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TOLERANCES AND COMMUTATORS ON LATTICES

DIETMAR SCHWEIGERT

The commutator has the following order theoretic properties: $[\alpha,\beta] \leqq \alpha \wedge \beta, [\alpha,\beta] = [\beta,\alpha], [\alpha_1 \vee \alpha_2, \beta] = [\alpha_1,\beta] \vee [\alpha_2,\beta]$ for congruences $\alpha,\beta \in$ Con A of an algebra A in a congruence modular variety generalising the original concept in group theory. A tolerance of a lattice L is a reflexive and symmetric sublattice of L^2. We show that to every commutator [,] of Con A corresponds a $\wedge$-subsemilattice of the lattice of tolerances of Con A. It can be shown that A in a congruence modular variety is nilpotent if $|\text{Con } A| > 2$ and Con A is simple.

0. INTRODUCTION

What can be said about the commutator of a given algebra if one considers only the congruence lattice Con A of the algebra A? This paper deals with this question, confining itself to the purely lattice-theoretic properties of the commutator.

The commutator was introduced by Smith from group theory to congruence permutable varieties in universal algebra. Furthermore Hagemann and Hermann extended this concept to congruence modular varieties. In its polished form it can be found now in the work of several authors [**3**,6]. In this paper we use the following three properties of the commutator: $[\alpha,\beta] \leqq \alpha \wedge \beta, [\alpha,\beta] = [\beta,\alpha], [\alpha_1 \vee \alpha_2,\beta] = [\alpha_1,\beta] \vee [\alpha_2,\beta]$. Via these properties we introduce a rather general commutator for lattices.

A tolerance [**1**] on a lattice $(L;\wedge,\vee)$ is a reflexive, symmetric binary relation compatible with the operations $\vee,\wedge$. A transitive tolerance of L is a congruence of L. We show that to every commutator on L there corresponds a system of tolerances of L. This system is a $\wedge$-subsemilattice of the lattice of tolerances are of L. The properties of this system are used to derive a characterisation of nilpotency for algebras A in a congruence modular variety. This characterisation implies for example that congruence modular variety A is nilpotent if the congruence lattice Con A is simple and $|\text{Con } A| > 2$. Besides other implications of this kind we give some examples and illustrations for the use of tolerances in studying the commutator.

Throughout the paper we assume that every algebra A is a congruence modular variety and that Con A denotes the congruence lattice of A. A commutator [,] on Con A is the commutator defined in the usual way for congruence modular varieties.

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A commutator $[\ ,\ ]$ on a lattice L without refering to Con A is a commutator as given in Definition 1.1. Also we assume throughout that all lattices under consideration are complete.

1. Commutators and tolerances

Definition 1.1: Let L be a *complete lattice.* A map $[\ ,\ ]\colon L \times L \to L$ is called a commutator of L if the following hold:

1.1.a) $[\bigvee_{j\in J} \alpha_j, \beta] = \bigvee_{j\in J} [\alpha_j, \beta]$ for all $\beta, \alpha_j \in L, j \in J$;

1.1.b) $[\alpha, \beta] = [\beta, \alpha]$ for all $\alpha, \beta \in L$;

1.1.c) $[\alpha, \beta] \leqq \alpha \wedge \beta$ for all $\alpha, \beta \in L$.

Remark: If L is the congruence lattice of an algebra in a congruence modular variety then the commutator in the sense of Freese, McKenzie, Herrmann and Gumm has these three properties.

In the following we assume that L is a complete lattice with a least element 0 and a greatest element 1.

Definition 1.2: The map $[\ ,\ )^i : L \times L \to L$ for $i \in \mathbb{N}$ is defined recursively by $[\alpha, \beta)^1 = [\alpha, \beta]$ and $[\alpha, \beta)^i = [[\alpha, \beta)^{i-1}, \beta]$ for a given commutator $[\ ,\ ]$.

Proposition 1.3. $[\ ,\ )^i$ *has the following properties:*

1.3.a) $[\alpha, \beta)^{i+1} \leqq [\alpha, \beta)^i$ *for* $i \in \mathbb{N}$;

1.3.b) $\left[\bigvee_{j\in J} \alpha_j, \beta\right)^i = \bigvee_{j\in J} [\alpha_j, \beta)^i$.

Proof: 1.3.a) By 1.1.c) we have $[\alpha, \beta)^{i+1} = [[\alpha, \beta)^i, \beta] \leqq [\alpha, \beta)^i \wedge \beta \leqq [\alpha, \beta)^i$.

