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A return time invariant for finitary isomorphisms

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Abstract. Poincare's recurrence theorem says that, given a measurable subset of a space on which a finite measure-preserving transformation acts, almost every point of the subset returns to the subset after a finite number of applications of the transformation. Moreover, Kac's recurrence theorem refines this result by showing that the average of the first return times to the subset over the subset is at most one, with equality in the ergodic case. In particular, the first return time function to any measurable set is integrable. By considering the supremum over all $p \geq 1$ for which the first return time function is p-integrable for all open sets, we obtain a number for each almost-topological dynamical system, which we call the return time invariant. It is easy to show that this invariant is non-decreasing under finitary homomorphism. We use the invariant to construct a continuum number of countable state Markov shifts with a given entropy (and hence measure-theoretically isomorphic) which are pairwise non-finitarily isomorphic.

1. *Introduction*

Let $(X, \mathscr{B}, \mu)$ be a probability space, and let $T: X \to X$ be measurable and preserve μ. For any $B \in \mathscr{B}$ define

$$f_n(B) = \mu\left(B \cap \bigcap_{i=1}^{n-1} T^{-i}B^c \cap T^{-n}B\right),$$

where B^c denotes the complement of B in X. For each $p \geq 1$ the number (which may be $+\infty$)

$$s(p, B) = \sum_{n=1}^{\infty} n^p f_n(B)$$

is the pth moment of the return time function on B. We define

$$p(B) = \sup\{p \in \mathbb{R}: s(p, B) < \infty\},$$

and call $p(B)$ the *return time exponent* of B. Note that (by Kac's recurrence theorem)

$$s(1, B) = \mu\left(\bigcup_{i=0}^{\infty} T^{-i}B\right),$$

so that $p(B) \geq 1$ for each $B \in \mathscr{B}$.

Next, let $(X, \mathscr{B}, \mu, T)$ be a dynamical system as above, and suppose in addition that a topological structure is given on X, i.e. there is given a set τ of subsets of X

(the open sets), closed under arbitrary unions and finite intersections, such that

(i) $\tau \subset \mathscr{B}$, and the σ-algebra generated by τ is $\mathscr{B}$, modulo μ-null sets;

(ii) $T^{-1}\tau \subset \tau$ modulo μ-null sets;

(iii) for each $U \in \tau$, $\mu(U) > 0$.

Such a dynamical system is called *almost-topological.*

Definition. To each almost-topological dynamical system we associate a number $p = p(T)$, $1 \leq p(T) \leq \infty$, called the *return time invariant* of the system, by setting

$$p(T) = \inf\{p(U) \colon U \in \tau\}$$

The following result, which is easily verified, plays a central role in our further discussion.

PROPOSITION. *Let $(X_1, \mathscr{B}_1, \mu_1, T_1)$ and $(X_2, \mathscr{B}_2, \mu_2, T_2)$ be almost-topological dynamical systems with topologies τ_1 and τ_2 respectively. Suppose that ψ is a finitary homomorphism from the first system to the second, i.e. ψ is a measure-theoretical homomorphism such that $\psi^{-1}(\tau_2) \subset \tau_1$ modulo null sets (see* [**1**]*). Then*

$$p(T_1) \leq p(T_2).$$

Proof. If $U_2 \in \tau_2$, then $\psi^{-1}(U_2)$ differs from some $U_1 \in \tau_1$ by a μ_1-null set. Therefore for each $U_2 \in \tau_2$ there exists $U_1 \in \tau_1$ with $p(U_1) = p(U_2)$, and the proposition follows immediately from the definition of the return time invariant. □

It should be mentioned that the definition of the return time invariant does not require the almost continuity of T. To make this somewhat more clear, define

$$p(T, \mathscr{A}) := \inf\{p(A) \colon A \in \mathscr{A}\}$$

for a measure-theoretical dynamical system $(X, \mathscr{B}, \mu, T)$ and an arbitrary subsystem $\mathscr{A} < \mathscr{B}$ of measurable sets. Instead of the preceding proposition we obtain now the following statement:

Suppose $(X_i, \mathscr{B}_i, \mu_i, T_i)$ are measure-theoretical dynamical systems $i = 1, 2$, $\mathscr{A}_i \subset \mathscr{B}_i$, $i = 1, 2$, are subsystems of measurable sets and $\phi : T_1 \to T_2$ is a measure-theoretical homomorphism such that $\phi^{-1}\mathscr{A}_2 \subset \mathscr{A}_1 \bmod 0$. Then $p(T_1, \mathscr{A}_1) \leq p(T_2, \mathscr{A}_2)$.

