ON FUNCTIONS OF BOUNDED VARIATION RELATIVE TO A SET

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1. Introduction, Definitions and Notations

The present paper on functions of bounded variation relative to a set has its point of departure in the work of R. L. Jeffery [10]. Below we recapitulate Jeffery's class U of functions of bounded variation relative to a set, we state and prove a number of preliminary lemmas and theorems, we introduce a suitable pseudometric space (X, d) of such functions, and the analogous space $(\overline{X}, \rho)$, and prove that (X, d) is separable, that every closed sphere in (X, d) is compact and that $(\overline{X}, \rho)$ is complete. These results extend known results of C. R. Adams, and C. R. Adams and A. P. Morse for the space of usual BV functions.

Let S be a subset of the closed interval $[a, b]$ such that S is dense in $[a, b]$. We define the class U of functions $F(x)$ in the following way: $F(x)$ is defined on $[a, b]$ such that for every point x_0 in $[a, b]$, $F(x)$ tends to finite limits as x tends to x_0+ and to x_0- over the points of S; these limits will be denoted by $F(x_0+)$ and $F(x_0-)$ respectively.

We now introduce the following definition:

DEFINITION 1.1. Let $F(x)$ belong to the class U and E be a subset of $[a, b]$ with α and β as its *g.l.b.* and *l.u.b.* Let

$$D : (\alpha < x_1 < x_2 < \cdots < x_p < \beta)$$

be any subdivision of $[\alpha, \beta]$ with $x_i \in E$. The *l.u.b.* of the sums V_D defined by

$$V_D = |F(\alpha)-F(x_1-)|+\sum_{i=1}^{r-1}|F(x_i+)-F(x_{i+1}-)|+|F(x_r+)-F(\beta)|$$

for all possible subdivisions D is called the *total variation*, $V_S(F; E)$, of $F(x)$ on E relative to the set S. If $V_S(F; E) < +\infty$, then $F(x)$ is said to be $BV-S$ on E.

From theorem 3.1 and lemma 2.2 it follows that a function which is of bounded variation in the ordinary sense on $[a, b]$ is a $BV-S$ function and the $BV-S$ variation of a function $F(x)$ on a dense subset of $[a, b]$ is the same as that of the ordinary variation of the function $F(x-)$ [or $F(x+)$] on $[a, b]$. However,

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there are examples of functions that are $BV-S$ on an infinite subset E of $[a, b]$ but not BV on E. Therefore the set of functions which are $BV-S$ on $[a, b]$ {that is, the set of functions $F(x)$ for which the total variation of $F(x-)$ [or $F(x+)$] on $[a, b]$ is finite} includes as a proper subset the functions which are BV on $[a, b]$. It may be noted in this connection that various authors have studied the properties of BV functions on a set. These studies can be found in most of the references appended in the list of the bibliography.

Throughout our discussion we suppose that S is a fixed set which is dense in $[a, b]$, and consequently U becomes a fixed class of functions as defined above. We denote the set of points x of S for which $F(x-) = F(x) = F(x+)$ by S_F, where S is as above and $F(x)$ is any function belonging to the class U. From theorem 3.4 onwards we suppose that S is Lebesgue measurable and $mS = b-a$.

2. Preliminary lemmas

LEMMA 2.1. *Let $F(x)$ belong to the class U. Then the set of points for which $F(x-) \neq F(x+)$ is countable. Also the subset of S for which we do not have $F(x-) = F(x) = F(x+)$ is countable.*

PROOF. For each positive integer n, let E_n denote the set of numbers x such that

$$|F(x-)-F(x+)| > \frac{1}{n}, \quad a+\frac{1}{n} < x < b-\frac{1}{n}.$$

The set E_n cannot have a cluster point, and hence it is finite. The set $\bigcup_{n=1}^{\infty} E_n$ is therefore countable. Similarly for the second part of the lemma and this completes the proof.

