16

Gallery of definitions

16.1 Units

SI units are used throughout this book unless otherwise stated. Most books on modern field theory choose natural units in which $\hbar = c = \epsilon_0 = \mu_0 = 1$. With this choice of units, very many simplifications occur, and the full beauty of the covariant formulation is apparent. The disadvantage, however, is that it distances field theory from the day to day business of applying it to the measurable world. Many assumptions are invalidated when forced to bow to the standard SI system of measurements. The definitions in this guide are chosen, with some care, to make the dimensions of familiar objects appear as intuitive as possible.

Some old units, still encountered, include electrostatic units, rather than coulombs; ergs rather than joules; and gauss rather than tesla, or webers per square metre:

Old	SI
1 e.s.u.	$\frac{1}{3} \times 10^{-9}$ C
1 erg	10^{-7} J
1 eV	1.6×10^{-19} J
1 Å	10^{-10} m
1 G	10^{-4} Wb m^{-2}
1 γ	10^{-5} G

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16.2 Constants

Planck's constant	$\hbar = 1.055 \times 10^{-34}$ J s
speed of light in a vacuum	$c = 2.998 \times 10^{8}$ m s^{-1}
electron rest mass	$m_e = 9.1 \times 10^{-31}$ kg
proton rest mass	$m_p = 1.67 \times 10^{-27}$ kg
Boltzmann's constant	$k_B = 1.38 \times 10^{-23}$ J K^{-1}
Compton wavelength	$\hbar/mc$
structure constant	$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137.3}$
classical electron radius	$r_0 = \frac{e^2}{4\pi\epsilon_0 mc^2} = 2.2 \times 10^{-15}$ m
Bohr radius	$a_0 = \frac{4\pi\epsilon_0\hbar^2}{e^2 m_e} = 0.5292$ Å
electron plasma frequency	$\omega_p = \sqrt{\frac{Ne^2}{\epsilon_0 m_e}}$ s^{-1}
	$\omega_p \sim 56\sqrt{N}$ rad s^{-1}
cyclotron frequency	$\omega_c = \omega_B = \frac{eB}{m}$ s^{-1}

16.3 Engineering dimensions

In n spatial dimensions plus one time dimension, we have the following engineering dimensions for key quantities (note that square brackets denote the engineering dimension of a quantity):

velocity (of light) $[c]$	LT^{-1}
Planck's constant $[\hbar]$	$\mathrm{ML}^2\mathrm{T}^{-1}$
electric charge $[e]$	$\mathrm{L}^{(n+1)/2}\mathrm{T}^{-2}$
gravitational constant G	$\mathrm{M}^{-1}\mathrm{L}^3\mathrm{T}^{-2}$
permittivity $[\epsilon_0]$	$\mathrm{M}^{-1}\mathrm{LT}^{-2}$
permeability $[\mu_0]$	$\mathrm{MT}^4\mathrm{L}^{-3}$
structure constant $[\alpha]$	L^{n-3}

The dynamical variables have dimensions:

Schrödinger field $[\psi]$	$\mathrm{L}^{-n/2}$
Dirac field $[\psi]$	$\mathrm{L}^{-n/2}$
Klein–Gordon $[\phi]$	$\mathrm{L}^{-n/2}\mathrm{T}^{\frac{1}{2}}[\hbar]^{-\frac{1}{2}}$
	$= \mathrm{L}^{-(n+2)/2}\mathrm{T}^{-\frac{1}{2}}\mathrm{M}^{-\frac{1}{2}}$
Maxwell $[A_\mu]$	$\mathrm{L}^{1-n}\mathrm{T}[\hbar] = \mathrm{MTL}^{-(n-1)/2}$
electric current density $J_\mu = \frac{\delta S}{\delta A^\mu}$	$[e]\mathrm{L}^{1-n}\mathrm{T}^{-1}$
particle number density N	L^{-n}

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The plasma distributions are defined from the fact that their integral over a phase space variable gives the number density:

$$N(x) = \int \mathrm{d}\mathbf{v}\ f_v(\mathbf{v}, x) \tag{16.1}$$

$$= \int \mathrm{d}^n\mathbf{p}\ f_p(\mathbf{p}, x) \tag{16.2}$$

and so on. One is generally interested in the distribution as a function of velocity $\mathbf{v}$, the momentum $\mathbf{p}$ or the wavenumber $\mathbf{k}$. In common units, where $\hbar = c = 1$, the above may be simplified by setting $\mathrm{L} = \mathrm{T} = \mathrm{M}^{-1}$. Notice that all coupling constants scale with spacetime dimension.

