AN ISOPERIMETRIC INEQUALITY FOR CONVEX POLYHEDRA WITH TRIANGULAR FACES

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H. T. Croft [1] has conjectured that among all tetrahedra with fixed total edge length the regular one has the greatest surface area. In this note we prove the following result, which includes this conjecture as a special case:

THEOREM. Let P_T be a convex polyhedron with triangular faces, and let L_T and A_T denote its total edge length and its total surface area. Then

$$L_T^2 / A_T \geq 12\sqrt{3}$$

with equality if and only if P_T is a regular tetrahedron.

The proof of this theorem will be based on the following preliminary remarks:

A (finite) set M of polygons will be called a L_T-set if the following hold:

A. The sum of the perimeters of the polygons of M is $2L_T$.

B. Each side of any polygon in M may be paired with a side of equal length from a different polygon in M, with no two polygons containing more than one complete set of paired sides.

C. No polygon in M has more than 3 sides.

The maximum total area of the polygons in a L_T-set will be at least as great as the surface area of any convex polyhedron with triangular faces and fixed total edge length L_T. since the set of faces of such a polyhedron is a L_T-set.

Sets of polygons satisfying A and B were considered by O. Aberth in [2]. The proof of the existence of a L_T-set of convex polygons enclosing maximum total area $A^*(L_T)$ is completely analagous to that of Lemma 1 in [2]. Further, O. Aberth shows in [2] that a circle may be circumscribed

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about each polygon of the maximal set. The sum, $2L_\tau$, of the perimeters of all polygons of the maximal set satisfies

$$2L_\tau \geq \sum_{s \in P_1} G(s),$$

where P_1 denotes the polygon with the largest circumscribed circle and $G(s)$ is defined as the length of any side $s \in P_1$ plus the perimeter of that polygon P_q containing the side s' paired to s.

Using Lemma 3 of [2] and normalizing the maximal set by putting $2A^*(L_\tau)/L_\tau = 1$, O. Aberth obtains in [2]:

$$G(s) \geq \begin{cases} 2r \sin\theta \; (1 + \pi/\phi) & \text{for } 1/2 \leq d < 1 \\ 6r \sin\theta & \text{for } \quad d \geq 1 . \end{cases}$$

Here r is the radius of the circle circumscribing P_1 (with centre 0), 2θ is the angle $s \in P_1$ subtends at 0 and d is the distance from 0 to s defined in Lemma 3 of [2]. $2\phi = \tan^{-1} \frac{r \sin\theta}{1 - r\cos\theta}$ is the angle $s' \in P_q$ subtends at the centre of the circle circumscribing P_q.

<u>Proof</u>. The simple transformation of the function $\sin\theta$ in $2 \cdot \sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2} = 2 \cdot \tan\frac{\theta}{2} \cdot \cos^2\frac{\theta}{2}$ allows us to establish a lower bound for the sum $2L_\tau$ as a function of the number of the sides of P_1. Using the inequality of Jensen for the convex function $\tan\frac{\theta}{2}$, we obtain (since P_1, in our case, has 3 sides (which implies $r \geq \frac{1}{2\cos\pi/3} = 1$)):

$$2L_\tau \geq \sum_{s \in P_1} G(s) = 4 \sum_{s \in P_1} \tan\frac{\theta}{2} \cdot \frac{G(s)}{4\tan\frac{\theta}{2}} \geq 12 \tan\frac{\pi}{6} \left\{ \min_{s \in P_1} \frac{G(s)}{4\tan\frac{\theta}{2}} \right\}$$

with equality only if the number of the polygons of the maximal set is four and P_1 is a regular triangle.

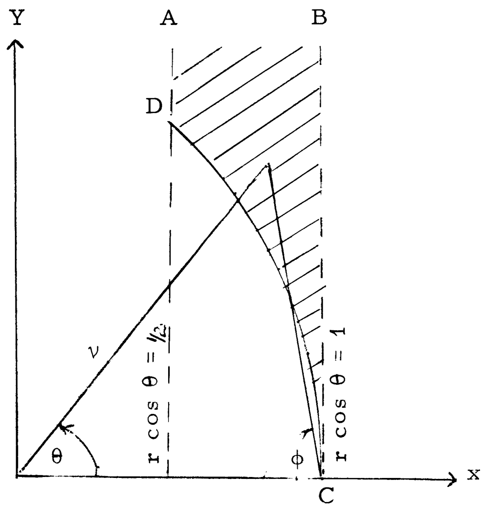
Now:

$$\frac{G(s)}{4\ \tan\frac{\theta}{2}} \geq \begin{cases} 3r\cdot\cos^2\frac{\theta}{2} \geq \frac{3(1+\cos\theta)}{2\cos\theta} > 3 & \text{for } d = r\ \cos\theta \geq 1. \\ r\cdot\cos^2\frac{\theta}{2}\ (1+\pi/\phi) = f(r,\theta) & \text{for } 1/2 \leq d < 1. \end{cases}$$

$$\text{with } \phi = \tan^{-1}(r\ \sin\theta/ 1 - r\cos\theta)\ .$$

We will show now that

$$f(r, \theta) \geq f(1, \frac{\pi}{3}) = 3 \quad \text{for} \quad 1 \leq r < (\cos\theta)^{-1},\ 0 < \theta < \frac{\pi}{2}:$$



Since $f(r, \theta)$ is continuous and has partial derivatives everywhere in its domain of definition shown as the shaded area in the figure, one of the following cases occurs:

1. $f(r, \theta)$ assumes its minimum at a point on AD or at an interior point of DC (i.e. $r = 1$).

