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PERMUTABLE DIAGONAL-TYPE SUBGROUPS OF $G \times H$

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Abstract. Two subgroups M and S of a group G are said to *permute*, or M *permutes with* S, if $MS = SM$. Furthermore, M is a *permutable subgroup* of G if M permutes with every subgroup of G. In this article, we provide necessary and sufficient conditions for a subgroup of $G \times H$, whose intersections with the direct factors are normal, to be a permutable subgroup.

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1. Introduction. Given that M and S are subgroups of a group G, M and S *permute*, or M *permutes with* S, if $MS = SM$. Recall that MS is itself a subgroup of G if and only if $MS = SM$. Furthermore, M is a *permutable subgroup* of G, or M is *permutable* in G, if M permutes with every subgroup of G. Permutable subgroups were first studied by Ore [**5**], who called them quasinormal, in 1939. While it is clear that a normal subgroup is permutable, Ore proved, among other results, that a permutable subgroup of a finite group is subnormal.

This article considers subgroup permutability in a direct product. There is a well known characterization of normal subgroups of a direct product. It states that N is a normal subgroup of $G \times H$ if and only if $\pi_G(N)/(N \cap G) \leq Z(G/(N \cap G))$ and $\pi_H(N)/(N \cap H) \leq Z(H/(N \cap H))$, where π_G and π_H are the natural projections of $G \times H$ onto G and H respectively. Additionally, P. Hauck [**4**] has studied subnormality in direct products. His results provide a connection between certain subnormal subgroups of a direct product and the Fitting subgroups of the direct factors. With respect to set containment, the set of permutable subgroups is between the set of subnormal and set of normal subgroups in a finite group. Hence it seems interesting to study the permutable subgroups of $G \times H$.

In a prior article [**3**], the permutability of a subgroup of $G \times H$ of type $A \times B$ where $A \leq G$ and $B \leq H$ is considered. Here we study a permutable subgroup of a direct product of two arbitrary groups whose intersections with the direct factors are normal; that is, a subgroup M of $G \times H$ for which $M \cap G \lhd G$ and $M \cap H \lhd H$. Since permutable subgroups behave well under homomorphism correspondence, we only need to concern ourselves with subgroups of $G \times H$ whose intersections with G and H are both trivial.

As a result, we will call a subgroup D of $G \times H$ a *diagonal-type* subgroup if $D \cap G = 1 = D \cap H$. Traditionally, a *diagonal subgroup* of $G_1 \times G_2$, where G_1 is isomorphic to G_2, is a subgroup S of $G_1 \times G_2$ for which there exists an isomorphism ϕ from G_1 to G_2 such that $S = \{(g, \phi(g)) | g \in G_1\}$. One can see that a diagonal-type subgroup D of $G \times H$ is actually a diagonal subgroup of $\pi_G(D) \times \pi_H(D)$.

It follows from the aforementioned characterization of normal subgroups of a direct product that a normal diagonal-type subgroup of $G \times H$ is contained in $Z(G \times H)$. Furthermore, a corollary to the work of Hauck [**4**] states that a subnormal diagonal-type subgroup of $G \times H$ must be contained in the Fitting subgroup of $G \times H$.

From this, one sees an interesting property of diagonal-type subgroups. Of course, every subgroup of the center is normal, and in the finite case, every subgroup of the Fitting subgroup is subnormal. Permutable subgroups behave in a similar way with respect to the *norm* of a group.

First studied by R. Baer [**1**], the *norm* of G, denoted by $N(G)$, is $\{g \in G | g \in N_G(X)$ for all $X \leq G\}$. It is clear that every subgroup of $N(G)$ is permutable in G. In Section 5, we prove that a diagonal-type subgroup of $G \times H$ is permutable if and only if it is contained in $N(G \times H)$.