The constants ϵ_0 and μ_0 are redundant scales; it is not possible to identify the dimensions of the fields and couplings between matter and radiation uniquely. Dimensional analysis of the action, allows one to determine only two combinations:

$$\begin{aligned} \left[eA_\mu\right] &= \mathrm{MLT}^{-1} \\ \left[\frac{e^2}{\epsilon_0}\right] &= \mathrm{L}^n\mathrm{MT}^{-2}. \end{aligned} \tag{16.3}$$

These may be determined by identifying $J^\mu A_\mu$ as an energy density and from Maxwell's equations, respectively. If we assume that ϵ_0 and μ_0 do not change their engineering dimensions with the dimension of space n, then we can identify the scaling relations

$$\begin{aligned} \left[A_\mu\right] &\sim \mathrm{L}^{-\frac{n}{2}} \\ [e] &\sim \mathrm{L}^{\frac{n}{2}}, \end{aligned} \tag{16.4}$$

where $\sim$ means 'scales like'. The former relation is demanded by the dimensions of the action; the latter is demanded by the dimensions of the coupling between matter and radiation, since the product eA_μ must be independent of the dimension of spacetime. Although the dimensions of e, A_μ, ϵ_0 and μ_0 are not absolutely defined, a solution is provided in the relations above.

From the above, one determines that the cyclotron frequency is independent of the spacetime dimension

$$\left[\frac{eB}{m}\right] = \mathrm{T}^{-1}, \tag{16.5}$$

and that the structure constant α has dimensions

$$\left[\frac{e^2}{4\pi\epsilon_0\hbar c}\right] = \mathrm{L}^{n-3}. \tag{16.6}$$

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The so-called plasma frequency is defined only for a given plasma charge density ρ, since

$$\left[\frac{e^2}{\epsilon_0 m}\right] = \mathrm{L}^n\mathrm{T}^{-2}. \tag{16.7}$$

Thus, $\omega_{\mathrm{p}}^2 = \frac{e\rho_N}{\epsilon_0 m} = \frac{eN}{\epsilon_0 m}$.

The Hall conductivity is a purely two-dimensional quantity. The dimensional equation $J = \sigma_{\mathrm{H}} E$ can be verified for $n = 2$ and $\sigma_{\mathrm{H}} = e^2/\hbar$, but it is noted that each of the quantities scales in a way which requires an additional scale for $n \neq 2$.

16.4 Orders of magnitude

16.4.1 Sizes

Planck length L_{p}	$\sqrt{G\hbar/c^3} = 1.6 \times 10^{-35}$ m
Planck time $T_{\mathrm{p}} = L_{\mathrm{p}}/c$	$\sqrt{G\hbar/c^5} = 5.3 \times 10^{-44}$ s
Planck mass M_{p}	$\sqrt{\hbar c/G} = 2.1 \times 10^{-8}$ kg
Planck energy $E_{\mathrm{p}} = M_{\mathrm{p}}c^2$	1.8×10^9 J $= 1.2 \times 10^{19}$ GeV
Hall conductance in $n = 2$	$\sigma_{\mathrm{H}} = e^2/\hbar$
Landau length at $k_{\mathrm{B}}T$	$l = e^2/(4\pi\epsilon_0 k_{\mathrm{B}}T) = 1.67 \times 10^{-5}/T$ m
Debye length	$h = \sqrt{\epsilon KT/Ne^2} = 69 \times \sqrt{T/N}$ m

The Landau length is that length at which the electrostatic energy of electrons balances their thermal energy.