We now define the function $G(x)$ on $[a, b]$ as follows:

$$G(a) = F(a), \quad G(b) = F(b),$$
$$G(x) = F(x-) \text{ for } a < x < b.$$

It is easy to verify that

$$F(x+) = \lim G(\eta) \text{ as } (\eta > x, \eta \to x),$$

and

$$G(x) = \lim F(\xi+) \text{ as } (\xi < x, \xi \to x).$$

Clearly $G(x) = F(x)$ at each point of the set S_F.

LEMMA 2.2. *If E is dense in $[a, b]$ and if $F(x)$ belongs to U, then*

$$V_S(F; E) = V_a^b(G).$$

PROOF. The symbol $\approx$ shall mean: can be made to differ by ε (> 0) by going far enough in the limiting process indicated by $\to$. The symbols x_0+ and $x_{r+1}-$ shall mean a and b respectively.

To prove $V_S(F; E) \leqq V_a^b(G)$ we consider any subdivision

$$D : (a = x_0 < x_1 < x_2 < \cdots < x_{r+1} = b)$$

of $[a, b]$ with $x_i \in E$ $(i = 1, 2, \cdots, r)$ and then take points η_i satisfying $x_i < \eta_i < x_{i+1}$ and $\eta_i \to x_i$ (except for $\eta_0 = a$, $\eta_{r+1} = b$). Denoting by V_D the sum

$$\sum_{i=0}^{r} |F(x_i+)-F(x_{i+1}-)|,$$

we have

$$V_D \approx \sum_{i=0}^{r} |F(\eta_i-)-F(x_{i+1}-)| = \sum_{i=0}^{r} |G(\eta_i)-G(x_{i+1})| \leqq V_a^b(G).$$

Hence

$$V_S(F; E) \leqq V_a^b(G).$$

To prove the reverse inequality we consider any subdivision D of $[a, b]$, then take points ξ_i satisfying $\xi_i \in E$, $x_{i-1} < \xi_i < x_i$ (except for $\xi_0 = a$, $\xi_{r+1} = b$). Then

$$\sum_{i=0}^{r} |G(x_i)-G(x_{i+1})| \approx \sum_{i=0}^{r} |F(\xi_i+)-F(\xi_{i+1}-)| \leqq V_S(F; E);$$

and hence

$$V_S(F; E) = V_a^b(G).$$

COROLLARY 2.2.1. *If E is dense in $[a, b]$, then $V_S(F; E) = V_S(F; [a, b])$.*

LEMMA 2.3. *Let $a < c < b$. If $F(x)$ is $BV-S$ on $[a, c]$ and on $[c, b]$, then it is so on $[a, b]$; further if $c \in S_F$ then*

$$V_S(F; [a, b]) = V_S(F; [a, c])+V_S(F; [c, b]).$$

LEMMA 2.4. *If $F(x)$ is $BV-S$ on $[a, b]$, then $F(x+)$ is bounded on $[a, b]$. The proofs of these results are straightforward.*

Let $F(x)$ be $BV-S$ on $[a, b]$. We define the function $\pi(x)$ on $[a, b]$ as follows:

$$\pi(a) = 0 \text{ and } \pi(x) = V_a^x(G) \text{ for } a < x \leqq b.$$

Clearly the function $\pi(x)$ is non-decreasing on $[a, b]$.

LEMMA 2.5. *If $F(x)$ is $BV-S$ on $[a, b]$, then $F(x)$ can be expressed as $F(x) = \pi(x)-\nu(x)$, where $\nu(x)$ is non-decreasing on S_F.*

PROOF. We define $\nu(x)$ by $\nu(x) = \pi(x)-F(x)$. Let x_1 and $x_2(> x_1)$ be any two points of S_F. Then

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$$\begin{aligned} v(x_2)-v(x_1) &= \{\pi(x_2)-F(x_2)\}-\{\pi(x_1)-F(x_1)\} \\ &= \pi(x_2)-\pi(x_1)-\{G(x_2)-G(x_1)\} \\ &\geqq 0 \end{aligned}$$

and the lemma is proved.