2. $f(r, \theta)$ assumes its minimum at an interior point with $\frac{\partial f}{\partial f} = \frac{\partial f}{\partial \theta} = 0$.

3. A sequence of points (r_ν, θ_ν) can be found, approaching a boundary point on BC, such that $f(r_\nu, \theta_\nu)$ is a monotone decreasing sequence and $\lim_{\nu\to\infty} f(r_\nu, \theta_\nu)$ is a strict lower bound for the function in the domain.

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Case 1: Along AD, $r \cos\theta = 1/2$ and $\phi = \theta \geq \pi/3$, so that

$$f(r, \theta) = f(\theta) = \frac{1}{4}\left(\frac{1}{\cos\theta} + 1\right)\left(1 + \frac{\pi}{\theta}\right).$$

Since $\theta \geq \frac{\pi}{3}$, we have:

$$\frac{df(\theta)}{d\theta} = \frac{1}{4}\left[\frac{1}{\cos\theta}\left(\tan\theta + \pi\frac{\tan\theta}{\theta} - \frac{\pi}{\theta^2}\right) - \frac{\pi}{\theta^2}\right]$$

$$> \frac{1}{2}\left[1.73 + 3.14 - 4.3\right] > 0,$$

i.e., along AD, $f(r, \theta) \geq f(1, \frac{\pi}{3}) = 3$.

Along DC (i.e., $r = 1$) with $\frac{\pi}{3} < \phi < \frac{\pi}{2}$, one has

$\theta = \pi - 2\phi$, $0 < \theta < \frac{\pi}{3}$, and therefore

$$f(r, \theta) = f(\phi) = \sin^2\phi\,(1 + \pi/\phi).$$

Further: $\frac{df(\phi)}{d\phi} = \sin\phi\left[2\cos\phi\left(1 + \frac{\pi}{\phi}\right) - \frac{\pi \sin\phi}{\phi^2}\right]$.

Since $f(\frac{\pi}{3}) = f(\frac{\pi}{2}) = 3$ and $\frac{df(\phi)}{d\phi} > 0$ for $\phi = \frac{\pi}{3}$, we have

$f(r, \theta) > 3$ along DC with $0 < \theta < \frac{\pi}{3}$.

Case 2:

$$\frac{\partial f}{\partial r} = \cos^2\frac{\theta}{2}\,\frac{\partial\left[r\,(1 + \pi/\phi)\right]}{\partial r}$$

$$= \cos^2\frac{\theta}{2}\left[\left(1 + \frac{\pi}{\phi}\right) - \frac{\pi r}{\phi^2}\cdot\frac{\sin\theta}{1 - 2r\cos\theta + r^2}\right],$$

$$\frac{\partial f}{\partial\theta} = \frac{r}{2}\left(1 + \frac{\pi}{\phi}\right)(-\sin\theta) - r\cdot\cos^2\frac{\theta}{2}\cdot\frac{\pi}{\phi^2}\cdot\frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2}$$

$\frac{\partial f}{\partial r} = 0$ implies $1 + \frac{\pi}{\phi} = \frac{\pi r}{\phi^2}\cdot\frac{\sin\theta}{1 - 2r\cos\theta + r^2}$.

Together with $\frac{\partial f}{\partial\theta} = 0$ one obtains:

$\frac{-\sin^2\theta}{2} = \cos^2\frac{\theta}{2}(\cos\theta - r)$ which implies $r = 1$.

But along $r = 1$ with $0 < \theta < \frac{\pi}{3}$ we have $f(r, \theta) > 3$.

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Case 3: Along BC, $r \cdot \cos\theta = 1$, and $\phi = \frac{\pi}{2}$, so that

$$f(r, \theta) = 3r \cdot \cos^2\frac{\theta}{2} > 3 \quad \text{for} \quad \theta > 0 .$$

However, $\lim_{\substack{\nu \to 1 \\ \theta \to 0}} (3r \cdot \cos^2\frac{\theta}{2}) = 3.$

Thus $f(r, \theta) \geq f(1, \frac{\pi}{3}) = 3$ in ABCD, and therefore

$$2L_T \geq 12\sqrt{3} \quad \text{in the case} \quad \frac{2A^*(L_T)}{L_T} = 1$$

with equality if and only if the number of the polygons of the maximal set is four and the four polygons are congruent regular triangles.

In the general case, since $\frac{2A^*(L_T)}{L_T}$ varies directly with L_T,

$$L_T^2 \geq 12\sqrt{3} \cdot A^*(L_T)$$

with equality only for the set of polygons consisting of the faces of a regular tetrahedron. This completes the proof of the theorem.

REFERENCES

1. H. T. Croft, Review Article No. 879, Math. Reviews 35 (1968) 169.

2. O. Aberth, An isoperimetric inequality. Proc. London Math. Soc. (3) 13 (1963) 322-336.

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