r + \underline{v}_z) \qquad (2)$$

$$\rho = \rho^{(0)} + 0 + \rho^{(2)}$$

$$P = P^{(0)} + 0 + p^{(2)}$$

Hence, for the m=0 mode the first order perturbations are azimuthal both in the velocity and in the magnetic field. The second order perturbations occur in density, pressure and in two directions for the velocity as well as the magnetic field.

Introducing equation (2) in (1) the following expression is obtained

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for the second order perturbation, b_z, in the magnetic field (for details, see Andreassen, 1982)

$$\Phi \cdot b_z = \frac{B}{\rho^{(0)}} \left(\frac{\partial^2}{\partial t^2} - s^2 \frac{\partial^2}{\partial z^2}\right) \frac{1}{r} \frac{\partial}{\partial r} \left\{\frac{b_\theta^2}{\mu} - \rho^{(0)} v_\theta^2\right\} + \frac{B}{\rho^{(0)}} \frac{1}{r} \frac{\partial}{\partial r} \left\{r \frac{\partial}{\partial r} \left[\frac{\partial^2}{\partial t^2} \left(\frac{b_\theta^2}{2\mu}\right)\right]\right\} \tag{3a}$$

where the operator,

$$\Phi = \frac{\partial^4}{\partial t^4} - (s^2 + a^2) \left[\frac{\partial^4}{\partial t^2 \partial z^2} + \frac{\partial^2}{\partial t^2} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r}\right)\right] + s^2 a^2 \left[\frac{\partial^4}{\partial z^4} + \frac{\partial^2}{\partial z^2} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r}\right)\right] \tag{3b}$$

and s and a are sound speed and Alfvén speed, respectively. It appears from the right hand side of (3a) that the second order magnetic field perturbation b_z is caused by a combination of the magnetic tension, the centripetal force and the magnetic pressure.

Let the m=0, first order solution be written as

$$b^{(1)} = b_\theta = \tilde{b}_\theta f(r) \exp[i\omega(t \mp z/a)], \tag{4a}$$

$$v^{(1)} = v_\theta = \mp \frac{\tilde{b}_\theta}{B} a f(r) \exp[i\omega(t \mp z/a)], \tag{4b}$$

where f(r) is a function of the distance from the centre of the flux tube and the minus sign corresponds to propagation in the positive z-direction. We note that the magnetic tension and the centripetal force (see 3a) do balance in this mode.

Equation (3) will have solutions with twice the original frequency,

$$b_z^{(2)}(r,z,t) = b_z = \tilde{b}_z(r) \exp\left[2i\omega(t \mp z/a)\right]. \tag{5}$$

Introducing this solution in (3), taking (4) into account we find

$$\frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial b_z(r)}{\partial r}\right] = - \frac{\tilde{b}_\theta^2}{2B_z} \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial f^2}{\partial r}\right]. \tag{6}$$

Integration of (6) gives

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$$\tilde{b}_z\,(r) = -\frac{\tilde{b}_\theta{}^2}{2B_z}\,[f^2 + c], \qquad c = -2B_z b_z(0)/\tilde{b}_\theta^2 \tag{7}$$

Here c is a constant of integration and we have set the term $c_1\int dr/r = 0$, i.e. only finite values of $b_z^{(2)}$ at r=0 are accepted. The second order solutions may be summarized as follows:

$$b_z = -\frac{\tilde{b}_\theta^2}{2B_z}\,[f^2 + c]\,\exp[2i\omega(t \mp z/a)],$$

$$b_r = \mp\frac{i\omega}{a}\frac{\tilde{b}_\theta^2}{B_z}\,[\frac{1}{r}\int_0^r r\,f^2\,dr + cr/2]\,\exp[2i\omega(t\mp z/a)],$$

$$v_z = \mp\frac{\tilde{b}_\theta^2}{2\mu\,\rho^{(0)}a}\,[f^2 + \frac{c\,s^2}{s^2-a^2}]\,\exp[2i\omega(t\mp z/a)], \tag{8}$$

$$v_r = i\omega\frac{\tilde{b}_\theta^2}{B_z^2}\,[\frac{1}{r}\int_0^r r\,f^2 dr + cr/2]\,\exp[2i\omega(t\mp z/a)],$$

$$p^{(2)}(r,z,t) = \frac{c\,s^2}{s^2 - a^2}\frac{\tilde{b}_\theta^2}{2\mu}\exp[2i\omega(t\mp z/a)].$$

We note that the perturbations in pressure and density are independent of r, whereas the perturbations in velocity and magnetic induction vary with r and z. The magnitude of the perturbations depend on the boundary conditions.

4. BOUNDARY CONDITIONS

In the field-free medium ($r > r_o$) the wave solution may be written:

$$v_{r,e}^{(2)}\,(r,z,t) = v_{r,e} = \tilde{v}_{r,e}\,R_o'(\nu r)\,\exp[2i(\omega t\mp kz)],$$
$$p_e^{(2)}\,(r,z,t) = p_e = -\frac{2i\omega\rho_e^{(0)}}{\nu}\,R_o(\nu r)\,\exp[2i(\omega t\mp kz)], \tag{9}$$

where $R_o'(\nu r) = \partial R_o\,(\nu r)/\partial(\nu r)$ and R_o is a cylindrical function (Bessel, modified Bessel or Hankel function).