3. Some results on $BV-S$ functions

THEOREM 3.1. *If $F(x)$ is of bounded variation on $[a, b]$, then it is $BV-S$ on $[a, b]$ and in any case*

$$V_S(F; [a, b]) \leqq V_a^b(F), \quad F(x) \in U.$$

PROOF. We first suppose that $V_a^b(F)$ is finite. Let

$$D: (a = x_0 < x_1 < x_2 < \cdots < x_{r+1} = b)$$

be any subdivision of $[a, b]$. Take points ξ_i, η_i of S with $x_i < \xi_i < \eta_i < x_{i+1}$ (except for $\xi_0 = a$, $\eta_{r+1} = b$). Then

$$\sum_{i=0}^{r} |F(\xi_i)-F(\eta_i)| \leqq V_a^b(F).$$

Now letting $\xi_i \to x_i$, $\eta_i \to x_{i+1}$ over the points of S we obtain

$$|F(a)-F(x_1-)|+\sum_{i=1}^{r-1}|F(x_i+)-F(x_{i+1}-)|+|F(x_r+)-F(b)| \leqq V_a^b(F).$$

Since D is arbitrary, we have

$$V_S(F; [a, b]) \leqq V_a^b(F). \tag{1}$$

So $F(x)$ is $BV-S$ on $[a, b]$. If $V_a^b(F)$ is infinite, then clearly (1) holds.

NOTE. It is clear that if $F(x)$ is monotone or continuous on $[a, b]$ then

$$V_S(F; [a, b]) = V_a^b(F).$$

If (BV) denotes the set of all functions which are of bounded variation on $[a, b]$ and $(BV-S)$ the set of all functions which are $BV-S$, then by Theorem 3.1, $(BV) \subset (BV-S)$. The following example shows that (BV) is a proper subset of $(BV-S)$.

EXAMPLE. Let S be a dense subset of $[a, b]$ and let E be an infinite subset of $[a, b]$ with a and b as lower and upper bounds. Let $\phi(x)$ be any non-decreasing function on $[a, b]$ and $\sum \beta_n$ be a divergent series of positive terms with $\lim \beta_n = 0$. Choose a strictly monotone sequence $\{\alpha_n\}$ from E. Suppose that $\{\alpha_n\}$ is increasing. We define the function $F(x)$ on $[a, b]$ as follows:

$$F(x) = \phi(x)+\beta_n \text{ for } x = \alpha_{2n}(n = 1, 2, 3, \cdots),$$
$$= \phi(x) \text{ elsewhere.}$$

It is easy to see that $F(x\pm) = \phi(x\pm)$ for all $x \in [a, b]$. For any two points c, $d(> c)$ of E

$$|F(c+)-F(d-)| = |\phi(c+)-\phi(d-)| \leqq \phi(d)-\phi(c)$$

which shows that $F(x)$ is $BV-S$ on E. Now consider the subdivision $a \leqq \alpha_1 < \alpha_2 < \cdots < \alpha_{2m} < b$ and denote by V the sum

$$|F(a)-F(\alpha_1)| + \sum_{i=1}^{2m-1} |F(\alpha_i)-F(\alpha_{i+1})|+|F(\alpha_{2m})-F(b)|.$$

Then

$$\begin{aligned} V &\geqq \sum_{i=1}^{m} |F(\alpha_{2i-1})-F(\alpha_{2i})| \\ &= \sum_{i=1}^{m} \{\phi(\alpha_{2i})+\beta_i-\phi(\alpha_{2i-1})\} \\ &\geqq \beta_1+\beta_2+\cdots+\beta_m. \end{aligned}$$

Since $\sum \beta_n$ is divergent, it follows that $F(x)$ is not of bounded variation on E.

THEOREM 3.2. (cf. [2], Th. 2; [4], Lemma 1; [9], § 7).