This shows also that the number p makes sense for a measure preserving system without topology, by taking $\mathscr{A} = \mathscr{B}$, but one can show that it is then always equal to one. On the other hand, if $(X, \mathscr{B}, \mu, T)$ is a finite state Bernoulli scheme with the usual product topology, then for any open set U the sequence $f_n(U)$ is easily seen to be exponentially decaying, so that in this case $p(T) = +\infty$. (This property, called exponential recurrence, is of importance in [**2**]).

2. *Statement of results*

The first theorem shows that in the case of Markov shifts the return time invariant can be finite:

THEOREM 1. *Let μ be an ergodic Markov measure on the countable state shift space $(\mathbb{N}^{\mathbf{Z}}, \mathscr{B}, T)$. Denote by τ the natural topology of this space. Then*

$$p(\mu, T, \tau) = p({}_0[x]) \qquad \textit{for all } x \in \mathbb{N}$$

where ${}_0[x] := \{(w_n)_{n \in \mathbf{Z}} \in \mathbb{N}^{\mathbf{Z}} \mid w_0 = x\}$.

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By an easy construction one can derive from this theorem:

COROLLARY 1. *For every real number $k>1$ there are countable state mixing Markov shifts with return time invariant k and arbitrarily small entropy.*

Furthermore:

COROLLARY 2. *If $(\mathbb{N}^{\mathbf{Z}}, \mathcal{B}, \mu, T)$ is an ergodic Markov shift and $(\{1,\ldots,s\}^{\mathbf{Z}}, \Sigma, \nu, \sigma)$ is a Bernoulli shift, both systems with natural topology, then*

$$p(T)=p(T\times\sigma).$$

The next theorem follows easily from the above corollaries.

THEOREM 2. *For every two real numbers $h>0$ and $k>1$ one can construct a countable state mixing Markov shift with entropy h and return time invariant k.*

This enables us to construct a continuum number of measure-theoretically isomorphic countable state Markov shifts which are pairwise non-finitarily isomorphic.

3. *Proofs*

Before passing to the proof of theorem 1 we state two propositions:

PROPOSITION 1. *If T is a measure-preserving transformation on the probability space $(X, \mathcal{B}, \mu)$ and $B_1, B_2 \in \mathcal{B}$ such that B_1 is a sweep-out-set, i.e. $\bigcup_1^\infty T^{-i}B_1 = X$, and $B_1 \subset B_2$, then $p(B_1)\le p(B_2)$.*

Proof. Using integral estimates one can establish the following. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers, such that a_n decreases monotonically to zero and $a_n - a_{n+1}$ is also monotonically decreasing. Then for $p>1$

$$\sum_{n=1}^{\infty} n^{p-2}a_n<\infty \Leftrightarrow \sum_{n=1}^{\infty} n^p[(a_n-a_{n+1})-(a_{n+1}-a_{n+2})]<\infty, \tag{1}$$

We leave the simple proof to the reader.

Now define $g_n(B) := \mu\,(\bigcap_1^{n-1} T^{-i}B^c)$. With $a_n := g_n(B)$ it is easy to see that (1) implies, for every sweep-out-set B, the equivalence

$$\sum_1^\infty n^{p-2}g_n(B)<\infty \Leftrightarrow \sum_1^\infty n^p f_n(B)<\infty \quad \text{for all } p>1.$$

This equivalence, together with the fact that $g_n(B_2)\le g_n(B_1)$ for all n, implies the proposition. □

Putting the proposition together with:

every open set in a shift space is a union of thin cylinders; and

every set in an ergodic system is sweep-out;

we obtain the

COROLLARY. *For an ergodic finite or countable state shift system $(S^{\mathbf{Z}}, \mathcal{B}, \mu, T)$ with natural topology,*

$$p(T) := \inf_{A\in\bigcup_{n=1}^\infty S^n} p({}_0[A]),$$

where ${}_0[A] := \{(w_n)_{n\in\mathbf{Z}} \in S^{\mathbf{Z}} \mid (w_0,\ldots,w_n)=A\}$.

