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HOMOLOGY INVARIANTS

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There have been few published results concerning the relationship between the homology groups of branched and unbranched covering spaces of knots, despite the fact that these invariants are such powerful invariants for distinguishing knot types and have long been recognised as such [**8**]. It is well known that a simple relationship exists between these homology groups for cyclic covering spaces (see Example 3 in § 3), however for more complicated covering spaces, little has previously been known about the homology group, $H_1(M)$ of the branched covering space or about $H_1(U)$, U being the corresponding unbranched covering space, or about the relationship between these two groups.

In this paper, a number of specific results are given concerning these homology groups and their interrelationship, and techniques are developed which may be applied to yield information about arbitrary covering spaces, many illustrative examples being given. The question has great practical importance, since any 3-manifold can be obtained as a branched covering space of S^3, and therefore, information about the homology groups and homotopy groups of branched covering spaces of knots gives information about the homology and homotopy groups of 3-manifolds. In particular, a possible method of attacking the Poincaré conjecture is to find a covering space which is a homotopy sphere, and to demonstrate that it is not a sphere. It is shown in this paper that the degree $p + 1$ $PSL(2, p)$ covering spaces of knots can not give homotopy spheres except in very few cases.

The method used is quite general and provides a way of obtaining representations of the homotopy or homology groups of large classes of covering spaces onto non-trivial groups.

Particular attention is given throughout this paper to the PSL covering spaces first considered by Riley. These covering spaces are important in that they are the first class of covering spaces to be considered which correspond to representations of the knot group on unsolvable groups. For this reason, they have great value in the study of knots with trivial Alexander polynomial for which the usual metabelian invariants are useless. In this paper, specific information is given about the PSL homology invariants and conjectures A, B, and C of Riley [**9**] are proved, sometimes in slightly modified form. (Actually the parts concerning the Betti numbers have already been proved in the paper [**5**].)

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In the first section of this paper, we consider the problem of lifting a homomorphism of a knot group. If $\omega: H \twoheadrightarrow K$ is an epimorphism, then we say that *ω lifts to a group F* if there are homomorphisms $\psi: H \rightarrow F$ and $\eta: F \rightarrow K$ such that $\omega = \psi\eta$. If ψ is an epimorphism, we say that ω lifts *onto* F. Some of the results proved in this section will be used later on in the study of PSL coverings, however the subject of lifting of homomorphisms is sufficiently interesting to be treated here in a general context. Recently, Perko proved that any representation of a knot group onto S_3 lifts to a representation onto S_4 [**7**], and Riley considered representations onto $PSL(2, p)$ which lift to $PSL(2, Z)$ [**9**, p. 609]. (These are also considered in the last section of this paper.) Furthermore a metabelian representation of a knot group may be thought of as a lifting of a cyclic representation; thus the method of lifting is valuable in finding new representations of knot groups. Corollary 1.3 shows how an arbitrary representation may be lifted. This result does not overlap with the previous results quoted above. Lemma 4.1 may also be thought of as giving a necessary condition for a representation to be lifted.

The following notation will be used:

K is a knot in S^3
$G = G(K)$ is a knot group, $\pi_1(S^3 - K)$
m is a meridian of K
l is a longitude of K
$N(K)$ is a regular neighbourhood of K
J_n is a set of n elements
$S(J_n)$ is the group of all permutations of J_n
ϕ is a transitive representation $\phi: G \rightarrow S(J_n)$
$\mathrm{St}_\phi(a)$ is the stabilizer of $a \in J_n$ under ϕ.
$M = M_\phi$ is the branched covering space corresponding to ϕ.
$\tilde{K}$ is the branch curve in M. $\tilde{K} = \tilde{K}_1 \cup \tilde{K}_2 \cup \ldots \cup \tilde{K}_r$.
$U = U_\phi$ is the unbranched covering space, $M_\phi - \tilde{K}$
$\mathrm{br}\,(\tilde{K}_i)$ is the branching index of $\tilde{K}_i$
$|K_i|$ is the order of K_i in $H_1(M)$
$|x|$, where x is an element of a group, is the order of x.
$|A|$, where A is a set or a group, is the number of elements in A.

If A is an abelian group, then:
$\tau(A)$ is the torsion subgroup of A.
$B(A)$ is the Betti number of A, that is, the rank of $A/\tau(A)$
$T(A)$ is the order of $\tau(A)$.

If X is a relation matrix for an abelian group, A, then
$B(X)$ is the column nullity of X, equal to $B(A)$
$T(X)$ is a generator of the first non-zero elementary ideal of X, and is equal to $T(A)$.

If M is a manifold, then

$T(M)$ is abbreviated notation for $T(H_1(M))$

$B(M)$ is abbreviated notation for $B(H_1(M))$

$PSL(2, p)$ and $SL(2, p)$ are defined in § 5

$A \twoheadrightarrow B$ is an epimorphism from A to B.

1. The lifting problem. The present section deals with the problem of lifting a homomorphism from one group to another. The relevance of this section to the rest of the paper lies in the results of Example 2 and Theorem 1.7 below, which will be used in Section 5, to deduce properties of PSL covering spaces of knots.

An important tool in the study of liftings is the pull-back diagram [**6**, p. 71]. Given a diagram of groups and epimorphisms

$$\begin{tikzcd} & B \arrow[d, two heads, "\beta"] \\ A \arrow[r, two heads, "\alpha"'] & C \end{tikzcd} \tag{1.1}$$

then the pull-back exists, is unique up to isomorphisms and may be realised as the subgroup $\{(a, b): a \in A, b \in B, a\alpha = b\beta\}$ of $A \times B$. The following lemma is useful for recognising pull-backs.

LEMMA 1.1 *The diagram*

$$\begin{tikzcd} H \arrow[r, "\eta"] \arrow[d, "\xi"'] & B \arrow[d, two heads, "\beta"] \\ A \arrow[r, two heads, "\alpha"'] & C \end{tikzcd}$$

is a pull-back diagram if and only if ξ and η are onto, ker $(\xi) \cap$ ker $(\eta) = \{1\}$, *and* (ker $(\eta))\xi$ = ker (α). (*That is ξ maps* ker (η) *isomorphically onto* ker (α)).

Proof. ("If" part) Given homomorphisms ν: $V \rightarrow A$ and μ: $V \rightarrow B$ such that $\nu\alpha = \mu\beta$ one sees by diagram chasing that for any $x \in V$ there is a unique element $x' \in H$ such that $x'\eta = x\mu$ and $x'\xi = x\nu$. Define a homomorphism ψ: $V \rightarrow H$ such that $x\psi = x'$. This shows that H is the pull-back.