Let $\{F_n(x)\}$ be a sequence of functions in the class U and $S_0 = \cap \{S_{F_n}; n = 1, 2, \cdots\}$. If $F_n(x) \to F(x) \in U$ at each point of the set $E \cup \{a, b\}$ such that $E \subset S_0 \cap S_F$ and E is dense in $[a, b]$, then

$$\liminf_{n\to\infty} V_S(F_n; [a, b]) \geqq V_S(F; [a, b]).$$

PROOF. We suppose that $V_S(F; E)$ is finite. If $V_S(F; E)$ is infinite the proof is analogous. Let $\varepsilon > 0$ be arbitrary. There exists a subdivision

$$D : (a = x_0 < x_1 < x_2 < \cdots < x_{r+1} = b)$$

with $x_i \in E$ $(i = 1, 2, \cdots, r)$ such that

$$|F(a)-F(x_1-)|+\sum_{i=1}^{r-1}|F(x_i+)-F(x_{i+1}-)|+|F(x_r+)-F(b)| > V_S(F; E)-\varepsilon$$

or

$$V_D(F) = \sum_{i=0}^{r}|F(x_i)-F(x_{i+1})| > V_S(F; E)-\varepsilon.$$

Since $V_D(F_n) \to V_D(F)$ as $n \to \infty$; a positive integer n_0 exists such that for $n \geqq n_0$

$$V_S(F_n; [a, b]) \geqq V_D(F_n) > V_S(F; E)-\varepsilon.$$

So,

$$\liminf_{n\to\infty} V_S(F_n; [a, b]) \geqq V_S(F; E)-\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain by using corollary 2.2.1,

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$$\liminf_{n\to\infty} V_S(F_n; [a, b]) \geqq V_S(F; [a, b]).$$

DEFINITION 3.1. Let $F(x) \in U$ and $D : (a = x_0 < x_1 < x_2 < \cdots < x_{r+1} = b)$ be any subdivision of $[a, b]$ with $x_i \in S_F (i = 1, 2, \cdots, r)$. We denote by $B(x) = B(x; F, D)$ the function whose graph is the polygonal line joining the points $(x_i, F(x_i))$ $(i = 0, 1, 2, \cdots, r)$. $B(x)$ is said to be a *Polygonal function* associated with $F(x)$.

It is clear that

$$V_D(F) = \sum_{i=0}^{r} |F(x_i) - F(x_{i+1})| = V_D(B) = V_S(B; [a, b]).$$

So,

$$V_S(F; [a, b]) \geqq V_S(B; [a, b]).$$

THEOREM 3.3. (cf. [3], § 2). *If $F(x)$ is $BV-S$ on $[a, b]$ and S_F is dense in $[a, b]$, then it is possible to choose a sequence $\{B_n(x)\}$ of polygonal functions such that $B_n(x) \to F(x)$ at each point of S_F and*

$$\lim_{n\to\infty} V_S(B_n; [a, b]) = V_S(F; [a, b]).$$

PROOF. Let $\{D_n\}$ be a sequence of subdivisions

$$D_n : (a = x_0^{(n)} < x_1^{(n)} < x_2^{(n)} < \cdots < x_{r_n+1}^{(n)} = b)$$

of $[a, b]$ with $x_i^{(n)} \in S_F$ $(i = 1, 2, \cdots, r_n)$ such that $D_n \subset D_{n+1}$ for each n and the set $E = \bigcup \{D_n;\ n = 1, 2, \cdots\}$ is dense in $[a, b]$. Writing $B_n(x) = B(x; F, D_n)$ we have

$$V_S(B_n; [a, b]) \leqq V_S(F; [a, b]). \tag{2}$$

Let $\varepsilon > 0$ be arbitrary. Then a subdivision $D : (a = x_0 < x_1 < \cdots < x_{r+1} = b)$ with $x_i \in E$ $(i = 1, 2, \cdots, r)$ exists such that

$$\sum_{i=0}^{r} |F(x_i) - F(x_{i+1})| > V_S(F; E) - \varepsilon.$$

Since x_i's are points of E, we can choose a positive integer n_0 such that $D \subset D_n$ for all $n \geqq n_0$.