PROPOSITION 2. *If* $(\mathbb{N}^{\mathbb{Z}}, \mathcal{B}, \mu, T)$ *is an ergodic shift system with natural topology and* $\mu({}_0[x]) > 0$ *for all* $x \in \mathbb{N}$, *we define*

$$\mathfrak{z}_x := \{(x, x_1, \ldots, x_l, x) \mid x \in \mathbb{N},\ x_i \in \mathbb{N} \quad \text{for all } i;\ x \neq x_i,\ i = 1, \ldots, l,\ l > 1;\ \mu({}_0[x, x_1, \ldots, x_l, x]) > 0\}.$$

Then the following statements are equivalent:

(i) $p(T) = p({}_0[x])$ *for all* $x \in \mathbb{N}$;

(ii) $p({}_0[x]) \leq p({}_0[Z])$ *for all* $Z \in \mathfrak{z}_x$, *for all* $x \in \mathbb{N}$;

(iii) $1 < s < p({}_0[x]) \Rightarrow \sum_{n=1}^{\infty} n^s f_n({}_0[Z]) < \infty$ *for all* $Z \in \mathfrak{z}_x$ *for all* $x \in \mathbb{N}$.

Proof. It is clear that (iii) is a restatement of (ii).

(i) $\Rightarrow$(ii) is trivial.

(ii) $\Rightarrow$(i): take any element $A \in \bigcup_{n=1}^{\infty} S^n$. Because A is finite and the process is ergodic one can find $x \in \mathbb{N}$ and $Z \in \mathfrak{z}_x$ with $T^n({}_0[Z]) \subset {}_0[A]$ for some $n \in \mathbb{N}$. Proposition 1 implies $p({}_0[x]) \leq p({}_0[A])$, so we have $p(T) = \inf_{x \in \mathbb{N}} p({}_0[x])$ by the preceding corollary. But taking $A = (x)$ and $x^1 \neq x$ the same reasoning shows that $p({}_0[x^1]) \leq p({}_0[x])$ for all x, x^1 with $x \neq x^1$. Therefore we have $p(T) = p({}_0[x])$ for all $x \in \mathbb{N}$. □

Proof of theorem 1. It suffices to show part (iii) of proposition 2. Take a fixed $x \in \mathbb{N}$, a fixed number s with $1 < s < p({}_0[x])$ and a fixed $Z \in \mathfrak{z}_x$. Let X_n be the projection on the nth coordinate.

Then

$$Y_1 := \inf\{m \geq 0 \mid X_m = x\},$$
$$Y_n := \inf\{m \geq Y_{n-1} \mid X_m = x\};$$

and

$$W_1 := Y_1,$$
$$W_n := Y_n - Y_{n-1} \qquad \text{for } n \geq 2.$$

From the Markov property it follows that the random variables $(W_i)_{i \geq 1}$ are independent and also that the $(W_i)_{i \geq 2}$ are identically distributed with

$$\mu(W_i = n) = \mu({}_0[x])^{-1} \cdot f_n({}_0[x]).$$

Let $l \geq 2$ be the number which appears in $Z = (x, x_1, \ldots, x_{l-1}, x)$. Because $\mu({}_0[Z]) > 0$ we can choose a fixed α with $0 < \alpha < \mu(W_i = l)$. Put $r = [EW_2] + 3$. Now we want to estimate

$$g_{nr}({}_0[Z]) = \mu\left(\bigcap_0^{nr-1} T^{-i}\,{}_0[Z]^c\right)$$

(see proposition 1). Therefore define

$$K_n := \{w \mid T^j w \notin {}_0[Z] \quad \text{for } j = 0, \ldots, n([EW_2] + 2)\}.$$