("Only if" part) Just observe that the conditions are true for the particular realisation of the pull-back as a subset of $A \times B$.

THEOREM 1.2 *Given a commuting diagram*

$$\begin{tikzcd} F \arrow[r, two heads, "\mu"] \arrow[d, two heads, "\nu"'] & B \arrow[d, two heads, "\beta"] \\ A \arrow[r, two heads, "\alpha"'] & C \end{tikzcd}$$

then there exists a homomorphism of F onto the pull-back of diagram (1.1) *if and only if μ maps* ker (ν) *onto* ker (β) (*or ν maps* ker (μ) *onto* ker (α)).

Proof. From the realisation of the pull-back as a subgroup of $A \times B$, we see that the map ψ from F to the pull-back is onto if and only if, given $a \in A$ and $b \in B$ such that $a\alpha = b\beta$, then there exists $x \in F$ such that $x\mu = b$ and $x\nu = a$. This is equivalent to the given condition.

Note. $x\psi$ is the unique element such that $x\psi\eta = x\mu$ and $x\psi\xi = x\nu$.

The most important case occurring in the study of knot groups is of the pull-back of a diagram

(1.2) $\quad C \twoheadrightarrow B/B' \overset{\beta}{\twoheadleftarrow} B \quad$ where C is a cyclic group.

Theorem 1.2 takes the form:

COROLLARY 1.3. *If μ: $G(K) \rightarrow B$ is a homomorphism of G onto B, then μ lifts to a mapping ψ onto the pull-back of the diagram* (1.2).

Proof. There is an epimorphism ν: $G \rightarrow C$ whose kernel contains G'. Then μ maps G' onto B'. So ψ is onto.

The pull-back of such a diagram as (1.2) is also easy to identify. As a corollary to Lemma 1.1 we get:

COROLLARY 1.4. *If the diagram* (1.2) *is completed to the diagram*

$$\begin{array}{ccc} H & \xrightarrow{\eta} & B \\ {\scriptstyle\xi}\downarrow & & \downarrow \\ C & \twoheadrightarrow & B/B' \end{array}$$

such that ker $(\eta) \cap$ ker $(\xi) = \{1\}$, *then H is the pull-back.*

Proof. Once again the condition that (ker $(\xi))\eta$ = ker (β) is automatic.

Example 1. Any epimorphism ω: $G(K) \rightarrow PSL(2, Z)$ can be lifted to an epimorphism ψ: $G(K) \rightarrow SL(2,Z)$. (Also see Example 2). This follows from Corollary 1.3, since $SL(2, Z)$ is the pull-back of the following diagram: $Z_{12} \twoheadrightarrow Z_6 \twoheadleftarrow PSL(2, Z)$.

Example 2. Let g.c.d. $(m, n) = 1$. Then any epimorphism, ω, of $G(K)$ onto the free product, $Z_m * Z_n$ of Z_m and Z_n can be lifted to an epimorphism ψ: $G(K) \rightarrow G(K_{m,n}) = \langle s, t\colon s^m = t^n \rangle$ where $K_{m,n}$ represents the torus knot of type (m, n) and $G(K_{m,n})$ is its knot group. To prove this, one simply observes that $G(K_{m,n})$ is the pull-back of the diagram $Z \twoheadrightarrow Z_{mn} \twoheadleftarrow Z_m * Z_n$.

Example 2 yields a necessary condition for a knot group to have a representation on $Z_m * Z_n$.

PROPOSITION 1.5. *Let $\Delta_K(t)$ and $\Delta_{m,n}(t)$ denote the Alexander polynomials of a knot K and of $K_{m,n}$ respectively. If $G(K)$ has a representation on $Z_m * Z_n$ with* g.c.d. $(m, n) = 1$, *then $\Delta_K(t)$ is divisible by $\Delta_{m,n}(t)$.*

The proof is straight-forward and so is omitted.

Since $PSL(2, Z) = Z_2 * Z_3$ and $\Delta_{2,3}(t) = 1 - t + t^2$, we obtain the following.

COROLLARY 1.6. *If a knot group, $G(K)$, has a representation on $PSL(2, Z)$ then $\Delta_K(t)$ is divisible by $1 - t + t^2$.*

Example 3. If n is a square free integer and n divides $\Delta_K(-1)$ then $G(K)$ maps onto the semidirect product $Z_{2q} \circledS Z_n$ given by $\langle s, t: s^{2q}, t^n, sts^{-1}t\rangle$ for any $q = 0, 1, 2 \ldots$.

Proof. According to [**4**, p. 163; **8**, p. 715] the knot group $G(K)$ can be represented on $D_n = Z_2 \circledS Z_n$. Then observe that $Z_{2q} \circledS Z_n$ is the pull-back of the diagram $Z_{2q} \twoheadrightarrow Z_2 \twoheadleftarrow Z_2 \circledS Z_n$.

We now consider under what conditions a representation of a knot group onto $PSL(2, p)$, $p \geqq 3$, may be lifted to $SL(2, p)$. Since there are two elements in $SL(2, Z)$ which are mapped onto each element of $PSL(2, Z)$, it is possible for a mapping to be lifted in more than one way.

THEOREM 1.7. *If ω is a mapping from a knot group G onto $PSL(2, p)$ such that $|m\omega| \neq 2$, then ω may be lifted in two different ways to $SL(2, p)$.*

Proof. A simple calculation shows that the elements of $PSL(2, p)$ which have trace equal to zero are exactly the elements of order 2.

The group $G(K)$ is generated by a set of conjugate elements $\{x_i\}$, meridians of the knot. We define a homomorphism ψ: $G(K) \to SL(2, p)$ by specifying its value on the generators, extending and verifying that the relators of $G(K)$ are mapped to the identity matrix. Denote the natural homomorphism from $SL(2, p)$ onto $PSL(2, p)$ by a star, *. For each of the generators, x_i define $x_i\psi$ to be one of the two elements of $SL(2, p)$ such that $x_i\psi^* = x_i\omega$ in such a way that $x_i\psi$ has the same trace for all i. This can be done in two possible ways. (Note that conjugate matrices have the same trace). A Wirtinger relator of the group $G(K)$ is of the form $x_i^\epsilon x_j x_i^{-\epsilon} x_k^{-1}$. Now $(x_i^\epsilon x_j x_i^{-\epsilon} x_k^{-1})\psi = \pm$ id, since $(x_i^\epsilon x_j x_i^{-\epsilon} x_k^{-1})\psi^* =$ id. Thus $(x_i^\epsilon x_j x_i^{-\epsilon})\psi = \pm x_k\psi$. However, as remarked, $(x_i^\epsilon x_j x_i^{-\epsilon})\psi$ and $x_k\psi$ have the same trace. We deduce that $(x_i^\epsilon x_j x_i^{-\epsilon})\psi = x_k\psi$, whence $(x_i^\epsilon x_j x_i^{-\epsilon} x_k^{-1})\psi =$ id.