This result demonstrates that permutable subgroups of a direct product are in some way forced towards being normal. While the norm of a group is not always equal to the center, it is close. E. Schenkman [**7**] proved that $N(G)$ is always contained in the second center of G. Furthermore, a paper of R.A. Bryce and J.L. Rylands [**2**] places bounds on the nilpotency class of certain p-groups with non-central norms.

Yet the norm of $G \times H$ is still closer to the center. In Theorem 4.1, we describe $N(G \times H)$, and then, in Example 4.2, we demonstrate that $N(G \times H)$ can be properly contained in $N(G) \times N(H)$. Thus, when proving in Section 5 that a permutable diagonal-type subgroup D of $G \times H$ is contained in $N(G \times H)$, it is necessary to do more than show that $\pi_G(D)$ and $\pi_H(D)$ are contained in $N(G)$ and $N(H)$ respectively.

Some of the results presented here are a portion of my Ph.D. dissertation at SUNY-Binghamton under direction from Ben Brewster. In a future article, we will report on results about arbitrary permutable subgroups of $G \times H$.

2. Notation. We use $\mathbb{N}$ to represent the natural numbers. If m and n are integers, then (m, n) is the greatest common divisor of m and n.

As mentioned previously, for groups G and H, π_G is the natural projection of $G \times H$ onto G, and π_H is defined similarly. Functions are always written on the left.

For G, an arbitrary group, $N(G)$ stands for the norm of G. If x is an element of G, then $o(x)$ is the order of x. Group theoretical notations not explained here are consistent with those used in [**6**].

3. Basic results. The following two well known results are applied in this article.

(3.1) *Let G be an arbitrary group. If $N(G)$ contains an element of infinite order, then $N(G) = Z(G)$.*

(3.2) *Let M be a subgroup of the group G, and assume that $N \lhd G$ and $N \leq M \leq G$. Then M is a permutable subgroup of G if and only if M/N is a permutable subgroup of G/N.*

Statement (3.1) is a result from [**1**], and a proof of (3.2) is outlined in [**8**, p. 202].

4. The norm of a direct product. Theorem 4.1 describes the norm of $G \times H$ when all elements of G and H have finite order.

THEOREM 4.1. *Let G and H be groups in which all elements have finite order. Then $(g, h) \in N(G \times H)$ if and only if*

(1) $g \in N(G)$,

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(2) $h \in N(H)$,
(3) (g, h) *normalizes each* $\langle (x, y)\rangle$ *where* $o(x) = o(y)$.

Proof. It is clear that the forward direction holds, and so we only prove the converse. Let $(g, h) \in G \times H$ which satisfies properties 1–3. In order to complete the proof, it is sufficient to show that if $(v, w) \in G \times H$ such that $o(v) = p^i$ and $o(w) = p^j$ where $i, j \in \mathbb{N}$, then $(g, h) \in N_{G\times H}(\langle (v, w)\rangle)$. Furthermore, without loss of generality, we may assume $i \geq j$.

Then $o(v^{p^{i-j}}) = p^j$, and there is $n \in \{1, \ldots, p^j - 1\}$ such that $(g^{-1}, h^{-1})(v^{p^{i-j}}, w)(g, h) = ((v^{p^{i-j}})^n, w^n)$, which is equal to $(v^{n\cdot p^{i-j}}, w^n)$. But notice that $g^{-1}(v^{p^{i-j}})g = (g^{-1}vg)^{p^{i-j}}$. Since $g \in N(G)$, $g^{-1}vg = v^k$ for some $k \in \mathbb{N}$. So, $g^{-1}(v^{p^{i-j}})g = v^{k\cdot p^{i-j}}$. As a result, $k \bmod p^j = n$. Now $(g^{-1}, h^{-1})(v, w)(g, h) = (v^k, w^n)$, which equals $(v, w)^k$. Hence, we conclude that $(g, h) \in N_{G\times H}(\langle (v, w)\rangle)$. □

Examples 4.2 and 4.3 involve a group of prime power order with a modular subgroup lattice. These groups are of significant interest in the study of permutable subgroups because a group of prime power order has a modular subgroup lattice if and only if all of its subgroups are permutable. This follows from a well known result, Theorem 5.1.1 in [**8**], stating that a subgroup of a finite group is permutable if and only if it is both subnormal and modular.