16.4.2 Densities and pressure

number density ('particles' per unit volume)	$N(x)$ or ρ_N
number current $N^0 = Nc$, $N^i = Nv^i$.	N_μ or $J_{N\,\mu}$
mass density current	$J^m_\mu = mN_\mu$
charge density current	$J^e_\mu = eN_\mu$
charge density or other sources	J_μ

interstellar gas	10^6 m^{-3}
ionosphere	10^8–10^{12} m^{-3}
solar corona	10^{13} m^{-3}
solar atmosphere	10^{18} m^{-3}
laboratory plasma	10^{18}–10^{24} m^{-3}
mean density of Earth	5520 kg m^{-3}
mean density of Jupiter/Saturn	1340/705 kg m^{-3}
solar wind particle number density	3–20 cm^{-3}
magneto-pause number density	10^5–10^6 m^{-3}

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Pressure is denoted by P and has the dimensions of energy density or force per unit area.

16.4.3 Temperatures

interstellar gas	10^2 K
Earth ionosphere	10^4 K
solar corona	10^6 K
solar atmosphere	10^4 K
laboratory plasma	10^6 K
super-conducting transition	0–100 K
Bose–Einstein condensation	μK–nK

16.4.4 Energies

first ionization energies	~ 10 eV
Van der Waals binding energy	2 keV
covalent binding energy	20 keV
hydrogen bond binding energy	20 keV
plasma energies, solar wind	1–100 keV
Planck energy $E_{\rm p} = M_{\rm p}c^2$	1.8×10^9 J $= 1.2 \times 10^{19}$ GeV

Lorentz energy–momentum tensor	$\theta_{\mu\nu}$
conformal energy–momentum tensor	$T_{\mu\nu}$

16.4.5 Wavelengths

radio waves	$> 10^{-2}$ m
microwaves	10^{-2} m
infra-red (heat)	10^{-3}–10^{-6} m
visible light	10^{-6}–10^{-7} m
ultra-violet	10^{-7}–10^{-9} m
X-rays	10^{-9}–10^{-13} m
gamma rays	$< 10^{11}$ m
thermal de Broglie wavelength	$\lambda = \sqrt{\frac{\hbar^2}{2mk_{\rm B}T}}$
hydrogen atom at 273 K	2.9×10^{-11} m
hydrogen atom at 1 K	4.9×10^{-10} m
electron at 273 K	1.27×10^{-9} m

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16.4.6 Velocities

speed of light in vacuum c	2.9×10^8 m s^{-1}
solar wind	300–800 km s^{-1}
phase velocity	$\frac{\omega}{k_i} = v^i_{\mathrm{ph}}(k)$
group velocity	$\frac{\partial\omega}{\partial k_i} = v^i_{\mathrm{g}}(k)$
energy transport velocity	$\frac{T_{0i}}{T_{00}} = v^i_{\mathrm{en}}$

16.4.7 Electric fields

geo-electric field at surface (fine weather)	100 V m^{-1}
geo-electric field at surface (stormy weather)	1000 V m^{-1}
auroral field	10^{-3}–10^{-2} V m^{-1}

16.4.8 Magnetic fields

intense laboratory field **H**	$H \sim 10^6$ A m^{-1} [102]
highest coercive field **H** in minerals	$H \sim 10^6$ A m^{-1} [102]
geo-magnetic field	$H \sim 10$ A m^{-1} [102]
geo-magnetic field	$B_0 = 1.88 \times 10^{-5}$ tesla
vertical geo-magnetic field	$B_v = B_0 \tan\delta$, $\delta =$ declination from north
Earth dipole moment	7.95×10^{22} A m^{-2}

16.4.9 Currents

atmospheric current from ionosphere to ground	10^{-12} A m^{-2} [50]
auroral current aligned with field	10^{-7} A m^{-2}
ionospheric dynamo	500 000 A (eastward)

16.5 Summation convention

Einstein's summation convention is used throughout this book. This means that repeated indices are summed over implicitly:

$$\sum_A \phi_A \phi_A \to \phi_A \phi_A, \tag{16.8}$$

and

$$\sum_\mu A^\mu A_\mu \to A^\mu A_\mu. \tag{16.9}$$

In other words, summation signs are omitted for brevity.