As a result of Theorem 1.7, we can see that if $m\omega$ is of odd order, then ω can be lifted to an epimorphism ψ: $G(K) \to SL(2, p)$ such that $|m\omega| = |m\psi|$.

2. Homology groups of 3-manifolds. In this section, we consider a link L of r components, $K_1, \ldots, K_r$ in an orientable closed 3-manifold M paying particular attention to relationships between $H_1(M - L)$ and $H_1(M)$. Brody [**1**] showed that for links of one component, $H_1(M - L)$ depends only on the homology class of L. The following theorem gives more specific information for links of any number, r, of components.

THEOREM 2.1. *Let $H_1(L)i_*$ be the subgroup of $H_1(M)$ generated by the K_j. Then*

$$B(M - L) = B(M) + r - B(H_1(L)i_*)$$

and

$$\tau(H_1(M)/H_1(L)i_*) \cong \tau(H_1(M - L)).$$

Proof. The formula for the Betti numbers is simply a restatement of Proposition 1.3 of [**5**]. Thus we turn our attention to the torsion subgroups.

Considering the following section of the homology exact sequence:

$$H_1(L) \xrightarrow{i_*} H_1(M) \rightarrow H_1(M, L) \rightarrow H_0(L)$$

one sees that the following sequence is exact:

$$0 \rightarrow H_1(M)/H_1(L)i_* \rightarrow H_1(M, L) \rightarrow H_0(L).$$

Since $H_0(L)$ is torsion free, $\tau(H_1(M)/H_1(L)i_*) \cong \tau(H_1(M, L))$. By duality, $H_1(M, L) \cong H_1(M, N(L)) \cong H_1(M - N(L)^0, \partial N(L)) \cong H^2(M - N(L)^0)$ where $N(L)^0$ represents the interior of $N(L)$. Furthermore, the Universal Coefficient Theorem for cohomology [**11**, p. 243] gives, with X representing $M - N(L)^0$:

$$0 \rightarrow \text{Ext } (H_1(X), Z) \rightarrow H^2(X, Z) \rightarrow \text{Hom } (H_2(X), Z) \rightarrow 0.$$

Since Hom $(H_2(X), Z)$ is torsion free, and $\tau(\text{Ext } (H_1(X), Z)$ is isomorphic to $\tau(H_1(X))$, we see that $\tau(H_1(X)) \cong \tau(H^2(X))$. Therefore, $\tau(H_1(M - L)) \cong \tau(H_1(M)/H_1(L)i_*)$.

The case where each $|K_i|$ is finite is the one which concerns us most. As a consequence of the first part of Theorem 2.1 we obtain the following.

COROLLARY 2.2 *Each K_i is of finite order in $H_1(M)$ if and only if the set of meridians $\{m_i\colon i = 1, \ldots, r\}$ generates a free abelian group of rank r in $H_1(M - L)$.*

For the special case where $r = 1$ we get the following.

COROLLARY 2.3. *If K is a knot in a 3-manifold M and m is its meridian, then K is of finite order in $H_1(M)$ if and only if m is of infinite order in $H_1(M - K)$.*

The following corollary follows straight from Theorem 2.1 in the case where each $|K_i|$ is finite.

COROLLARY 2.4. *If each K_i is of finite order in $H_1(M)$, then $B(M - L) = B(M) + r$ and $\tau(H_1(M - L)) \cong \tau(H_1(M)/H_1(L)i_*)$.*

A simple estimate of the order of $H_1(L)i_*$ leads to the following theorem.

THEOREM 2.5. *Let $L = K_1 \cup \ldots \cup K_r$ be a link in a 3-manifold M, and suppose that each K_i is of finite order in $H_1(M)$. Then*

$$\text{l.c.m. } (|K_i|) \mid T(M)/T(M-L) \mid \prod_{i=1}^{r} |K_i|.$$

In the study of covering spaces of knots one can often derive relations between the components K_i (see next section). In such a case, the following lemma applies.

LEMMA 2.6. *If there exist integers a_{ij} such that $\sum_{j=1}^{r} a_{ij}K_j \sim 0$ for $i = 1, \ldots, s$, and if $|K_j| = q_j \neq 0$, then*

$$\text{l.c.m. } (q_i) \mid T(M)/T(M-L) \mid T(X),$$

where X is the matrix $\begin{bmatrix} Y \\ Z \end{bmatrix}$ *and* $Y = ||\delta_{ij}q_j||_{r\times r}$, $Z = ||a_{ij}||_{s\times r}$.

Proof. Clearly the order of $H(L)i_*$ divides $T(X)$.

The case where we are given a single relation is the most important. In this case the previous lemma can be simplified.

LEMMA 2.7. *If $\sum_{j=1}^{r} a_jK_j \sim 0$ and if s is the least positive integer such that $sa_i \equiv 0 \pmod{|K_i|}$ for all i, then*

$$\text{l.c.m. } (q_i) \mid T(M)/T(M-L) \mid \left(\prod_{i=1}^{r} q_i\right)/s.$$

Since $\tau(H_1(M-L))$ depends on the relationship between $H_1(M)$ and $H_1(L)$, it is not surprising that the orders of each K_i can be read from a certain matrix for $H_1(M-L)$. For each i, let m_i be a meridian of K_i. The set $\{m_i\}$ can be extended to a set $\{m_1, \ldots, m_r, u_1, \ldots, u_n\}$ of generators for $H_1(M-L)$. One obtains a relation matrix for $H_1(M-L)$ of the form $(A|B)$, where A has r columns, the ith column corresponding to the generators m_i, and B has n columns corresponding to the generators u_i. Thus B is a relation matrix for $H_1(M)$. If c is a column vector, then we say $c \equiv 0 \pmod B$ if c is an integral linear combination of the columns of B. The order of $c \pmod B$ is the least positive integer a such that $a\,c \equiv 0 \pmod B$. In the following lemma, c_i denotes the ith column of A corresponding to a meridian m_i, and the a_i are integers.

LEMMA 2.8. *The following statements are equivalent.*

i) $\sum_{i=1}^{r} a_ic_i \equiv 0 \pmod B$.

ii) $\sum_{i=1}^{r} a_iK_i \sim 0$ *in* $H_1(M)$.

iii) *There is a homomorphism* $\Lambda: H_1(M-L)$ *to* Z *such that* $m_i\Lambda = a_i$ *for all* i.

Proof. Clearly i) and iii) are equivalent. We now show ii) and iii) are equivalent.