1.3.b) For $i = 1$ this holds by 1.1.a). For $i = k+1$ we have by induction using 1.1.a) again that

$$\left[\bigvee_{j\in J} \alpha_j, \beta\right)^{k+1} = [\left[\bigvee_{j\in J} \alpha_j, \beta\right)^k, \beta] = [\bigvee_{j\in J} [\alpha_j, \beta)^k, \beta]$$
$$= \bigvee_{j\in J} [[\alpha_j, \beta)^k, \beta] = \bigvee_{j\in J} [\alpha_j, \beta)^{k+1}.$$

∎

The following concept is used to connect a commutator with a set of tolerances.

Definition 1.4: For a fixed element $\beta \in L$ the relation $d^i_\beta \subseteq L \times L$ is given by $(\theta_1, \theta_2) \in d^i_\beta$ if and only if $[\theta_1, \beta)^i \leqq \theta_2$ and $[\theta_2, \beta)^i \leqq \theta_1$. We write $d^i_\beta = d_\beta$.

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THEOREM 1.5. *For $\beta \in L$ the relation d^i_β is a tolerance.*

PROOF: d^i_β is reflexive. By 1.3.a) it is obvious that $[\theta,\beta)^i \leqq \theta$ and hence $(\theta,\theta) \in d^i_\beta$.
d^i_β is symmetric because of the symmetry in the definition of d^i_β.
d^i_β is compatible with $\vee$. Let $(\theta_1,\theta_2) \in d^i_\beta$ and $(\eta_1,\eta_2) \in d^i_\beta$. We have by 1.3.b) and the definition of d^i_β that $[\theta_1 \vee \eta_1,\beta)^i = [\theta_1,\beta)^i \vee [\eta_1,\beta)^i \leqq \theta_2 \vee \eta_2$ and similarly $[\theta \vee \eta_2,\beta^i) \leqq \theta_1 \vee \eta_1$. Hence $(\theta_1 \vee \eta_1, \theta_2 \vee \eta_2) \in d^i_\beta$.
d^i_β is compatible with $\wedge$. By 1.3.b) it is obvious that $[x,\beta)^i$ is a monotone map. Hence $[\theta_1 \wedge \eta_1)^i \leqq [\theta_1,\beta)^i \wedge [\eta_1,\beta)^i \leqq \theta_2 \wedge \eta_2$ and similarly $[\theta_2 \wedge \eta_2,\beta)^i \leqq \theta_1 \wedge \eta_1$. Hence $(\theta_1 \wedge \eta_1, \theta_2 \wedge \eta_2) \in d^i_\beta$. ∎

The next result shows that $[\alpha,\beta)^i$ is the least element in α-block of the tolerance d^i_β.

PROPOSITION 1.6. $[\alpha,\beta)^i = \inf\{x \in L \mid (x,\alpha) \in d^i_\beta\}$

PROOF: Write $\gamma = \inf\{x \in L \mid (x,\alpha) \in d^i_\beta\}$. We have $[\gamma,\beta)^i \leqq \alpha$ and $[\alpha,\beta)^i \leqq \gamma$ by definition. On the other hand $\left[[\alpha,\beta)^i,\beta\right)^i \leqq \alpha$ and $[\alpha,\beta)^i \leqq [\alpha,\beta)^i$. Hence $\left([\alpha,\beta)^i,\alpha\right) \in d^i_\beta$. It follows that $\gamma \leqq [\alpha,\beta)^i$ and therefore $\gamma = [\alpha,\beta)^i$. ∎

To some extent we would like to study the set of tolerances $\{d_\beta \mid \beta \in L\}$ with respect to the lattice of tolerances of L.

LEMMA 1.7. *For $i \in \mathbb{N}$ and $\alpha,\beta \in L$ the following hold*

1.7.1 *if $\alpha \leqq \beta$ then $d^i_\beta \leqq d^i_\alpha$;*
1.7.2 $d_\alpha \wedge d_\beta = d_{\alpha\vee\beta}$.