It is clear that $g_{nr}({}_0[Z]) \leq \mu(K_n)$. To estimate $\mu(K_n)$ we use the events

$$A_n := \left(\sum_2^{n+1} W_i \geq n([EW_2] + 1)\right);$$

$$B_n := \left(\sum_2^{n+1} 1_{\{l\}} \circ W_i < \alpha n\right);$$

$$C_n := (Y_1 \geq n)$$

$$E_n := \left(\sum_{j=0}^{n[EW_2]+2n} 1_{\{_0[\mathfrak{z}_x]\}} \circ T^j \geq \alpha n \right) \qquad \text{where } {}_0[\mathfrak{z}_x] := \bigcup_{Z \in \mathfrak{z}_x} {}_0[Z].$$

Then we have

$$\begin{aligned} P(K_n) &= P(K_n \cap E_n) + P(K_n \cap E_n^c) \\ &\leq P(K_n \cap E_n) + P(A_n) + P(B_n) + P(C_n). \end{aligned}$$

Now we will check, for every term in the above sum, that multiplying by n^{s-2} and summing over n gives a finite value. Then we shall have $\sum_1^\infty n^{s-2} g_{nr}({}_0[Z]) < \infty$; this implies $\sum n^{s-2} g_n({}_0[Z]) < \infty$ because $g_n({}_0[Z])$ is monotonically decreasing, and with the proof of proposition 1 we can conclude $\sum_{n=1}^\infty n^s f_n({}_0[Z]) < \infty$ and the theorem is proved.

(i) $\mu(K_n \cap E_n) \leq (1-\beta)^{\alpha n}$ with $\beta := \mu({}_0[Z]) \cdot f_1({}_0[x])^{-1} > 0$.

Proof. Without loss of generality assume $\mu(E_n) > 0$. Then $\mu(K_n \cap E_n) \leq \mu(K_n | E_n)$ and by splitting E_n into sets according to the first $q = \inf\{m \in \mathbb{N} | m \geq \alpha n\}$ numbers $j_1, \ldots, j_q$ with $T^{-j_i}(w) \in {}_0[\mathfrak{z}_x]$ and using the Markov property the above inequality follows.

(ii) To show $\mu(A_n) = o(n^{-r+1})$ for all r with $s < r < p({}_0[x])$ we use theorem 28 of [3] which states if $(X_j)_{j\in\mathbb{N}}$ is a sequence of i.i.d., centred, random variables in L_r, $(r \geq 1)$, then

$$\text{prob.}\left(\left| \frac{1}{n} \sum_{j=1}^n X_j \right| \geq \varepsilon \right) = o(n^{-r+1}).$$

Taking $X_j := W_{j+1} - EW_2$, $j \in \mathbb{N}$ and any $s < r < p({}_0[x])$ it is easy to check the mentioned conditions. Setting $\varepsilon = 1 + [EW_2] - EW_2$ shows $\mu(A_n) = o(n^{-r+1})$, so $\sum_{n=1}^\infty n^{s-2} \mu(A_n) < \infty$.

(iii) Put $U_i := 1_{\{l\}} \circ W_i$; then

$$\mu(B_n) = \mu\left(\sum_{i=2}^{n+2} U_i < \alpha n \right) \leq \mu\left(\left| \frac{1}{n} \sum_{i=2}^{n+2} (U_i - EU_2) \right| \geq EU_2 - \alpha \right)$$

with $EU_2 - \alpha > 0$ according to the choice of α. Because $(U_i - EU_2)_{i\geq 2}$ is an i.i.d. centred process with finite exponential moment it is well known that $\mu(B_n)$ is exponentially decreasing (see [3]).

(iv) $\mu(C_n) = g_n({}_0[x])$, so using the proof of proposition 1 and the fact $1 < s < p({}_0[x])$ we conclude that $\sum_{n=1}^\infty n^{s-2} \mu(C_n) < \infty$. □

Proof of corollary 1. Every sequence $(c_n)_{n\in\mathbb{N}}$ of real numbers with

(i) $c_n \geq 0$ for all n, $\sum c_n = 1$;
(ii) $c_n > 0$ infinitely often;
(iii) $\sum_{n=1}^\infty nc_n < \infty$;
(iv) $-\sum_{n=2}^\infty (\sum_{i=n}^\infty c_i) \log (\sum_{i=n}^\infty c_i) < \infty$

leads via the stochastic matrix

$$P(i,j) := \begin{cases} c_n & i = 1, j = n, n \in \mathbb{N}; \\ 1 & j - i = 1; \\ 0 & \text{otherwise}; \end{cases}$$

to an ergodic Markov measure μ on the shift space $(\mathbb{N}^{\mathbb{Z}}, \mathscr{B}, T)$ with finite entropy

$$H(\mu) = -\left(\sum_{n=1}^{\infty} c_n \log c_n\right)\left(\sum_{n=1}^{\infty} nc_n\right)^{-1} \le -\sum_{n=1}^{\infty} c_n \log c_n.$$