Suppose $\sum_{i=1}^{r} a_i K_i \sim 0$ in $H_1(M)$. Let v_2 be a 2-chain with boundary $\sum_{i=1}^{r} a_i K_i$. Then for α in $H_1(M - L)$ we define $\alpha\Lambda = \text{Int}(v_2, \alpha)$, where Int is the intersection number. Then Λ is a homomorphism of $H_1(M - L)$ into Z such that $m_i \Lambda = a_i$.

Conversely, suppose the existence of such a Λ. By an adaptation of Proposition 1.1 of [**5**], one sees that there exists a 2-chain v_2 with boundary $\sum_{i=1}^{r} q_i K_i$ such that $a_i = m_i \Lambda = \text{Int}(v_2, m_i)$. It follows that $q_i = a_i$.

It follows from Lemma 2.8 that $|K_i|$ is the order of column $c_i (\text{mod } B)$.

3. Branched and unbranched coverings. In this section, we apply the results of the previous section to the covering space of a knot in S^3. If K is a knot in S^3 and $\tilde{K} = \tilde{K}_1 \cup \ldots \cup \tilde{K}_r$ is the covering link in a branched covering space, M, of S^3 we will refer to $M - \tilde{K}$ as U. If F is a Siefert surface spanning the knot K, then F lifts to a surface $\tilde{F}$ in M with boundary $\sum_{i=1}^{r} a_i \tilde{K}_i$ where a_i is the branching index of $\tilde{K}_i$. Thus

$$\sum_{i=1}^{r} a_i \tilde{K}_i \sim 0 \quad \text{in } H_1(M). \tag{3.1}$$

The examples given in this section apply Lemma 2.6 to various covering spaces to obtain relationships between $T(U)$ and $T(M)$. It is clear that one could instead use Corollary 2.4 to obtain relationships between the torsion coefficients themselves with little extra work.

Example 1. (Irregular S_3 coverings) If M is an irregular S_3 covering of a knot, then $\tilde{K} = \tilde{K}_1 \cup \tilde{K}_2$ where $\tilde{K}_i$ is of index i and $|K_i| = q_i \neq 0$. The matrix X of Lemma 2.6 is

$$\begin{bmatrix} q_1 & 0 \\ 0 & q_2 \\ 1 & 2 \end{bmatrix}$$

which has torsion g.c.d. $(2q_1, q_2)$.

Hence

$$\text{l.c.m. } (q_1, q_2) \quad \Big| \quad T(M)/T(U) \quad \Big| \quad \text{g.c.d. } (2q_1, q_2).$$

Therefore, $T(M) = q_2 T(U) = |\tilde{K}_2| T(U)$.

(Since link $(\tilde{K}_1, \tilde{K}_2) = -2$ link $(\tilde{K}_2, \tilde{K}_2)$ it follows that link $(\tilde{K}_1, \tilde{K}_2)$ is of the form $2a/b$ where $b = T(M)/T(U)$.)

Example 2. ($PSL(2, \text{p})$ coverings of degree $p + 1$) The same argument applies to the $PSL(2, p)$ representations called by Riley "reps" [**9**, p. 608] (see next section for more details on $PSL(2, p)$ representations). These are $PSL(2, p)$ representations into the permutation group $S(J_{p+1})$ such that $m\phi$ is of order p. We obtain $T(M)/T(U) = |\tilde{K}_p|$ where $\tilde{K}_p$ is the knot of index p. In this case, link$(\tilde{K}_1, \tilde{K}_p)$ is of the form pa/b where $b = T(M)/T(U)$.

Often one can obtain more restrictive homology relationships than (3.1). In fact, the surface $\tilde{F}$, which is a lifting of a Seifert surface, may split into several connected components, in which case we obtain one homology relation for each connected component. Let the base point b in S^3 be on the intersection of F with the boundary of a regular neighbourhood N of K. Then b is covered by n points $\tilde{b}_i$. There exists a path $\tilde{x}$ on $\tilde{F}$ from $\tilde{b}_i$ to $\tilde{b}_j$ if and only if there exists a path x on F such that $i(x\phi) = j$. Since any element of G represented by a path on F must lie in the commutator subgroup, G', we deduce that $\tilde{b}_i$ and $\tilde{b}_j$ lie on the same component of $\tilde{F}$ only if i and j lie in the same orbit of J_n under the action of $(G\phi)'$. Hence, if $\{P_i\}, i = 1, \ldots, s$ and $\{Q_i\}, i = 1, \ldots, t$ are respectively the orbits of J_n under $(G\phi)'$ and $\langle m\phi\rangle$ and $a_{ik} = |P_i \cap Q_j|$ where Q_j is any orbit of $m\phi$ contained in O_k, then we have i relations of the form $\sum_{k=1}^{r} a_{ik}\tilde{K}_k \sim 0$. Here O_k is one of the orbits of J_n under $\langle m\phi, l\phi\rangle$. These orbits are in a one-to-one relationship with the components K_i (see [**5**]).

In the case of regular coverings, the situation is slightly more simple. ϕ is called a *regular representation* if $\mathrm{St}_\phi(i) = \ker \phi$. The corresponding covering space is also called *regular*. It is easily seen that if ϕ is a transitive regular representation of G into $S(J_n)$, then the order of $G\phi$ is n. In a regular covering space, the group of covering translations acts transitively on the points $\tilde{b}_i$ above b, and hence on the branch curves. Since all the branch curves are therefore equivalent, we can write $|\tilde{K}_i| = t$ and $\mathrm{br}\ (\tilde{K}_i) = a$. Then $\sum_{i=1}^{r} a\tilde{K}_i \sim 0$. If $|G\phi/(G\phi)'| = s$ then there are s orbits of $(G\phi)'$ in J_n. The relation (3.1) then splits up into s identical relations of the form $\sum_{i=1}^{r} a/s\tilde{K}_i \sim 0$.

Example 3. (Cyclic covering spaces) If M is the n-fold cyclic branched covering space, and ϕ the corresponding representation, then $\tilde{K}$ consists of a single branch curve of branching index n. Also $|G\phi/(G\phi)'| = n$. Hence $\tilde{K} \sim 0$ and so $H_1(U) = Z + H_1(M)$ [**4**, p. 149].

Example 4. (Regular S_3 coverings) If M is the regular covering space corresponding to ϕ: $G \to S_3 \to S(J_6)$, then K consists of 3 knots of branching index 2. Since $|S_3/S_3'| = 2$, we obtain $\tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3 \sim 0$. So $|\tilde{K}_i| \,\big|\, T(M)/T(U) \,\big|\, |\tilde{K}_i|^2$. If K is a 2-bridged knot, it can be shown further that $|\tilde{K}_i| = 1$. (The proof depends on the fact that M is a 2-fold branched cover of the irregular S_3 covering space, which is S^3 [**2**]. Hence $H_1(U) = Z + Z + Z + H_1(M)$ and $H_1(M) \cong Z_d$, $d = \frac{1}{3}|\Delta(-1)|$.)