PROOF: 1.7.1. If $(\theta_1,\theta_2) \in d^i_\beta$ then $[\theta_1,\beta)^i \leqq \theta_2$ and $[\theta_2,\beta)^i \leqq \theta_1$. If $\alpha \leqq \beta$ then $[\theta_1,\alpha)^i \leq [\theta_1,\beta)^i$, since $[\theta_1,x)^i$ is a monotone map. We have $[\theta_1,\alpha)^i \leqq \theta_2$ and by the same argument $[\theta_2,\alpha)^i \leqq \theta_1$. Therefore $(\theta_1,\theta_2) \in d^i_\alpha$.
1.7.2. As $\alpha \leqq \alpha \vee \beta$ and $\beta \leqq \alpha \vee \beta$ we have by 1.7.1 that $d_{\alpha\vee\beta} \leqq d_\alpha \wedge d_\beta$. On the other hand if $(\theta_1,\theta_2) \in d_\alpha \wedge d_\beta$ we have $(\theta_1,\theta_2) \in d_\alpha$ and hence $[\theta_1,\alpha] \leqq \theta_2$. From this it follows that $[\theta_1,\alpha] \vee [\theta_1,\beta] \leqq \theta_2$ and therefore $[\theta_1,\alpha \vee \beta] \leqq \theta_2$. Similarly we have $[\theta_2,\alpha \vee \beta] \leqq \theta_1$ and therefore $(\theta_1,\theta_2) \in d_{\alpha\vee\beta}$. ∎

PROPOSITION 1.8. *The set $D = \{d_\beta \mid \beta \in L\}$ forms a $\wedge$-subsemilattice of the tolerance lattice of L with least element d_1 and greatest element $d_o = L \times L$.*

In the following we study relational products of tolerances.

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LEMMA 1.9. *For $i \in \mathbb{N}$ and $\beta \in L$ the following hold:*

1.9.1 $d_\beta^i \leqq d_\beta^{i+1}$;

1.9.2 $d_\beta^i \circ d_\beta^j \leqq d_\beta^{i+j}$;

1.9.3 $d_\beta^{i+1} \leqq d_\beta^1 \circ d_\beta^i$.

PROOF: 1.9.1. Let $(\theta_1, \theta_2) \in d_\beta^i$. By definition we have $[\theta_1, \beta)^i \leqq \theta_2$ and because of 1.3.a) we have $[\theta_1, \beta)^{i+1} \leqq [\theta_1, \beta)^i$ and hence $[\theta_1, \beta)^{i+1} \leqq \theta_2$. Together with $[\theta_2, \beta)^{i+1} \leqq \theta_1$ we have $(\theta_1, \theta_2) \in d_\beta^{i+1}$.
1.9.2. Let $(\theta_1, \theta_2) \in d_\beta^i \circ d_\beta^i$. Then there exists $\eta \in L$ such that $(\theta_1, \eta) \in d_\beta^i$ and $(\eta, \theta_2) \in d_\beta^j$. We have $[\theta_1, \beta)^i \leqq \eta$ and $[\eta, \beta)^j \leqq \theta_2$. From this we have $[\theta_1, \beta)^{i+j} = [\theta_1, \beta)^i, \beta)^j \leqq [\eta, \beta)^j \leqq \theta_2$. Similarly we have $[\theta_2, \beta)^{i+j} = [\theta_2, \beta)^{j+1} \leqq \theta_1$. Therefore we have $(\theta_1, \theta_2) \in d_\beta^{i+j}$.
1.9.3. Let $(\theta_1, \theta_2) \in d_\beta^{i+1}$ and assume $\theta_1 \leqq \theta_2$. Write $\varepsilon = [\theta_2, \beta)^i$. Then we have $(\theta_2, \varepsilon) \in d_\beta^i$ as we have $[\theta_2, \beta)^i \leqq \varepsilon$ and $[\varepsilon, \beta)^i = \left[[\theta_2, \beta)^i, \beta\right)^i \leqq \theta_2$. By Proposition 1.6 we have $(\varepsilon, [\varepsilon, \beta]) \in d_\beta^1$. From this we have that $(\varepsilon \vee \theta_1, [\varepsilon, \beta] \vee \theta_1) \in d_\beta^1$. But $[\varepsilon, \beta] = [[\theta_2, \beta)^i, \beta] = [\theta_2, \beta)^{i+1} \leqq \theta_1$ and hence (*) $(\varepsilon \vee \theta_1, \theta_1) \in d_\beta^1$. We have already $(\theta_2, \varepsilon) \in d_\beta^i$. From this we derive $(\theta_2 \vee \theta_1, \varepsilon \vee \theta_1) \in d_\beta^i$ and finally (**) $(\theta_2, \varepsilon \vee \theta_1) \in d_\beta^i$. Taking (*) and (**) together we have $(\theta_1, \theta_2) \in d_\beta^1 \circ d_\beta^i$. The assumption $\theta_1 \leqq \theta_2$ does not restrict the generality (see [**1**] p. 369). ∎

THEOREM 1.10. *For $i \in \mathbb{N}$ and $\alpha, \beta \in L$ the following are equivalent:*

1.10.1 d_β^i *is transitive;*

1.10.2 $d_\beta^{i+k} = d_\beta^i$ *for every* $k \in \mathbb{N}$;

1.10.3 $[\alpha, \beta)^{i+k} = [\alpha, \beta)^i$ *for every* $k \in \mathbb{N}$.