Then for $n \geqq n_0$

$$V_S(B_n; [a, b]) \geqq \sum_{i=0}^{r} F(x_i) - F(x_{i+1})| > V_S(F; E) - \varepsilon. \tag{3}$$

Combining (2) and (3) and noting corollary 2.2.1, we obtain

$$\lim_{n\to\infty} V_S(B_n; [a, b]) = V_S(F; [a, b]).$$

It is clear that $B_n(x) \to F(x)$ at each point of the set E. Let ξ be any point of $S_F - E$. Choose points ξ', ξ'' of E with $\xi' < \xi < \xi''$ such that $\pi(\xi'') - \pi(\xi') < \frac{1}{2}\varepsilon$.

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Let m be a positive integer such that $\xi', \xi'' \in D_n$ for all $n \geqq m$. Then for all $n \geqq m$

$$\begin{aligned} |F(\xi)-B_n(\xi)| &\leqq |F(\xi)-F(\xi')|+|F(\xi')-B_n(\xi')|+|B_n(\xi')-B_n(\xi)| \\ &\leqq |G(\xi)-G(\xi')|+V_S(F; [\xi', \xi'']) \\ &\leqq 2[\pi(\xi'')-\pi(\xi')] < \varepsilon. \end{aligned}$$

This proves the theorem.

THEOREM 3.4. (cf. [13], p. 222). *Let* $\mathscr{F} = \{F(x)\}$ *be a sequence of functions in the class* U. *If there is a positive* K *such that* $|F(x\pm)| \leqq K$, $a < x < b$; $|F(a)|$, $|F(b)| \leqq K$ *and* $V_S(F; [a, b]) \leqq K$ *for every* $F(x) \in \mathscr{F}$, *then there exist a subsequence in* $\mathscr{F}$ *which converges to a function* $\phi(x)$ *almost everywhere in* $[a, b]$, *where* $\phi(x)$ *is of bounded variation in* $[a, b]$.

To prove the theorem we require the following lemma:

LEMMA 3.1. (cf. [13], p. 221). *Let* $\mathscr{F} = \{F(x)\}$ *be a sequence of functions in the class* U *and* $S_0 = \cap \{S_F; F \in \mathscr{F}\}$. *If each* $F(x)$ *is non-decreasing on* S_0 *and if there is a positive* K *such that* $|F(x\pm)| \leqq K$, $a < x < b$; $|F(a)|, |F(b)| \leqq K$ *for each* $F \in \mathscr{F}$, *then there is a subsequence* $\{F_n(x)\}$ *of functions in* $\mathscr{F}$ *which converges to a function* $\phi(x)$ *almost everywhere in* $[a, b]$, *where* $\phi(x)$ *is non-decreasing on* $[a, b]$.

The lemma can be proved in the usual way.

PROOF OF THEOREM 3.4. By lemma 2.5, each $F(x)$ in $\mathscr{F}$ can be expressed as $F(x) = \pi(x)-v(x)$, where $\pi(x)$ is non-decreasing on $[a, b]$ and $v(x)$ is non-decreasing on $S_0 = \cap \{S_F; F \in \mathscr{F}\}$. Clearly $\pi(x)$ belongs to the class U. Also $V_S(\pi; [a, b]) = \pi(b)$. So $\pi(x) \leqq k$ for all $x \in [a, b]$. Since $v(x) = \pi(x)-F(x)$, $v(x)$ belongs to the class U and $|v(x\pm)| \leqq 2k$, $a < x < b$; $|v(a)|, |v(b)| \leqq 2k$. By lemma 2 ([13], p. 221) there is a subsequence $\{\pi_n(x)\}$ of $\{\pi(x)\}$ which converges to a non-decreasing function $\alpha(x)$ everywhere in $[a, b]$.