Finally we consider covering spaces in which there is a single branch curve. This will usually mean that $m\phi$ is a single n cycle, but not always (see [**5**]). In any case, $m\phi$ is a product of cycles of the same length, $b = \mathrm{br}\ (\tilde{K}) = |m\phi|$. If $\tilde{N}$ is a regular neighbourhood of $\tilde{K}$, then l lifts to a torus link $\tilde{l}$ in the boundary of $\tilde{N}$. If we choose a longitude $\tilde{\lambda}$ for $\partial\tilde{N}$ (this is uniquely determined only up to a multiple of $\tilde{m}$), then $\tilde{l} \sim a\tilde{m} + b\tilde{\lambda}$, where g.c.d. (a, b) is the number of components of $\tilde{l}$. Each component is a torus knot which is homologous in M to $(b/\mathrm{g.c.d.}(a, b))\tilde{K}$. We write $v = b/\mathrm{g.c.d.}(a, b)$, and observe that v is easily determined from the representation ϕ. In fact, if $C_1, \ldots, C_s$ are the cycles of $m\phi$ and $D_1, \ldots, D_t$ are the cycles of $l\phi$, it is clear that all the C_i have the same length and so do all the D_i. Furthermore $|C_i \cap D_j|$ does not depend on the particular i and j. Then $v = |C_i \cap D_j|$. It follows that $st|C_i \cap D_j| = n$. Also $s|m\phi| = n$ and $t|l\phi| = n$. Hence $v = |C_i \cap D_j| = |m\phi|\cdot|l\phi|/n$.

THEOREM 3.1. *If M is an n sheeted branched covering space in which the covering link has just one component, then*

$$|m\phi|\cdot|l\phi|/n \mid |\tilde{K}| \mid |m\phi|.$$

Furthermore $|m\phi| \mid n$ *and* $|l\phi| \mid n$.

Proof. Firstly, since $l \sim 0$ in S^3, it follows that $\tilde{l} \sim 0$ in M. Thus br $(\tilde{K}) \cdot \tilde{K} \sim 0$ and so $|\tilde{K}| \big|$ br $(\tilde{K})$. Let $d = |\tilde{K}|$ and $b =$ br $(\tilde{K})$. Then $\tilde{l} \sim a\tilde{m} + b\tilde{\lambda}$ in $H_1(M - K)$ where g.c.d. $(a, b) = b/v$. Now, since $d\tilde{K} \sim 0$, we see that $d\tilde{\lambda} \sim d\tilde{K} \sim 0$ in $H_1(M)$. Hence $d\tilde{\lambda} \sim \alpha\tilde{m}$ in $H_1(M - \tilde{K})$. Thus from $a\tilde{m} + b\tilde{\lambda} \sim 0$ and $d\tilde{\lambda} - \alpha\tilde{m} \sim 0$ comes $(a + b\alpha/d)\tilde{m} \sim 0$. Since $\tilde{m}$ must be of infinite order, this means $a + b\alpha/d = 0$. So $ad + b\alpha = 0$. Dividing by g.c.d. (a, b) we obtain $ad/\text{g.c.d.}(a, b) + b\alpha/\text{g.c.d.}(a, b) = 0$. Since $a/\text{g.c.d.}(a, b)$ and $b/\text{g.c.d.}(a, b)$ have no common factor greater than 1, it follows that $b/\text{g.c.d.}$ $(a, b) = v$ divides d.

Example 5. ($PSL(2, p)$ representations; $p = 5, 7, 11$) If M is a p sheeted covering and $m\phi$ is a p cycle (p prime) and $l\phi \neq$ id, then $|\tilde{K}| = p$ and so $T(M) = pT(U)$. Actually $\tau(U) \cong \tau(M)/\langle\tilde{K}\rangle$ where $\langle\tilde{K}\rangle \cong Z_p$. Also $B(M) = B(U) - 1$. This case occurs in certain $PSL(2, p)$ coverings when $p = 5, 7, 11$. See [**9**] and also the next section. This proves Conjecture C of Riley [**9**] under slightly modified hypotheses.

4. Epimorphisms of knot groups. A basic tool in the investigation of of homology invariants is a notation which was introduced in [**5**] but is redefined below for convenience.

Let F be a group and x some element of F. Let ϕ: $F \rightarrow S(J_n)$ be a transitive representation and let $S = \text{St}_\phi(a)$ where a is some element of J_n. Define the elements $\{x_i: i \in J_n\}$ of S as follows. Let v_{ai} be some element of F such that $a(v_{ai}\phi) = i$. Then x_i is to be the element $v_{ai}x^{\sigma(i)}v_{ai}^{-1}$ where $\sigma(i)$ is the smallest positive integer such that $x^{\sigma(i)} \in \text{St}_\phi(i)$. Of course, x_i depends on the choice of elements v_{ai}, but the conjugacy class of x_i in $\text{St}_\phi(a)$ is independent of this choice.

Define the group $\pi(F, \phi, x)$ to be $S/\langle\{x_i\}\rangle^S$ and define $H(F, \phi, x)$ to be the commutator quotient group of $\pi(F, \phi, x)$. Now let G be a knot group and let l and m be longitude and meridian of the knot. Let ϕ be a transitive representation of G and M and U the corresponding branched and unbranched covering spaces. It is well known that $\text{St}_\phi(a)$ is isomorphic to $\pi_1(U, \tilde{b}_a)$. In fact the projection map, p: $U \rightarrow S^3 - K$ induces an isomorphism $p_a{}^*$ of $\pi_1(U, \tilde{b}_a)$ onto

$\mathrm{St}_\phi(a)$. Hence, $\pi(G, \phi, \mathrm{id}) \cong \mathrm{St}_\phi(a) \cong \pi_1(U)$. The elements m_i are just the projections of appropriately chosen meridians of the components of the covering link $\tilde{K}$. It follows that $\pi(G, \phi, m) \cong \pi_1(M)$.

LEMMA 4.1. *In the following diagram let ψ be an epimorphism and ξ a transitive permutation representation (not necessarily faithful).*

$$G \xrightarrow{\psi} H \xrightarrow{\xi} S(J_n)$$

Let x be any element of G; then ψ induces epimorphisms from $\pi(G, \psi\xi, x)$ onto $\pi(H, \xi, x\psi)$ and from $H(H, \psi\xi, x)$ onto $H(H, \xi, x\psi)$.

