A QUESTION ON MCCOY RINGS

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Nelsen [J. Algebra **298** (2006) 134–141] asked whether there is a natural class of McCoy rings which includes all reversible rings and all rings R such that $R[X]$ is semi-commutative. In this paper, some new equivalent conditions of McCoy rings are given. One of them is used to answer this question in the affirmative. Finally, an example is given which is McCoy and semi-commutative, but it is not reversible and does not have the property that $R[x]$ is semi-commutative.

Throughout this paper all rings are associative with identity. For a ring R, let $R[x]$ be the polynomial ring and $M_n(R)$ be the ring of $n \times n$ matrices over R. $E_{ij}(r)$ denotes a matrix of which the (i, j) entry is r and others are 0. Recall that a ring R is said to be reversible if $ab = 0$ implies $ba = 0$ for all $a, b \in R$, and R is said to be semi-commutative if $ab = 0$ implies $aRb = 0$ for all $a, b \in R$. In [**4**], a ring R is called a McCoy ring if the equation $f(x)g(x) = 0$, where $f(x), g(x) \in R[x] \backslash \{0\}$, implies there exist $s, t \in R \backslash \{0\}$ such that $f(x)s = tg(x) = 0$.

Let $\mathcal{C}$ denote the class of rings R which have the property that R is both McCoy and semi-commutative, and let $\mathcal{D}$ denote the class of rings R which have the property that $R[x]$ is semi-commutative. In [**4**], Nielsen asked whether there is a natural class of McCoy rings which includes all reversible rings and $\mathcal{D}$. In this paper, some equivalent conditions of McCoy rings are proven and two examples are given to show that $\mathcal{C}$ is this natural class, giving a positive answer to the question of Nielsen. At last, an example is given which is McCoy and semi-commutative, but it is not reversible and does not have the property that $R[x]$ is semi-commutative.

More equivalent conditions of McCoy rings are proven in the following.

THEOREM 1.   *A ring R is McCoy if and only if $R[x]$ is McCoy.*

PROOF: "$\Rightarrow$". Let $F(y) = \sum_{i=0}^{n} f_i y^i$, $G(y) = \sum_{j=0}^{m} g_j y^j$ be nonzero polynomials in $R[x][y]$ such that $F(y)G(y) = 0$, where $f_i = \sum_{s=0}^{p_i} a_{is} x^s$, $g_j = \sum_{t=0}^{q_j} b_{jt} x^t \in R[x]$. As in the proof of

[1, Theorem 2], we let $k = \sum \deg f_i + \sum \deg g_j$, where the degree is as polynomial in x and the degree of the zero polynomial is taken to be 0. Then $F(x^k) = \sum_{i=0}^{n} f_i x^{ik}$, $G(x^k) = \sum_{j=0}^{m} g_j x^{jk} \in R[x]$ and the set of coefficients of the $f_i's$ (respectively $g_j's$) equals the set of coefficients of $F(x^k)$ (respectively $G(x^k)$). Since $F(y)G(y) = 0$ and x commutes with elements of R, $F(x^k)G(x^k) = 0$. Because R is McCoy, there exist $u, v \in R \backslash \{0\}$ such that $F(x^k)u = 0, vG(x^k) = 0$. Thus, $F(y)u = 0, vG(y) = 0$. That is, $R[x]$ is McCoy.

"$\Leftarrow$". Let $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x] \backslash \{0\}$ such that $f(x)g(x) = 0$. Let $f(y) = \sum_{i=0}^{n} a_i y^i$, $g(y) = \sum_{j=0}^{m} b_j y^j$. Then $f(y), g(y) \in R[x][y]$ and $f(y)g(y) = 0$. Since $R[x]$ is McCoy, there exists $h(x) = \sum_{k=0}^{p} h_k x^k \in R[x] \backslash \{0\}$ such that $f(y)h(x) = 0$. Hence $a_i h(x) = 0$ for every $i = 0, 1, \ldots n$. Thus, there exists $h_k \in R \backslash \{0\}$ such that $f(x)h_k = 0$. Similarly, there exits $t \in R \backslash \{0\}$ such that $tg(x) = 0$. Hence R is McCoy. $\square$

THEOREM 2.   *A ring R is McCoy if and only if the ring*

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \ldots & a_{1n} \\ 0 & a & a_{23} & \ldots & a_{2n} \\ 0 & 0 & a & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a \end{pmatrix} : a, a_{kl} \in R \right\}$$

is McCoy for any $n \geqslant 1$.