Let E_n denote the set of points in $[a, b]$, where

$$v_n(x-) = v_n(x) = v_n(x+) \text{ and } E_0 = \cap \{E_n; \quad n = 1, 2, \cdots\}.$$

Then Lebesgue measure of E_0 is $b-a$. Applying lemma 3.1 to the sequence $\{v_n(x)\}$ (where S_0 is to be replaced by $S_0 \cap E_0$) we obtain a subsequence $\{v_{n_i}(x)\}$ which converges to a non-decreasing function $\beta(x)$ almost everywhere in $[a, b]$. Let $\phi(x) = \alpha(x)-\beta(x)$. Then $\phi(x)$ is of bounded variation on $[a, b]$ and the sequence $\{F_{n_i}(x)\}$ converges to $\phi(x)$ almost everywhere in $[a, b]$. This proves the theorem.

4. The space (X, d)

Let X denote the set of all functions $x(t)$ in the class U which are $BV-S$ on $[0, 1]$. To each pair x, y of functions in X we associate the real number $d(x, y)$

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defined by

$$d(x, y) = \int_0^1 |x(t)-y(t)|dt+|T(x)-T(y)|, \tag{4}$$

where the integral is taken in Lebesgue sense and $T(x)$ stands for $V_S(x; [0, 1])$. Since $d(x, y) = 0$ implies $T(x) = T(y)$ and $x(t) = y(t)$ almost everywhere in $[0, 1]$, it follows that d is a pseudo-metric for X and therefore (X, d) is a pseudo-metric space.

The pseudo-metric (4) is analogous to that introduced by C. R. Adams [1] and C. R. Adams & A. P. Morse [4] to study the properties of the space (BV) of functions of bounded variation. The above two papers contain interesting and elaborate discussions of the space (BV). Here we wish to mention only two properties of the space (X, d) leaving out, of course, possible scope of further study.

THEOREM 4.1. *The space (X, d) is separable.*

PROOF. Let E denote the set of all polygonal functions in X with rational corners. Then clearly E is countable. Let $x(t)$ be any function in X. By theorem 3.3, it is possible to choose a sequence of polygonal functions $\{B_n(t)\}$ in X such that $B_n(t) \to x(t)$ almost everywhere in [0, 1] and $T(B_n) \to T(x)$. For each $B_n(t)$ we can choose a polygonal function $P_n(t)$ in X with rational corners such that $|B_n(t)-P_n(t)| < 1/n$ for all $t \in [0, 1]$ and $|T(B_n)-T(P_n)| < 1/n$. So the sequence $\{P_n(t)\}$ converges to $x(t)$ almost everywhere in [0, 1] and $T(P_n) \to T(x)$. Therefore $d(P_n, x) \to 0$ as $n \to \to$ and hence x is an accumulation point of E. Thus the set E is dense in X. This completes the proof.

THEOREM 4.2. *Every closed sphere in (X, d) is compact.*

PROOF. Let x_0 be an element of X and

$$Y = \{x; x \in X \text{ and } d(x, x_0) \leqq r\},$$

where r is a positive number. Since X is separable, Y considered as a subspace of X is a Lindelöf space. Let $\theta(t) \equiv 0$ in [0, 1]. Then θ is an element of X. For any x in Y,

$$d(x, \theta) \leqq d(x, x_0)+d(x_0, \theta) \leqq r+d(x_0, \theta).$$

So

$$d(x, \theta) = \int_0^1 |x(t)|dt+T(x) \leqq M, \tag{5}$$

where M denotes the constant $r+d(x_0, \theta)$. If t is a point in (0, 1), then

$$|x(t\pm)| \leqq \max\{|x(0)|, |x(1)|\}+T(x). \tag{6}$$

We show that $\max\{|x(0)|, |x(1)|\} \leqq 2M$. If possible, assume that $\max\{|x(0)|, |x(1)|\} > 2M$. If $|x(t)| \geqq M$ almost everywhere in [0, 1], then $\int_0^1|x(t)|dt \geqq M$

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which with (5) gives $T(x) = 0$. So $|x(t\pm)| = |x(0)| = |x(1)| > 2M$. This contradicts (5). Hence there is a subset E of $[0, 1]$ of positive measure such that $|x(t\pm)| < M$ for all $t \in E$. Let t be any point of $E \cap S_x$. Then

$$|x(0)-x(t)|+|x(t)-x(1)| > M.$$

So, $T(x) > M$ which contradicts (5). Therefore $\max\{|x(0)|, |x(1)|\} \leqq 2M$. Combining this with (6) we get $|x(t\pm)| \leqq 3M$ for all $t \in (0, 1)$.

