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ON SEMIPERFECT FPF-RINGS

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ABSTRACT. We show that a semiperfect right FPF-ring is right self-injective if and only if $J(R) = Z(R_R)$, extending a well-known result due to Carl Faith on semiperfect right FPF-rings with nil Jacobson radical.

A result of H. Tachikawa [6] asserts that a left perfect right FPF-ring is right self-injective. This result was extended and Tachikawa's proof was simplified by Carl Faith [2] who showed that a semiperfect right FPF-ring with nil Jacobson radical is right self-injective. However, there is an example due to B. Osofsky [5] of a right PF-ring with non-nil Jacobson radical. In this note we show that if R is a semiperfect right FPF-ring, then R is right self-injective if and only if $J(R) = Z(R_R)$. We also show that Faith's result can be regarded as a corollary of our result.

Throughout this paper all rings considered are associative with identity and all modules are unitary right R-modules. We write $J(M)$, $Z(M)$ and $E(M)$ for the Jacobson radical, the singular submodule and the injective hull of the right R-module M_R respectively. $N(R)$ will denote the nil radical of R. For any subset X of R, $r_R(X)$ represents the right annihilator of X in R. A ring R is called *right* (F) PF if every (finitely generated) faithful right R-module M generates the category of all right R-modules.

THEOREM 1. *Suppose R is a semiperfect right* FPF-*ring. Then R is right self-injective if and only if $J(R) = Z(R_R)$.*

PROOF. Suppose $J(R) = Z(R_R)$ and let $\{e_1, \ldots, e_m\}$ be a basic set of primitive idempotents for R. As in [1] and [2], if $E_1 = E(e_1R)$ and $\mu \in E_1$ then $(\mu R + e_1R)$ is uniform and $M = (\mu R + e_1R) \oplus e_2R \oplus \cdots \oplus e_mR$ is finitely generated and faithful. Hence M is a generator. By [3, Theorem 1.2B], $(\mu R+e_1R)\oplus e_2R\oplus\cdots\oplus e_mR \cong e_1R\oplus\cdots\oplus e_mR\oplus X$, for some module X_R. By Krull-Schmidt Theorem, since $\mathrm{End}_R(e_1R)$ is local and $e_jR \not\cong e_1R$ $\forall j \geq 2$, it follows that $\mu R + e_1R \cong e_1R \oplus T$ for some module T_R. Since $\mu R + e_1R$ is uniform, $\mu R + e_1R \cong e_1R$ and hence $\mu R + e_1R$ is a local module. Let σ be an R-isomorphism between $\mu R+e_1R$ and e_1R. If $e_1R \neq \mu R+e_1R$, then $e_1R \subseteq J(\mu R+e_1R)$ and $\sigma(e_1R) \subseteq J(e_1R) = e_1J(R) = e_1Z(R_R) \subseteq Z(R_R)$. Now $r_R(e_1) = r_R\big(\sigma(e_1)\big)$ which is right essential in R_R, a contradiction. Thus $\mu \in e_1R$, and so e_1R is injective. This completes the proof.

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Next we show that Faith's result is a corollary of Theorem 1.

LEMMA 2. *Suppose R is a semiperfect ring. Then $Z(R_R) \subseteq J(R)$ and $Z({}_RR) \subseteq J(R)$.*

PROOF. Since R is semiperfect, $Z(R_R)$ lies over a direct summand, *i.e.* $R = eR \oplus (1-e)R$ such that $e^2 = e \in R$, $eR \subseteq Z(R_R)$ and $Z(R_R) \cap (1-e)R$ is small in R. Since $Z(R_R)$ does not contain non-zero idempotents, it follows that $Z(R_R)$ is small in R. Thus $Z(R_R) \subseteq J(R)$. Similarly $Z({}_RR) \subseteq J(R)$.

LEMMA 3. *If R is a right* FPF*-ring then $N(R) \subseteq Z(R_R)$.*

PROOF. This is Lemma 1.2(a) of [4].

COROLLARY 4. *Suppose R is a semiperfect right* FPF*-ring with nil Jacobson radical. Then R is right self-injective.*

PROOF. By Lemma 1 and Lemma 2, $Z(R_R) = J(R)$. Now the result follows from Theorem 1.

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