PROOF: "$\Rightarrow$". Let

$$A_i = \begin{pmatrix} a_i & a_{12}^{(i)} & \ldots & a_{1n}^{(i)} \\ 0 & a_i & \ldots & a_{2n}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_i \end{pmatrix}, B_j = \begin{pmatrix} b_j & b_{12}^{(j)} & \ldots & b_{1n}^{(j)} \\ 0 & b_j & \ldots & b_{2n}^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_j \end{pmatrix} \in R_n.$$

Then

$$F(x) = \sum_{i=0}^{p} A_i x^i = \begin{pmatrix} f(x) & f_{12}(x) & \ldots & f_{1n}(x) \\ 0 & f(x) & \ldots & f_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & f(x) \end{pmatrix},$$

$$G(x) = \sum_{j=0}^{q} B_j x^j = \begin{pmatrix} g(x) & g_{12}(x) & \ldots & g_{1n}(x) \\ 0 & g(x) & \ldots & g_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & g(x) \end{pmatrix},$$

where $f(x) = \sum_{i=0}^{p} a_i x^i$, $f_{kl}(x) = \sum_{i=0}^{p} a_{kl}^{(i)} x^i$, $g(x) = \sum_{j=0}^{q} b_j x^j$, $g_{kl}(x) = \sum_{j=0}^{q} b_{kl}^{(j)} x^j$ for any $k = 1, 2, \ldots, n$, $l = 2, 3, \ldots, n$ and $k < l$.

Suppose $F(x)G(x) = 0$, and $F(x), G(x) \neq 0$. Set $H(x) = F(x)G(x) = (h_{pq}(x))$ for $p, q = 1, 2, \ldots, n$.

CASE 1. If $f(x) \neq 0, g(x) \neq 0$, then $h_{11}(x) = f(x)g(x) = 0$. Since R is McCoy, there exist $s, t \in R\backslash\{0\}$ such that $f(x)s = 0$, and $tg(x) = 0$. Let $A = E_{1n}(s), B = E_{1n}(t)$. Then $F(x)A = 0$ and $BG(x) = 0$.

CASE 2. If $f(x) \neq 0, g(x) = 0$, then there exists $g_{kl}(x) \neq 0$, such that $g_{k+u,l}(x) = 0$ for some k, l, and $1 \leqslant u \leqslant n - k$ since $G(x) \neq 0$. So $h_{kl}(x) = f(x)g_{kl}(x) = 0$. There exists $s \in R\backslash\{0\}$ such that $f(x)s = 0$ because R is McCoy. Let $A = E_{1n}(s)$. Then $F(x)A = AG(x) = 0$.

CASE 3. If $f(x) = 0, g(x) \neq 0$, then there exist $A, B \in R_n\backslash\{0\}$ such that $F(x)A = BG(x) = 0$. The proof is similar to Case 2.

CASE 4. If $f(x) = 0, g(x) = 0$, then for any $s \in R\backslash\{0\}$, let $A = E_{1n}(s)$. It is obvious that $F(x)A = AG(x) = 0$.

Therefore, R_n is McCoy for any case.

"$\Leftarrow$". Assume that $f(x)g(x) = 0$, where

$$f(x) = \sum_{i=0}^{n} a_i x^i \neq 0, g(x) = \sum_{j=0}^{m} b_j x^j \neq 0, a_i, b_j \in R.$$

Let $F(x) = \sum_{i=0}^{n} A_i x^i, G(x) = \sum_{j=0}^{m} B_j x^j$, where

$$A_i = \begin{pmatrix} a_i & a_i & \cdots & a_i \\ 0 & a_i & \cdots & a_i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_i \end{pmatrix}, B_j = \begin{pmatrix} b_j & b_j & \cdots & b_j \\ 0 & b_j & \cdots & b_j \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_j \end{pmatrix}$$

for any $i = 0, 1, \ldots n, j = 0, 1, \ldots m$. Then

$$F(x)G(x) = \begin{pmatrix} f(x) & f(x) & \cdots & f(x) \\ 0 & f(x) & \cdots & f(x) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(x) \end{pmatrix} \begin{pmatrix} g(x) & g(x) & \cdots & g(x) \\ 0 & g(x) & \cdots & g(x) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g(x) \end{pmatrix} = 0.$$