PROOF: 1.10.1 $\Rightarrow$ 1.10.2.
By 1.9.3 we have $d_\beta^{i+1} \leqq d_\beta^1 \circ d_\beta^i \leqq d_\beta^i \circ d_\beta^i$. As d_β^i is transitive we have $d_\beta^{i+1} \leqq d_\beta^i$. By 1.9.1 it follows that $d_\beta^{i+1} = d_\beta^i$. Using induction we have

$$d_\beta^{i+k} \leqq d_\beta^1 \circ d_\beta^{i+k-1} = d_\beta^1 \circ d_\beta^i \leqq d_\beta^i \circ d_\beta^i = d_\beta^i.$$

1.10.2 $\Rightarrow$ 1.10.3.
Let $\alpha \in L$ and write $\gamma = [\alpha, \beta)^{i+k}$. By Proposition 1.6 we have $(\alpha, \gamma) \in d_\beta^{i+k}$. By 1.10.2 we have $(\alpha, \gamma) \in d_\beta^i$ which means $[\alpha, \beta)^i \leqq \gamma$ and therefore $[\alpha, \beta)^i \leqq [\alpha, \beta)^{i+k}$. By 1.3.a) we have $[\alpha, \beta)^{i+k} = [\alpha, \beta)^i$.
1.10.3 $\Rightarrow$ 1.10.1.
Let $(\theta_1, \theta_2) \in d_\beta^{i+k}$ for some $k \in \mathbb{N}$. Then we have $[\theta_1, \beta)^{i+k} \leqq \theta_2$ and by 1.10.3

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$[\theta_1,\beta)^i \leqq \theta_2$. Therefore we conclude $d_\beta^{i+k} \leqq d_\beta^i$. Now we have $d_\beta^i \circ d_\beta^i \leqq d_\beta^{i+i}$ by 1.9.2 and hence $d_\beta^i \circ d_\beta^i \leqq d_\beta^i$. Therefore d_β^i is transitive. ∎

2. Nilpotency

In the following we assume that the algebra A is in a congruence modular variety and that the congruence lattice Con A is of finite length. The algebra A is called nilpotent if there is an $n \in \mathbb{N}$ such that the (usual) commutator $[1,1) = A \times A = 0_A$.

Lemma 2.1. *The algebra A is nilpotent if and only if the transitive hull of d_1 is Con $A \times$ Con A.*

Proof: Let A be nilpotent and let θ be a congruence of the lattice Con A such that $d_1 \leqslant \theta$. We have $[1,1)^n \leqq 0$ and $[0,1)^n \leqq 1$; hence $(1,0) \in d_1^n$. As $d_1^n \leqslant \theta$ we have that $\theta =$ Con $A \times$ Con A. Now assume that the transitive hull of d_1 is Con $\times$ Con A. The transitive hull of d_1 is a finite power $d_1^n = d_1 \circ \cdots \circ d_1$. We have $(0,1) \in d_1^n$ and hence $[1,1)^n = 0$. ∎

Theorem 2.2. *For the algebra A let Con A be simple and $|$Con $A| > 2$. Then A is nilpotent*

Proof: Consider the solvability congruence $\overset{s}{\sim}$ in [**6**, Definition 7.3]. If $\overset{s}{\sim}$ is the identity relation then we have from $[\alpha,\alpha] \overset{s}{\sim} \alpha$ that $[\alpha,\alpha] = \alpha$. From this it follows that $[\alpha \wedge \beta, \alpha \wedge \beta] = \alpha \wedge \beta$ and furthermore we have $[\alpha \wedge \beta, \alpha \wedge \beta] \leqq [\alpha,\beta] \leqq \alpha \wedge \beta$. Now from $[\alpha,\beta] = \alpha \wedge \beta$ it follows that Con A is distributive. As Con A is simple we have $|$Con $A| = 2$, a contradiction. Therefore we conclude that $\overset{s}{\sim}$ is not the identity relation on Con A and therefore $[1,1] < 1$. As d_1 is not trivial its transitive hull is Con $A \times$ Con A. By Lemma 2.1 the algebra A is nilpotent.