K. Iwasawa classified groups of prime power order with modular subgroup lattices in 1941. A complete discussion of this classification can be found in [**8**, chapter 2].

In Example 4.2, the norm of the direct product is not the direct product of the norms. One simply shows this by applying the characterization of the norm given in Theorem 4.1 and then carrying out the appropriate calculations.

Example 4.2. *Let* $E_{27} = \langle a, b | a^9 = 1 = b^3$ *and* $b^{-1}ab = a^4\rangle$. *Then* $N(E_{27} \times E_{27}) = \{(xb^i, yb^i) | (x, y) \in \langle a^3\rangle \times \langle a^3\rangle$ *and* $i \in \{0, 1, 2\}\}$.

Just as every subgroup of the center is normal, every subgroup of the norm is permutable. Thus, as a result of Example 4.2, we get the following example of a permutable diagonal-type subgroup.

Example 4.3. *Let* $E_{27} = \langle a, b | a^9 = 1 = b^3, b^{-1}ab = a^4\rangle$, *and suppose that* $D = \{(a^ib^j, a^ib^j)\}$ *where* $i \in \{0, 3, 6\}$ *and* $j \in \{0, 1, 2\}$. *Then* D *is a permutable diagonal-type subgroup of* $E_{27} \times E_{27}$, *but* D *is not contained in* $Z(E_{27} \times E_{27})$. *Furthermore, the projection of* D *onto each of the direct factors is a normal subgroup of* E_{27}.

Example 4.3 is particularly interesting in the study of permutability in direct products because even though $\pi_G(D) \lhd G$, $D \cap G \lhd G$, $\pi_H(D) \lhd H$, and $D \cap H \lhd H$, D is a permutable subgroup which is not normal.

5. Permutability of diagonal-type subgroups of $G \times H$. Our goal in this section is to provide necessary and sufficient conditions for a diagonal-type subgroup of a direct product of two groups to be permutable. We begin with a lemma concerning an arbitrary permutable subgroup of $G \times H$.

Lemma 5.1. *If* M *is a permutable subgroup of* $G \times H$, *then, for all* $g \in G$ *and* $h \in H$, $\pi_G(M) \leq N_G((M \cap G)\langle g\rangle)$ *and* $\pi_H(M) \leq N_H((M \cap H)\langle h\rangle)$.

Proof. Let $g \in G$ and $(v, w) \in M$. Since M is permutable in $G \times H$, $(g, 1)(v, w) = (\bar{v}, \bar{w})(g^i, 1)$ for some $(\bar{v}, \bar{w}) \in M$ and $i \in \mathbb{Z}$. Thus, $w = \bar{w}$. But then $\bar{v} = vx$ for some

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$x \in M \cap G$. So, $v^{-1}gv = xg^i$. Of course, $\pi_G(M) \leq N_G(M \cap G)$, and therefore $\pi_G(M) \leq N_G((M \cap G)\langle g \rangle)$ for all $g \in G$. Similarly, $\pi_H(M) \leq N_H((M \cap H)\langle h \rangle)$ for all $h \in H$. $\square$

By applying Lemma 5.1 and (3.1), in Lemma 5.2 we observe that a permutable diagonal-type subgroup that contains an element of infinite order must be normal. The main result of the section is then Theorem 5.3, where for arbitrary groups G and H, we characterize a permutable diagonal-type subgroup of $G \times H$.