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16.6 Symbols and signs

16.6.1 Basis notation

$g_{\mu\nu}$	the spacetime metric with signature $- + + + \cdots$
$\eta_{\mu\nu}$	the constant Minkowski spacetime metric with value $\mathrm{diag}(-1, 1, 1, 1, \ldots)$
$g = -\det g_{\mu\nu}$	the unsigned determinant of the metric which appears in volume measures
$\mu, \nu, \lambda, \rho \ldots$	Greek indices are spacetime-covariant and run from $0, \ldots, n$ in $n + 1$ dimensions
$i, j, k = 1, \ldots, n$	Latin indices refer to spatial dimensions
∂_t	shorthand for $\frac{\partial}{\partial t}$ etc.
$A, B = 1, \ldots, d_R$	upper case Latin indices are the components of a group multiplet for non-spacetime groups, e.g. charge, colour, in a general representation G_R
$a, b = 1, \ldots, d_G$	lower case Latin indices are group indices which belong to the adjoint representation G_{adj}
σ	signifies space
$\mathrm{d}\sigma = \mathrm{d}x^1 \ldots \mathrm{d}x^n$	the spatial volume element
$\mathrm{d}\sigma^\mu$	volume element for a spacelike hyper-surface
∂_σ	derivative normal to a spacelike hyper-surface has the canonical interpretation ∂_0
$U_\mu^{\ \nu}$ or $U_A^{\ B}$	matrix element of a transformation group

Some books make the abbreviation, $(\partial_\mu\phi)^2$, when – in fact – they mean $(\partial_\mu\phi)(\partial^\mu\phi)$. In this text $(\partial_\mu\phi)^2$ means only $(\partial_\mu\phi)(\partial_\mu\phi)$ which differs by a factor of the metric. Note that, because of the choice of metric above, $(\partial_i\phi)^2 = (\partial^i\phi)(\partial_i\phi) = (\partial_i\phi)(\partial_i\phi)$.

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16.6.2 Volume elements

$\mathrm{d}V_x$	invariant volume element in $(n+1)$ dimensional spacetime; $\mathrm{d}V_x = \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\ldots\mathrm{d}x^n\sqrt{g}$
$\mathrm{d}V_t = \frac{1}{c}\mathrm{d}V_x$	the volume element which appears in in most dynamical contexts, such as the action
$(\mathrm{d}x) = \mathrm{d}V_t$	alternative notation for $\mathrm{d}V_t$
$\mathrm{d}\sigma^\mu$	the volume element on spacelike hyper-surface with a unit normal n^μ parallel to $\mathrm{d}\sigma^\mu$
$\mathrm{d}\sigma \equiv (\mathrm{d}\mathbf{x})$	an abbreviation for $\mathrm{d}\sigma^0$, the 'canonical' spacelike hyper-surface; $\mathrm{d}\sigma = \mathrm{d}x^1\ldots\mathrm{d}x^n\sqrt{-\det g_{ij}}$

The volume element appearing in the action, and in most transformations, is $(\mathrm{d}x)$, which differs from the spacetime volume element by a factor of $1/c$. This is because the action has dimensions of energy $\times$ time. Had the action been defined with an extra factor of c one could have avoided this blemish, but that is not traditionally the case. In natural units, $\hbar = c = 1$, this problem is concealed.

16.6.3 Symmetrical and anti-symmetrical combinations

A bar (like a mean value) is used for objects which are symmetrical, e.g.

$$\overline{x} = \frac{1}{2}(x_1 + x_2), \tag{16.10}$$

whereas a tilde is used to signify anti-symmetry:

$$\tilde{x} = (x_1 - x_2). \tag{16.11}$$

Similarly, tensor parts $\overline{T}_{ij}$ and $\tilde{T}_{ij}$ are, by assumption, symmetrical and anti-symmetrical parts.

16.6.4 Derivatives

Field theory abounds with derivatives. Since we often have use for more symbols than are available, some definitions depend on context.

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$\frac{\mathrm{d}}{\mathrm{d}x^\mu}$	the total derivative (this object is seldom used)
$\frac{\partial}{\partial x^\mu} = \partial_\mu$	the partial derivative acting on x, e.g. $\overset{x}{\partial_\mu}\, G(x, x')$
D_μ	a generic derivative; it commonly denotes the gauge-covariant derivative $D_\mu = \partial_\mu - \mathrm{i}eA_\mu$
∇_μ	the Lorentz-covariant derivative, which includes the 'affine connection' ∇_μ is the same as ∂_μ when acting on scalar fields, but for a vector field $\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\lambda{}_{\mu\nu} A_\lambda$
$\nabla^2 = \nabla^i \nabla_i$	the spatial Laplacian
$\Box = \nabla^\mu \nabla_\mu$	the d'Alambertian operator; in Cartesian coordinates, $\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \partial_i^2$, but generally $\Box = \frac{1}{\sqrt{g}}\partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu\right)$
$\hat{\partial}_\mu$	a partial derivative in which the speed of light is replaced by an effective speed of light; also used for higher-dimensional indices in Kaluza–Klein theory.