Hence, there exists $A = \begin{pmatrix} s & s_{12} & \cdots & s_{1n} \\ 0 & s & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s \end{pmatrix} \in R_n\backslash\{0\}$ such that $F(x)A = 0$ because R_n

is McCoy. If $s \neq 0$, then $f(x)s = 0$. If $s = 0$, then there exists $s_{ij} \neq 0$ for some i, j, such

that $s_{i+v,j} = 0$ for any $1 \leqslant v \leqslant n - i$. We also have $f(x)s_{ij} = 0$. Similarly, there exists $t \in R\backslash\{0\}$ such that $tg(x) = 0$. Thus, R is McCoy. $\qquad\square$

EXAMPLE 3. Let R be a McCoy ring. By Theorem 2, R_4 is McCoy. Since $E_{12}(1)E_{34}(1)$ $= 0$ and $E_{12}(1)E_{23}(1)E_{34}(1) \neq 0$, R_4 is not semi-commutative.

REMARK 4. In [2, Theorem 2], it is proven that R is McCoy if $R \in \mathcal{D}$. It is obvious that the subring of a semi-commutative ring is also. So that $R[x]$ is semi-commutative implies R is semi-commutative. Thus, $\mathcal{D} \subseteq \mathcal{C}$. Nielsen proven that all reversible rings are McCoy in [4], and it is a known result that a reversible ring is semi-commutative. So $\mathcal{C}$ includes the class of all reversible rings. Moreover, there is an example of a ring that is semi-commutative but not McCoy in [4], and R_4 in Example 3 is McCoy but not semi-commutative. Hence, $\mathcal{C}$ is a natural class of McCoy rings (and of semi-commutative rings) which includes all reversible rings and $\mathcal{D}$. It gives a positive answer to the question of Nielsen in [4].

The following example is constructed by Huh, Lee and Smoktunowicz in [3, Example 2]. We further investigate more properties of the ring in this example.

EXAMPLE 5. Let Z_2 be the field of integers modulo 2 and $A = Z_2[a_0, a_1, a_2, b_0, b_1, b_2, c]$ be the free algebra of polynomials with zero constant terms in noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over Z_2. Note that A is a ring without identity and consider an ideal I of $Z_2 + A$ generated by $a_0b_0, a_1b_2 + a_2b_1, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1$ $+ a_2b_0, a_2b_2, a_0rb_0, a_2rb_2, (a_0 + a_1 + a_2)r(b_0 + b_1 + b_2)$ with $r \in A$ and $r_1r_2r_3r_4$ with $r_1, r_2, r_3, r_4 \in A$. Let $R = (Z_2 + A)/I$. Then R is a semi-commutative and McCoy ring. But R is not reversible, and $R[x]$ is not semicommutative.

PROOF: It was proven that R is semicommutative but $R[x]$ does not have the same property in [3]. R is not reversible since $a_0b_0 \in I$ but $b_0a_0 \notin I$. We use the notion as [3, Example 2]. Each product of indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ is called a monomial and α is a monomial of degree n if it is a product of exactly n generators. Let H_n be the set of all linear combinations of monomials of degree n over Z_2. Then H_n is finite for any n and I is homogeneous (that is, if $\sum_{i=1}^{n} r_i \in I$ with $r_i \in H_i$ then every $r_i \in I$).