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We intend to fit the outside solution to an inside solution with m=0 and $\omega/k = a$. The boundary conditions at $r = r_o$ are:

$$v_{r,i} = v_{r,i}^{(2)} = v_{r,e}^{(2)} = v_{r,e} \tag{10}$$

$$p_i^{(2)} + \frac{\underline{B} \cdot \underline{b}^{(2)}}{\mu} + \frac{b^{(1)2}}{2\mu} = p_e^{(2)}. \tag{11}$$

Or alternatively,

$$i\omega \frac{\tilde{b}_\theta^{\,2}}{B_z^2} \left[\frac{1}{r_o} \int_0^{r_o} r\, f^2\, dr + c r_o/2\right] = \tilde{v}_{r,e}\, R_o'(\nu r_o), \tag{12}$$

$$c\, \frac{\tilde{b}_\theta^{\,2}}{2\mu} \frac{a^2}{s_i^{\,2} - a^2} = p_e^{(2)}\, R_o(\nu r_o). \tag{13}$$

Noting that $\tilde{v}_{r,e} = -\nu p_e^{(2)}/(2i\omega\rho_e^{(0)})$ these boundary conditions may be combined. Taking the dispersion relation for the external medium into account we find

$$\frac{1}{r_o c} \int_0^{r_o} r\, f^2\, dr = -\frac{r_o}{2} - \frac{R_o'(\nu r_o)}{4\nu R_o(\nu r_o)} \frac{a^2}{s_i^2 + \frac{1}{2}\gamma a^2} \frac{s_e^2 - a^2}{s_i^2 - a^2}. \tag{14}$$

For a given f(r) and ω the constant c is determined. Hence, given $\tilde{b}_\theta$, f(r) and ω the second order perturbation b_z is determined from equations (8) and (14). The perturbation in velocity and pressure in the external medium follows from equations (12) and (13). We find that $p_e^{(2)}$ is finite for $s_i \to a$, provided $s_e \neq a$. The waves in the external medium may occur as surface waves if $a < s_e$. The flux tube may lose energy by wave radiation if $a > s_e$, the progressive waves will travel in the direction $\cos\phi = s_e/a$ with respect to the z-direction. Hence, the sound radiation from the flux tube is in this case very similar to the sound radiation from an infinitely long cylinder excited in a single mode (e.g. Skudrzyk, 1971).

5. DISCUSSION

The present results supplement the numerical study of nonlinear Alfvén waves by Hollweg et al. (1982), where $v_r^{(2)}$ was set equal to zero, but the effect of gravity was taken into account. Our result implies that a weakly nonlinear torsional (m=0) Alfvén mode may generate sound waves in the field-free surroundings. The generated waves will have a fre-

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quency twice that of the torsional Alfvén wave. In contrast to the results of the linear theory (e.g. Spruit, 1982) we find that the flux tube may lose energy by wave radiation similar to sound radiation from an infinitely long cylinder excited in a single mode.

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DISCUSSION

IONSON: Since nonlinear effects are important to order $v_{wave}/v_{Alfvén}$, I would expect that they are unimportant, since $v_{wave} \approx 20$ km s^{-1} if one associates it with anomalous coronal line broadening.

MALTBY: The numerical study by Hollweg et al. (1982, *Solar Phys*. **75**, p. 35) shows that the nonlinearity of Alfvén waves may be important already at chromospheric heights.

MOUSCHOVIAS: If I understood your equation for b_r correctly, you always generate an r component of the magnetic field if the wave has z and θ components. We know, however, that a torsional Alfvén wave can propagate in the $\hat{z}$ direction without generating an r component of the magnetic field, even in a compressible medium. Why do you get the result I referred to? In particular, if you do not put in a b_r initially, will you generate one?

MALTBY: The results you mentioned are correct for a homogeneous medium. However, for a fluxtube with finite diameter a second-order perturbation in b_r will be generated, the magnitude of which depends on the boundary conditions.

ROBERTS: A point of clarification. Are the boundary conditions applied on the undisturbed boundary of the fluxtube or on the (first-order) disturbed boundary?

MALTBY: In our case the tube boundary is not perturbed in the first-order calculations. Hence, the boundary conditions could be applied to the undisturbed tube boundary.

CAMPOS: The Alfvén waves in an atmosphere without dissipation grow initially exponentially, by a factor of four per scale height, and later grow linearly. If there is resistive dissipation there are opposing effects, but the velocity is ultimately bounded. Since the Alfvén speed grows exponentially with height, the Alfvén number v/c_A is small at high altitudes. So I would expect non-linear effects to be small in the corona for plane, incompressible Alfvén waves. The conclusion could be modified by magnetosonic waves, which propagate a compression.

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