Proof. ψ maps $\mathrm{St}_{\psi\xi}(a)$ onto $\mathrm{St}_\xi(a)$, and $v_{ai}x^s v_{ai}^{-1}$ onto $(v_{ai}\psi)(x\psi)^s(v_{ai}\psi)^{-1}$.

As a simple application of Lemma 4.1 we can prove the "only if" part of a proposition due to Perko and Fox [**7**, Lemma 1]:

PROPOSITION 4.2. *Let ξ be the homomorphism of $S(J_4)$ onto $S(J_3)$ with kernel $A = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$. Then a knot group G has a representation ψ onto $S(J_4)$ such that $|m\psi| = 2$ only if $H_1(M_{\psi\xi})$ maps onto Z_2.*

Proof. We may assume $m\psi = (12)$. Then $m\psi\xi = (12)$. Lemma 4.1 states that $H_1(M_{\psi\xi}) = H(G, \psi\xi, m)$ maps onto $H(S(J_4), \xi, (12))$. Now $S = \mathrm{St}_\xi(1) = A \cup A(23)$. Then $H(S(J_4), \xi, (12)) = S/\langle(23)\rangle^S \cong Z_2$.

The following lemma will prove useful.

LEMMA 4.3. *Given ξ: $H \to S(J_n)$, suppose $|x| = |x\xi|$ and that $x\xi$ is a regular element of $S(J_n)$, that is, the cycles of $x\xi$ are all of the same length. Then $\pi(H, \xi, x) = \pi(H, \xi, \mathrm{id}) = \mathrm{St}_\xi(a)$.*

Proof. One simply observes that all x_i are the identity.

Under certain circumstances, one knot group may map onto another knot group, by an epimorphism taking meridian to meridian. (Such an example will be considered in § 5). Thus we consider the homomorphisms

$$\phi: G \xrightarrow{\omega} G^* \xrightarrow{\psi} S(J_n)$$

where $G = G(K)$ and $G^* = G(K^*)$ are knot groups. Let M, U, M^* and U^* be the covering spaces corresponding to ϕ and ψ respectively. Let $\tilde{K}$ consist of s components $\tilde{K}_1, \ldots, \tilde{K}_s$ with meridians $\tilde{m}_1, \ldots, \tilde{m}_s$ and let $\tilde{K}^*$ consist of r components with meridians $\tilde{m}_i^*$. Let $\{\tilde{m}_i\}$ generate a group F in $H_1(U)$ and let F^* be the group generated by $\{\tilde{m}_i^*\}$ in $H_1(U^*)$. As Lemma 4.1 indicates, ω induces an epimorphism $\omega_\#$: $H_1(U) \to H_1(U^*)$ which takes $\tilde{m}_i$ to some $\tilde{m}_j^*$, and takes F onto F^*. Since $\omega_\#$ maps the set $\{\tilde{m}_i\}$ onto the set $\{\tilde{m}_j^*\}$ we see that $r \leqq s$. If $r = s$, then we number the meridians (and hence the components) consistently such that $\tilde{m}_i\omega_\# = \tilde{m}_i^*$.

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THEOREM 4.4. *With the above notation, suppose that* $\tilde{K}$ *and* $\tilde{K}^*$ *have the same number, r, of components and that each* $|\tilde{K}_i^*|$ *is of finite order. Then*

i) *Each* $\tilde{K}_i$ *is of finite order, and* $|\tilde{K}_i|$ *divides* $|\tilde{K}_i^*|$;

ii) $B(M) - B(M^*) = B(U) - B(U^*) \geqq 0$ *and* $B(U) = B(M) + r$;

iii) $T(M)/T(U)$ *divides* $T(M^*)/T(U^*)$; *and*

iv) *if* $B(M) = B(M^*)$ *then* $T(M)/T(U) = T(M^*)/T(U^*)$.

Proof. Since each $\tilde{K}_i^*$ is of finite order, F^* is free abelian of rank r by Corollary 2.2. Since F is generated by r meridians and is mapped onto F^* by $\omega_\#$, it follows that F is free of rank r and so each $\tilde{K}_i$ is of finite order. Now suppose that $\sum_{i=1}^r a_i\tilde{K}_i^* \sim 0$ in $H_1(M^*)$. Then by Lemma 2.8 there is a homomorphism ρ: $H_1(U^*) \to Z$ such that $\tilde{m}_i^*\rho = a_i$. This means that $\omega_\#\rho$ is a homomorphism from $H_1(U)$ to Z such that $\tilde{m}_i\omega_\#\rho = a_i$. So $\sum_{i=1}^r a_i\tilde{K}_i \sim 0$ in $H_1(M)$. Therefore, the mapping $\tilde{K}_i^* \to \tilde{K}_i$ defines a homomorphism of $H_1(\tilde{K}^*)i_*$, onto $H_1(\tilde{K})i_*$ (notation as in Theorem 2.1). In particular $|\tilde{K}_i|$ divides $|\tilde{K}_i^*|$. This proves i). Part ii) follows from the facts that $H_1(M)$ maps onto $H_1(M^*)$ and $F \cong F^*$, and from Corollary 2.4. To prove iii) note that $T(M)/T(U) = |H_1(\tilde{K})i_*|$ divides $|H_1(\tilde{K}^*)i_*| = T(M^*)/T(U^*)$, where we use Corollary 2.4.

Proof of iv): Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & A & \xrightarrow{\gamma} & B & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F & \to & H_1(U) & \xrightarrow{\beta} & H_1(M) & \to & 0 \\
 & & \| \wr & & \downarrow \omega_\# & & \downarrow \omega_b & & \\
0 & \to & F^* & \to & H_1(U^*) & \to & H_1(M^*) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where A is the kernel of $\omega_\#$ and B the kernel of ω_b and γ is induced by β. All the rows and columns are exact except the top row. However, this is exact by the 3×3 lemma [**6**, p. 204], and so $A \cong B$. If $B(M) = B(M^*)$, then $B(U) = B(U^*)$ and both A and B are torsion groups. Therefore $T(U^*) = T(U)/T(A)$ and $T(M^*) = T(M)/T(B)$ from which the desired result follows.

5. PSL coverings. The remainder of this paper is devoted to a detailed study of PSL coverings of knot groups. These coverings were studied by Riley [**9**] who made several conjectures concerning their homology invariants. Many of these conjectures are proved here.