(Condition (iv) ensures that the entropy of the zero partition is finite.) So taking a sequence $c_n := \alpha n^{-k}$ with $k>2$ and $\alpha := (\sum_{n=1}^{\infty} n^{-k})^{-1}$ one can check that conditions (i)–(iv) are always satisfied. Now change the sequence c_n by defining for every $m \ge 2$,

$$c_n^m := \begin{cases} \sum_{i=1}^{m} c_i & n=1; \\ 0 & 2\le n \le m; \\ c_n & m<n. \end{cases}$$

The conditions (i)–(iv) remain satisfied for every m and, denoting by μ_m the associated Markov measure, we have $H(\mu_m) \to 0$ when $m \to \infty$. But by looking at $_0[1]$ it is clear that according to theorem 1 the return time invariant is equal to $k-1$ for all m. Because $c_1^m > 0$ the associated measure is always mixing. The corollary is proved. □

Proof of corollary 2. From the proposition in § 1 it is clear that $p(T\times\sigma) \le \inf\{p(T), p(\sigma)\}$. But $p(\sigma)=\infty$, so $p(T\times\sigma)\le p(T)$. By the natural identification

$$\mathbb{N}^{\mathbb{Z}} \times \{1,\ldots,s\}^{\mathbb{Z}} \to (\mathbb{N}\times\{1,\ldots,s\})^{\mathbb{Z}}$$

we can look at $T\times\sigma$ as acting on the right mentioned sequence space with the natural topology. Because $T\times\sigma$ is then again an ergodic Markov shift, by using theorem 1 it is now sufficient to show $1<s<p(_0[1])$ in the T-process implies $\sum_{n=1}^{\infty} n^s f_n(_0[1,1)]) < \infty$ in the process $((\mathbb{N}\times\{1,\ldots,s\})^{\mathbb{Z}}, T\times\sigma)$ in order to obtain $p(T\times\sigma)\ge p(T)$. This is done by defining

$$Y_1 : {}_0[1] \to \mathbb{N} \text{ in the } T\text{-process } w \to \min\{m\ge 1 \mid T^m w \in {}_0[1]\};$$
$$Y_n : w \to \min\{m > Y_{n-1}(w) \mid T^m(w) \in {}_0[1]\}.$$

and

$$W_1 := Y_1,$$
$$W_n := Y_n - Y_{n-1} \qquad n \ge 2.$$

Because T is an ergodic automorphism it follows that the $(W_i)_{i\ge1}$ are identically distributed. With $\nu_1 := \nu(_0[1])$ we have:

$$f_n(_0[(1,1)]) = \sum_{l=1}^{n} \mu\left(\sum_{i=1}^{l} W_i = n\right)(1-\nu_1)^{l-1}\nu_1^2,$$

so, by exchanging summations

$$\sum_{n=1}^{\infty} n^s f_n(_0[1,1)]) = \sum_{l=1}^{\infty} \sum_{n\ge l} (1-\nu_1)^{l-1}\cdot \nu_1^2 \cdot n^s \cdot \mu\left(\sum_{i=1}^{l} W_i = n\right)$$
$$= \sum_{l=1}^{\infty} (1-\nu_1)^{l-1}\nu_1^2 \left\| \sum_{i=1}^{l} W_i \right\|_s$$
$$< \sum_{l=1}^{\infty} (1-\nu_1)^{l-1}\cdot\nu_1^2\cdot l \|W_1\|_s < \infty,$$

because $\|W_1\|_s := \sum_{n=1}^{\infty} n^s f_n(_0[1]) < \infty$. The corollary is proved. □

Proof of theorem 2. It should now be obvious how to construct a mixing countable state Markov shift with arbitrary entropy $h > 0$ and arbitrary return time invariant $k > 1$ by using the techniques of the above stated corollaries. □

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