The algebra A is called abelian if $[1,1] = 0$. ∎

Theorem 2.3. *Let A be an algebra such that Con A has only trivial tolerances and $|$Con $A| > 2$. Then A is abelian.*

Proof: The greatest element 1 of Con A can not be $\vee$-irreducible because otherwise Con A would not be simple. Hence $1 = \alpha \vee \beta$ for some $\alpha, \beta < 1$. Now d_α and d_β are tolerances which are not the identity. Hence $(0,1) \in d_\alpha$ and $(0,1) \in d_\beta$. Therefore $(0,1) \in d_\alpha \wedge d_\beta = d_{\alpha\vee\beta} = d_1$. As $(0,1) \in d_1$ we have $[1,1] = 0$. ∎

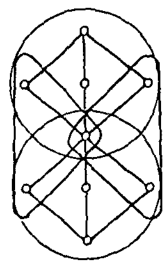
A lattice L is called tight if and only if L is finite, $|L| > 1$, and if ρ is any tolerance of L such that $(0,a) \in \rho$ for some $a > 0$ or $(1,b) \in \rho$ for some $b < 1$ then $\rho = L^2$.

Corollary 2.4. *Let A be an algebra such that Con A is a simple tight lattice and Con$|A| > 2$. Then A is abelian.*

PROOF: If L has a non-trivial tolerance ρ then by [**8**] ρ is contained in a non-trivial congruence or in a central relation. As L is simple there is no non-trivial congruence. As L is tight there is no central relation. Hence L has only trivial tolerances. The Corollary now follows from Theorem 2.3. ∎

3. ILLUSTRATIONS AND EXAMPLES

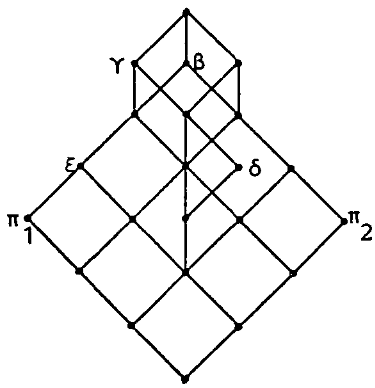
3.1. A commutator generated by a tolerance.
Let θ be a tolerance of a finite lattice L and write $d(\alpha) = \inf\{z \mid (z,\alpha) \in \theta\}$ for some $\alpha \in L$. As we have observed $d(\alpha \vee \beta) = d(\alpha) \vee d(\beta)$. By $d(0) = 0$ we have that $L(\theta) = \{d(\alpha) \mid \alpha \in L\}$ is a lattice and it is even a $\vee$-subsemilattice of L. If we assume that $L(\theta)$ is a distributive sublattice of L then a commutator can be constructed by defining $[\alpha,\beta] = d(\alpha) \wedge d(\beta)$. As an example consider the following tolerance θ.



One observes that $d(1)d(d(1)) = 0$ and $[1,1] > [[1,1],1] = 0$.
$L(\theta)$ is a distributive sublattice of L in this rather special case.
For a counterexample in the general case see [**4**].

3.2. Estimating the commutator on Con A.
Let β be a $\vee$-irreducible element of the lattice Con A of a finite algebra A. Define $r(\beta)$ as the tolerance generated by all pairs $(x, x \wedge \beta)$ fpr $x \in$ Con A. Write $]\alpha,\beta[= \inf\{x \mid (x,\alpha) \in r(\beta)\}$. Then we have for $]\ ,\ [$ that $]\alpha,\beta[\leqq \alpha \wedge \beta$ and $]\alpha_1 \vee \alpha_2, \beta[=]\alpha_1,\beta]\alpha_2,\beta[$. We extend this definition to every $\beta \in$ Con A with $\beta = \gamma_1 \vee \cdots \vee \gamma_n$ where γ_i is $\vee$-irreducible, $i = 1,\ldots,n$ by $]\alpha,\beta[=]\alpha,\gamma_1[\vee \cdots \vee]\alpha,\gamma_n[$. One can show that the commutator of the algebra A can be estimated by the upper bound $]\alpha,\beta[\wedge]\beta,\alpha[$.

3.3. A comparative example.
We consider the following lattice from [**5**].



Using the properties 1.1a) - c) we have $[\gamma,\beta] = [\pi_1, \vee\delta, \pi_1 \vee \pi_2]$ $= [\pi_1,\pi_1] \vee [\pi_1,\pi_2] \vee [\delta,\pi_1] \vee [\delta,\pi_2] \leqq \pi_1 \vee (\delta \wedge \pi_1) \vee (\delta \wedge_2) = \xi$ where ξ is the usual commutator. If we define [,] as the largest binary operation defined on Con A satisfying 1.1a) -c) we have $[\gamma,\beta] = \xi$.

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FB Mathematik
der Universität
D-6750 Kaiserslautern
Fed. Rep. Germany