The group $SL(2, p)$ consists of the 2×2 matrices over Z_p with determinant 1, and $PSL(2, p) = SL(2, p)/\langle E\rangle$ where $E = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. The natural projection of $SL(2, p)$ onto $PSL(2, p)$ will be represented by a star, * . $SL(2, p)$ is generated by two elements, $S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Since for $p \geqq 5$, the group $PSL(2, p)$ is simple, there is no apparent restriction on the image, $m\omega$, of a representation of a knot group G onto $PSL(2, p)$. $PSL(2, p)$ contains cyclic subgroups of orders p, $(p + 1)/2$ and $(p - 1)/2$, and any element of the group is contained in a cyclic subgroup of one of these orders [**3**, § 318]. Furthermore, for $n = p$, $(p + 1)/2$ or $(p - 1)/2$, all cyclic subgroups of order n are conjugate [**3**; § 315, § 319, § 316] and therefore, any two cyclic subgroups of the same order are conjugate. The order of any element x must divide p, $(p + 1)/2$ or $(p - 1)/2$.

The group $PSL(2, p)$ may be represented as a permutation group in various ways, since every subgroup of $PSL(2, p)$ gives rise to a representation of $PSL(2, p)$ into the group of permutations of its cosets. For a complete discussion of the subgroup structure of $PSL(2, p)$, see [**2**, Chapter 20]. We will be mainly interested in the degree $p + 1$ representation described below.

Let H be the group of lower triangular matrices in $SL(2, p)$, and H^* its image in $PSL(2, p)$. H and H^* are the internal product of two cyclic subgroups $F = \left\{\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}\right\}$ of order p, and D, the subgroups of diagonal matrices, of order $p - 1$. Furthermore, conjugation by a generator of D gives an automorphism of F of order $(p - 1)/2$. Thus H is a semi-direct product

$$Z_{p-1} \circledS Z_p = \langle S, V: S^p, V^{p-1}, VSV^{-1} = S^\alpha \rangle$$

where $V = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}$ and α is a primitive $(p - 1)/2$th root mod p. Then

$$H^* \cong Z_{(p-1)/2} \circledS Z_p \cong \langle S^*, V^*: S^{*p}, V^{*(p-1)/2}, V^*S^* = S^{*\alpha} V^* \rangle$$

Since H^* is of index $p + 1$, there is a representation η of $PSL(2, p)$ into $S(J_{p+1})$. In fact if J_{p+1} is taken as the set of $Z_p \cup \{\infty\}$, then η has a simple description:

$$i\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* \eta = (ai + c)/(bi + d) \quad \text{for } i \in J_{p+1},$$

with a natural convention regarding ∞ [**3**, § 315]. Then $H^* = \mathrm{St}_\eta(\infty)$. This simple formula makes this representation a particularly convenient one to deal with, and the one which will concern us most.

It follows from the first paragraph of [**3**, § 325] that there is only one conjugacy class of subgroups of index $p + 1$ in $PSL(2, p)$, and hence that all permutational representations of $PSL(2, p)$ of degree $p + 1$ are equivalent up to inner automorphism.

By a *PSL-representation of G* we mean a homomorphism of G onto a group of permutations isomorphic to $PSL(2, p)$.

THEOREM 5.1. *Let ϕ be a $PSL(2, p)$-representation of $G(K)$ of degree $p + 1$, and let o be the order of $m\phi$. Let $M = M_\phi$.*

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a) *If $o = p$, then $H_1(M)$ maps onto Z_{p-1}.*
b) *If o divides $(p+1)/2$ then $H_1(M)$ maps onto Z_n, where*
  $n = p - 1$ *if o is odd,*
  $n = (p-1)/2$ *if o is even.*
c) *If o divides $(p-1)/2$ then $H_1(M)$ maps onto Z_n, where*
  $n = (p-1)/o$ *if o is odd,*
  $n = (p-1)/2o$ *if o is even.*

Proof. We may assume that $\phi = \omega\eta$ where ω is a homomorphism $G \twoheadrightarrow PSL(2, p)$. Since all cyclic subgroups of the same order are conjugate we may assume in part a) that $m\omega$ is a power of $S^* = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^*$ of order p and in part c) that $m\omega$ is a power of V^* where $V = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}$ and α is a primitive root mod p. V^* has order $(p-1)/2$.

In the cases where o is odd, we apply Theorem 1.7 to lift ω to a representation ψ: $G(K) \rightarrow SL(2, p)$ such that $|m\psi| = |m\omega|$. Let λ: $SL(2, p) \rightarrow S(J_{p+1})$ be such that $\psi\lambda = \omega\eta$. Then $\mathrm{St}_\lambda(\infty) = H$, and $\pi_1(M) = \pi(G, \phi, m)$ maps onto $\pi(SL(2, p), \lambda, m\psi)$.

Part a). In this case o is odd. Suppose $m\psi = S^r = X$. Then $X_i = \mathrm{id}$ for all i except ∞, and $X_\infty = X = S^r$. So $\pi(SL(2, p), \lambda, S^r) = H/\langle S^r\rangle^H \cong Z_{p-1}$.

Part b). In this case, $m\phi$ is a regular element of $S(J_{p+1})$ [**3**, § 317], and so Lemma 4.3 applies. If o is odd, we lift to $SL(2, p)$. Lemma 4.3 still applies, and we see that $\pi_1(M)$ maps onto $H \cong Z_{p-1} \circledS Z_p$. In the case where o is even, we work in $PSL(2, p)$. Then $\pi_1(M)$ maps onto $Z_{(p-1)/2} \circledS Z_p$.

Part c). In this case $m\phi$ has two fixed points, 0 and ∞, and is regular in the other elements of J_{p+1}. Suppose that o is odd, then, if $m\psi = X$, then $X_i = \mathrm{id}$ for all i except for $i = 0$ and ∞, in which case, letting $v_{\infty 0} = T$, we see that $X_\infty = X$ and $X_0 = X^{-1}$. Thus $\pi(SL(2, p), \lambda, m\psi) \cong H/\langle X\rangle^H$. Since X is a power of V of order o, it is easily seen from the presentation of H that $H/\langle X\rangle^H \cong Z_n$ where $n = (p-1)/o$.

When o is even, we assume $m\omega = X^*$ is of order o. The details are similar to the case where o is odd, except that we work in $PSL(2, p)$. Then

$$\pi(PSL(2, p), \lambda, m\psi) \cong H^*/\langle X^*\rangle^{H^*} \cong Z_n$$

where $n = (p-1)/2o$.

For $p = 5, 7, 11$, there is a unique representation of $PSL(2, p)$ of degree p. This representation is particularly interesting when $p = 5$, for then $PSL(2, p) = A_5$, and the most natural representation of A_5 is in $S(J_5)$.

The following theorem lists all the possible representations of this type.