LEMMA 5.2. *Let G and H be arbitrary groups, and assume that D is a diagonal-type subgroup of $G \times H$ which contains an element of infinite order. Then D is a permutable subgroup of $G \times H$ if and only if $D \lhd G \times H$.*

THEOREM 5.3. *Let G and H be arbitrary groups, and let D be a diagonal-type subgroup of $G \times H$. Then D is a permutable subgroup of $G \times H$ if and only if $D \leq N(G \times H)$.*

Proof. Every subgroup of the norm is permutable, and hence we only need to prove the forward direction. Furthermore, it follows from Lemma 5.2 that we only need to consider the case that every element of D has finite order. As a result of Theorem 4.1, it is then sufficient to prove that if $(g, h) \in G \times H$ such that either $o(g, h)$ is infinite or $o(g) = p^n = o(h)$ for a prime p and $n \in \mathbb{N}$, then $D \leq N_{G\times H}(\langle (g, h)\rangle)$.

Let $(d_1, d_2) \in D$. By Lemma 5.1, $\pi_G(D) \leq N(G)$ and $\pi_H(D) \leq N(H)$. So, from the permutability of D in $G \times H$, we have that $(d_1, d_2)^{-1}(g, h)(d_1, d_2) = (x_1, x_2)(g, h)^z$ where $(x_1, x_2) \in D \cap (\langle g \rangle \times \langle h \rangle)$ and $z \in \mathbb{Z}$. It is clear that if $o(g, h)$ is infinite, then $(d_1, d_2) \in N_{G\times H}(\langle (g, h)\rangle)$. Hence, we suppose that $o(g)$ and $o(h)$ are p^n for a prime p and $n \in \mathbb{N}$.

Notice that $D \cap (\langle g \rangle \times \langle h \rangle)$ is itself a diagonal-type subgroup of $G \times H$. Thus, $D \cap (\langle g \rangle \times \langle h \rangle) = \langle (g^{p^a}, h^{kp^a})\rangle$ where $a \in \{0, 1, \ldots, n\}$ and $k \in \mathbb{N}$ such that $(k, p) = 1$.

By again applying the permutability of D in $G \times H$, we have that $(d_1, d_2)^{-1}(g, h^k)(d_1, d_2) = (y_1, y_2)(g, h^k)^w$ where $(y_1, y_2) \in D \cap (\langle g \rangle \times \langle h \rangle)$ and $w \in \mathbb{N}$. But $(y_1, y_2) = (g^{vp^a}, h^{vkp^a})$ for some $v \in \mathbb{N}$. Hence, $(d_1, d_2)^{-1}(g, h^k)(d_1, d_2) = (g, h^k)^{vp^a+w}$, and so (d_1, d_2) normalizes $\langle (g, h^k)\rangle$. Since $(k, p) = 1$, it follows that $(d_1, d_2) \in N_{G\times H}(\langle (g, h)\rangle)$, completing the proof. $\square$

Of course, we can apply (3.2) to extend our results to the case that the intersections of a permutable subgroup with the direct factors are normal.

COROLLARY 5.4. *Let G and H be arbitrary groups, and let M be a subgroup of $G \times H$ such that $M \cap G \lhd G$ and $M \cap H \lhd H$. Further, assume that $M/((M \cap G) \times (M \cap H))$ contains an element of infinite order. Then M is a permutable subgroup of $G \times H$ if and only if $M \lhd G \times H$.*

COROLLARY 5.5. *Let G and H be arbitrary groups, and let M be a subgroup of $G \times H$ such that $M \cap G \lhd G$ and $M \cap H \lhd H$. Then M is a permutable subgroup of $G \times H$ if and only if $M/((M \cap G) \times (M \cap H)) \leq N((G \times H)/((M \cap G) \times (M \cap H)))$.*

Finally, these results certainly do provide some insight into the permutability of an arbitrary subgroup of $G \times H$. While they do not entirely characterize a permutable subgroup M of $G \times H$, we do understand M as a subgroup of $N_G(M \cap G) \times N_H(M \cap H)$.

COROLLARY 5.6. *Let G and H be any groups. If M is permutable in $G \times H$, then $M/((M \cap G) \times (M \cap H)) \leq N((N_G(M \cap G) \times N_H(M \cap H))/((M \cap G) \times (M \cap H)))$.*

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