Write $\bar{r} = r + I \in R$ for every $r \in Z_2 + A$. Since $A^4 \subseteq I$, $\bar{r}$ can be written in the form $\bar{r} = \bar{a} + \bar{h}_1 + \bar{h}_2 + \bar{h}_3$ with $a \in Z_2, h_i \in H_i (i = 1, 2, 3)$. Write $f(x) = \sum_{i=0}^{n} \bar{p}_i x^i$ and $g(x) = \sum_{j=0}^{m} \bar{q}_j x^j$, where $\bar{p}_i = \bar{a}_p^{(i)} + \overline{f}_1^{(i)} + \overline{f}_2^{(i)} + \overline{f}_3^{(i)}$, $\bar{q}_j = \bar{a}_q^{(j)} + \overline{g}_1^{(j)} + \overline{g}_2^{(j)} + \overline{g}_3^{(j)}$ for some $a_p^{(i)}, a_q^{(j)} \in Z_2$ and $f_k^{(i)}, g_k^{(j)} \in H_k$, $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$, $k = 1, 2, 3$.
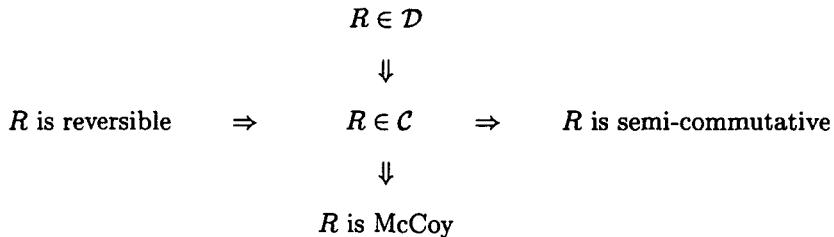
CLAIM. Suppose that $f(x)g(x) = 0$. If not all $\bar{a}_p^{(i)} = 0$, then $g(x) = 0$. If not all $\bar{a}_q^{(j)} = 0$, then $f(x) = 0$.

Suppose that s is the smallest index such that $\bar{a}_p^{(s)} \neq 0$. That is to say $\bar{a}_p^{(s)} = \bar{1}$. If not all $\bar{a}_q^{(j)} = 0$, then we can assume that j_1 is the smallest index such that $\bar{a}_q^{(j_1)} = \bar{1}$. Compare

the coefficient of $x^{(s+j_1)}$ in $f(x)g(x) = 0$. Then there exist $h_i \in H_i (i = 1, 2, 3)$ such that $\overline{a}_p^{(s)} \overline{a}_q^{(j_1)} + \overline{h}_1 + \overline{h}_2 + \overline{h}_3 = 0$. Thus, $\overline{h}_1 + \overline{h}_2 + \overline{h}_3 = \overline{1}$, a contradiction. Hence all $\overline{a}_q^{(j)} = 0$. The Claim is proven by induction on k. Suppose that all $\overline{g}_{k-1}^{(j)} = 0$ for $k \leqslant 3$. Next we prove $\overline{g}_k^{(j)} = 0$ for every j. Assume the contrary. Then we can find the smallest index j_2 such that $\overline{g}_k^{(j_2)} \neq 0$. From the equation $f(x)g(x) = 0$, there exist $h'_{k+1} \in H_{k+1}, \ldots, h'_3 \in H_3$ such that $\overline{g}_k^{(j_2)} + \overline{h}'_{k+1} + \cdots + \overline{h}'_3 = 0$. that is, $g_k^{(j_2)} + h'_{k+1} + \cdots + h'_3 \in I$. Because I is homogeneous, $g_k^{(j_2)} \in I$. This leads to a contradiction. Hence $g(x) = 0$. The remain is similar to proof.

Let $f(x), g(x) \in R[x]\backslash\{0\}$ with $f(x)g(x) = 0$. By the Claim, $\overline{a}_p^{(i)} = \overline{a}_q^{(j)} = 0$ for every i, j. Choose $t = \overline{c}^3$. Then $t \neq 0$ and $f(x)t = tg(x) = 0$. Therefore, R is a McCoy ring. $\square$

REMARK 6.   Example 5 shows that $\mathcal{C}$ properly includes the class of reversible rings and $\mathcal{D}$. The following implications hold.

$$R \in \mathcal{D}$$
$$\Downarrow$$
$$R \text{ is reversible} \quad \Rightarrow \quad R \in \mathcal{C} \quad \Rightarrow \quad R \text{ is semi-commutative}$$
$$\Downarrow$$
$$R \text{ is McCoy}$$

In the table above, each of these implications is irreversible, and there are no other implications between these conditions.

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