THEOREM 5.2. *Let ϕ be a $PSL(2, p)$-representation of $G(K)$ of degree p, and let o be the order of $m\phi$. Let $M = M_\phi$.*

Case 1. $p = 5$.
(a) *If* $o = 5$, *then* $\pi_1(M)$ *maps onto* A_4, *and so* $H_1(M)$ *maps onto* Z_3.
(b) *If* $o = 2$, *then* $\pi_1(M)$ *maps onto* Z_3.
(c) *If* $o = 3$, *we draw no conclusion.*
Case 2. $p = 7$.
(a) *If* $o = 7$, *then* $\pi_1(M)$ *maps onto* S_4, *and so* $H_1(M)$ *maps onto* Z_2.
(b) *If* $o = 3$, *then* $\pi_1(M)$ *maps onto* Z_2.
(c) *If* $o = 2$ *or* 4, *we draw no conclusion.*
Case 3. $p = 11$.
(a) *If* $o = 11$, *then* $\pi_1(M)$ *maps onto* A_5.
(b) *If* $o \neq 11$, *we draw no conclusion.*

Proof. Let ξ: $PSL(2, p) \to S(J_p)$ be a representation of degree p. Then the stabilizer $\mathrm{St}_\xi(1)$ is isomorphic to A_4, S_4 or A_5 according as $p = 5$, 7, or 11. Apply Lemma 4.1, or 4.3.

Of the cases considered in the previous theorems there are very few cases in which the covering space may be a homotopy sphere, or indeed even a homology sphere. For $p \geqq 5$, if ϕ is a $PSL(2, p)$-representation of degree $p + 1$, then M_ϕ may be a homology sphere only if $p \equiv 1 \pmod 4$ and $m\phi$ is of order $(p - 1)/2$.

This occurs when $p = 5$ and $|m\phi| = 2$. In this case, however, Riley's tables [**9**] always show $H_1(M)$ mapping onto Z_2, which suggest that it may be impossible to obtain a homology sphere as a degree $p + 1$ PSL-covering of a knot.

Theorem 5.2 shows a number of cases in which the degree p coverings for $p = 5$, 7, 11, may be homology spheres. In particular, when $p = 5$, one may obtain a homology sphere when $m\phi$ is of order 3. Riley's tables indeed show many examples of knots whose degree 5 covering spaces are homology spheres. For two-bridged knots it can be shown by the techniques of Burde [**2**] that these covering spaces are in fact spheres. However, the 3-bridged knots can not be so conveniently disposed of.

Finally, we prove sharpened Conjecture B of Riley [**10**; p. 25]. Riley observed that certain knot groups have a homomorphism onto $PSL(2, Z)$, and therefore onto $PSL(2, p)$ for any p. If mod_p is the homomorphism of $PSL(2, Z)$ onto $PSL(2, p)$ and η the permutation representation of $PSL(2, p)$ into $S(J_{p+1})$ let $\xi_p = \mathrm{mod}_p\eta$. Let M_p and U_p be the covering spaces, branched and unbranched, corresponding to the representation $\omega\xi_p$ of $G(K)$. (In fact M_p has $p + 1$ sheets). We assume that $m\omega = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then according to Example 2 in § 1, ω can be lifted to a homomorphism ψ of $G(K)$ onto $G(K_{2,3})$, and by the remark following Theorem 1.2, one sees that ψ takes m, a meridian of K onto m^*, a meridian of $K_{2,3}$. Now write $\omega = \psi\beta$, and define $M_p{}^*$ and $U_p{}^*$ to be the covering spaces of $K_{2,3}$ corresponding to $\beta\xi_p$.

Definition. When a prime $p \equiv 1$, 5, 7, 11 (mod 12), define respectively

$j_p = -13, -5, -7, 1$ and $r_p = 1, 3, 2, 6$. Write $b_p = (p + j_p)/6$ and $d_p = (p+1)/r_p$ and $t_p = 6/r_p$.

LEMMA 5.4. *The relation matrix of $H_1(U_p^*)$ is*

$$(5.1) \qquad (t_p\ t_p\ d_p t_p\ \underbrace{0\ 0\ \ldots\ 0}_{b_p})$$

with the first two columns corresponding to $\tilde{m}_1^*$ and $\tilde{m}_2^*$. Therefore, $B(M_p^*) = b_p$, $B(U_p^*) = b_p + 2$ and $T(M_p^*) = d_p t_p$, $T(U_p^*) = t_p$.

Proof. The matrix (5.1) can be computed by the well known Reidemeister-Schreier method. However, the computation is tedious, and hence is omitted. (Also, see [**10**, p. 26]).

The number of components of $\tilde{K}$ in M_p is always equal to two. Hence the hypotheses of Theorem 4.4 are satisfied in the case here considered. Thus as an immediate consequence of that lemma we obtain:

THEOREM 5.5. *Given*

$$\phi: G(K) \xrightarrow{\alpha} PSL(2, p) \xrightarrow{\eta} S(J_{p+1})$$

where α lifts to a homomorphism ω: $G(K) \to PSL(2, Z)$ such that $m\omega = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, then

i) *$B(M_p) \geqq b_p$ and $B(U_p) = B(M_p) + 2$;*
ii) *$T(M_p)/T(U_p)$ divides d_p;*
iii) *if $B(M_p) = b_p$, then $T(M_p) = T(U_p) \cdot d_p$.*

Remark. Riley proved in [**10**, Theorem 3] that $B(U_p) \geqq b_p + 2$.

REFERENCES

1. E. J. Brody, *The topological classification of the lens spaces*, Annals of Mathematics *71* (1960), 163–184.
2. G. Burde, *On branched coverings of S^3*, Can. J. Math. *23* (1971), 84–89.
3. W. Burnside, *Theory of groups of finite order* (Dover Publications, Inc. Second edition, 1911).
4. R. H. Fox, *A quick trip through knot theory*, Topology of three manifolds and related topics (Prentice Hall, 1962), 120–167.
5. R. Hartley and K. Murasugi, *Covering linkage invariants*, Can. J. Math. *29* (1977), 1312–1339.
6. S. MacLane, *Categories for the working mathematician* (Springer-Verlag, New York, 1971).
7. K. A. Perko, Jr., *Octahedral knot covers*, Knots, groups and 3-manifolds (Papers dedicated to the memory of R. H. Fox, ed. L. P. Neuwirth), Annals of Math. Studies *84* (1975), 47–50.
8. K. Reidemeister, *Knoten und Verkettungen*, Math. Zeit. *29* (1929), 713–729.
9. R. Riley, *Homomorphisms of knot groups on finite groups*, Mathematics of Computation, *25* (1971), 603–619.
10. ——— *Hecke invariants of knot groups*, Glasgow Math. J. *15* (1974), 17–26.
11. E. H. Spanier, *Algebraic topology* (McGraw-Hill Book Company, New York, 1966).

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