Let $\{x_n(t)\}$ be any sequence of points in Y. Then

$$|x_n(t\pm)| \leqq 3M, 0 < t < 1; \ |x_n(0)|, |x_n(1)| < 3M$$

and $V_S(x_n; [0, 1]) < 3M$. By theorem 3.4, there is a subsequence $\{x_{n_i}(t)\}$ which converges to a function $x(t)$ in X almost everywhere in $[0, 1]$. We may choose $\{x_{n_i}(t)\}$ and take the function $x(t)$ such that $\{x_{n_i}(t)\}$ converges to $x(t)$ also at $t = 0, 1$. Let $\tau = \lim_{i\to\infty} \inf T(x_{n_i})$. We choose a subsequence $\{T(x_{m_i})\}$ of $\{T(x_{n_i})\}$ which converges to τ. By theorem 3.2, $\tau \geqq T(x)$. Let $K = \tau - T(x)$. We define the function $y(t)$ on $[0, 1]$ as follows:

$$\begin{aligned} y(t) &= x(t), \ 0 < t \leqq 1, \\ &= x(0)+K \text{ for } t = 0 \text{ if } x(0) > x(0+), \\ &= x(0)-K \text{ for if } x(0) \leqq x(0+). \end{aligned}$$

It is clear that $y \in X$ and $T(y) = T(x)+K$. Further

$$d(x_{m_i}, y) = \int_0^1 |x_{m_i}(t)-x(t)|dt + |T(x_{m_i})-\tau|.$$

Then $d(x_{m_i}, y) \to 0$ as $i \to \infty$. We have

$$d(y, x_0) \leqq d(x_{m_i}, y)+d(x_{m_i}, x_0) \leqq r+d(x_{m_i}, y).$$

Letting $i \to \infty$ we get $d(y, x_0) \leqq r$. Thus every sequence in Y has a cluster point in Y. So by lemma ([12], Ch. 5, § 4) Y is compact.

5. The space $(\overline{X}, \rho)$

Let $\overline{X}$ denote the family of all sets

$$\{x\}^- = \{y; y \in X \text{ and } d(x, y) = 0\}$$

for $x \in X$. For convenience, we write $\bar{x}$ for $\{x\}^-$. For any two members $\bar{x}$, $\bar{y}$ of $\overline{X}$, let

$$\rho(\bar{x}, \bar{y}) = \inf\{d(\alpha, \beta); \alpha \in \bar{x} \text{ and } \beta \in \bar{y}\}.$$

Then $(\overline{X}, \rho)$ is a metric space ([12], Ch. 4, § 15).

THEOREM 5.1. *The space $(\overline{X}, \rho)$ is complete.*

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PROOF. Let $\{\bar{x}_n\}$ be any Cauchy sequence in $\bar{X}$. Then there is a positive number M such that $\rho(\bar{x}_n, \bar{\theta}) \leqq M$ for all n, where $\theta(t) = 0$ in $[0, 1]$. Let α_n be any member of $\bar{x}_n$. Then $d(\alpha_n, \theta) = \rho(\bar{x}_n, \bar{\theta}) \leqq M$ for all n. Following the method of theorem 4.2, we obtain a subsequence $\{\alpha_{n_i}(t)\}$ which converges to a function $x(t)$ in X almost everywhere in $[0, 1]$ such that $d(\alpha_{n_i}, x) \to 0$ as $i \to \infty$. Since $d(\alpha_{n_i}, x) = \rho(\bar{x}_{n_i}, \bar{x})$ it follows that the sequence $\{\bar{x}_{n_i}\}$ converges to $\bar{x}$ which implies that the sequence $\{\bar{x}_n\}$ converges to $\bar{x}$. This completes the proof.

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