16.6.5 Momenta

p_i	the *kinetic* momentum also denoted $\mathbf{p}$; the generalization of mass $\times$ velocity in classical mechanics Quantum theory replaces this by $-\mathrm{i}\hbar\partial_i$
p_μ	the $n+1$ dimensional spacetime momentum, a covariant representation of energy and momentum The $\mu = 0$ component is E/c and the spatial components are p_i
π_μ	the covariant momentum. It is the analogue of p_μ but includes any covariant connections, e.g. the electromagnetic vector potential, or the spacetime 'affine' connection e.g. π_μ is $-\mathrm{i}\hbar D_\mu$ or $-\mathrm{i}\hbar\nabla_\mu$
Π_σ	(or simply Π) is the *canonical* momentum, defined by the surface term of the variation of the action (see eqn. (4.23)) The covariant definition of the momentum conjugate to $q(x)$ is $\Pi_\sigma = \frac{\partial \mathcal{L}}{\partial(D^\sigma q(x))}$, where σ is a timelike direction; e.g. $\Pi = \frac{\partial \mathcal{L}}{\partial(D^0 q(x))}$. This quantity does not have the dimensions of

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momentum: it is referred to only as a momentum in the sense of being canonically conjugate to the field variable $q(x)$ (which does not have the dimensions of position)

$\hat{q}$, $\hat{p}_i$	coordinates and momenta which are re-scaled so as to have common engineering dimensions
(dk)	Schwinger notation for the integration measure $\frac{\mathrm{d}^{n+1}k}{(2\pi)^{n+1}}$
(d**k**)	Schwinger notation for the integration measure $\frac{\mathrm{d}^{n}k}{(2\pi)^{n}}$

16.6.6 Position, velocity and acceleration

$r_i = x_i - x'_i$	a ray between spatial position **x** and **x**′
$\hat{r}^i$	a hat can indicate a unit vector
v^i	components of the velocity of an object in the laboratory frame
β^i	same as above in units of the speed of light $\beta^i = v^i/c$
β^2	$\beta^i\beta_i$ where $\beta^i = v^i/c$
γ	relativistic contraction factor $1/\sqrt{1-\beta^2}$
U^μ, β^μ	velocity $(n+1)$-vector; $U^\mu = \gamma(c, v^i) = \gamma c \beta^\mu$ $U^\mu = \partial_\tau x^\mu$ is not directly measurable, but transforms as a vector under Lorentz transformations $\beta^\mu = \partial_t x^\mu$ does not because ∂_t is not invariant
$a^\mu = \partial_0 \beta^\mu$	components of the acceleration in the laboratory frame $a^i = \dot{v}^i/c^2$. This quantity does not transform as an $(n+1)$-vector
$A^\mu = \partial_\tau^2 x^\mu$	acceleration vector, transforms as a tensor of rank 1
$\frac{\omega_k}{k_i} = v^i_{\mathrm{ph}}(k)$	phase velocity
$\frac{\partial\omega_k}{\partial k_i} = v^i_{\mathrm{g}}(k)$	group velocity
$\frac{T_{0i}}{T_{00}}$	energy transport velocity

16.7 Limits

It is natural to check derived expressions in various limits to evaluate their reliability. Here we list a few special limits which might be taken in this context and caution against possibly singular limits. While it might be possible to set quantities such as the mass and magnetic field strength to zero without incurring any explicit singularities, one should not be surprised if these limits yield inconsistent answers compared with explicit calculations in their absence.

In many cases the singular nature of these limits is not immediately obvious, and one must be careful to set these to zero, only at the end of a calculation, or risk losing terms of importance:

$c \to \infty$	non-relativistic limit
$\hbar \to 0$	classical limit
$\mathbf{B} \to \mathbf{0}$	the limit of zero magnetic field is often singular
$m \to 0$	the limit of zero mass is often singular
$R \to 0$	the limit of zero curvature is often singular.

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