

A LIMIT THEOREM FOR DISCRETE GALTON–WATSON BRANCHING PROCESSES WITH IMMIGRATION

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Abstract

We provide a simple set of sufficient conditions for the weak convergence of discrete-time, discrete-state Galton–Watson branching processes with immigration to continuous-time, continuous-state branching processes with immigration.

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1. Introduction

Let (g, h) be a pair of probability generating functions. By a *discrete-time, discrete-state Galton–Watson branching process with immigration* (DBI process) corresponding to (g, h) we mean a discrete-time Markov chain $\{y(n), n = 0, 1, \dots\}$ with state space $\mathbb{N} := \{0, 1, \dots\}$ and one-step transition matrix $P(i, j)$ defined by

$$\sum_{j=0}^{\infty} P(i, j)z^j = g(z)^i h(z), \quad i = 0, 1, \dots, \quad 0 \leq z \leq 1. \tag{1.1}$$

The intuitive meaning of the process is clear from (1.1). In particular, if $h(z) \equiv 1$, we simply call $\{y(n), n = 0, 1, \dots\}$ a *discrete-time, discrete-state Galton–Watson branching process* (DB process).

Kawazu and Watanabe (1971) systematically studied the limit theorems of DBI processes and characterized the class of the limit processes completely as *continuous-time, continuous-state branching processes with immigration* (CBI processes). Let us consider a special class of the CBI processes introduced in Kawazu and Watanabe (1971). Suppose that R is a function on $[0, \infty)$ defined by

$$R(\lambda) = \beta\lambda - \alpha\lambda^2 - \int_0^{\infty} \left(e^{-\lambda u} - 1 + \frac{\lambda u}{1 + u^2} \right) \mu(du),$$

where $\beta \in \mathbb{R}$ and $\alpha \geq 0$ are constants and $\min\{1, u^2\}\mu(du)$ is a finite measure on $(0, \infty)$, and that F is a function on $[0, \infty)$ defined by

$$F(\lambda) = b\lambda + \int_0^{\infty} (1 - e^{-\lambda u})m(du), \tag{1.2}$$

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where $b \geq 0$ is a constant and $\min\{1, u\}m(du)$ is a finite measure on $(0, \infty)$. A Markov process $\{y(t), t \geq 0\}$ with state space $\mathbb{R}_+ := [0, \infty)$ is called a CBI process if it has transition semigroup $(P_t)_{t \geq 0}$ given by

$$\int_0^\infty e^{-\lambda y} P_t(x, dy) = \exp\left\{-x\psi_t(\lambda) - \int_0^t F(\psi_s(\lambda)) ds\right\}, \quad \lambda \geq 0, \tag{1.3}$$

where $\psi_t(\lambda)$ is the unique solution of

$$\frac{d\psi_t}{dt}(\lambda) = R(\psi_t(\lambda)), \quad \psi_0(\lambda) = \lambda. \tag{1.4}$$

Clearly, the transition semigroup $(P_t)_{t \geq 0}$ defined by (1.3) is stochastically continuous. In particular, if $F(\lambda) \equiv 0$, then we simply call $\{y(t), t \geq 0\}$ a *continuous-time, continuous-state branching process* (CB process).

A CBI process is said to be *conservative* if it does not explode, that is, $P_x\{y(t) < \infty\} = 1$ for every $t \geq 0$ and $x \in \mathbb{R}_+$, where P_x denotes the conditional law given that $y(0) = x$. By Kawazu and Watanabe (1971, Theorem 1.2), the process is conservative if and only if

$$\int_{0+} R^*(\lambda)^{-1} d\lambda = \infty,$$

where $R^*(\lambda) = \max\{0, R(\lambda)\}$. (This is a correction to Equation (1.21) of Kawazu and Watanabe (1971).)

Let $\{b_k\}$ and $\{c_k\}$ be sequences of positive numbers such that $b_k \rightarrow \infty$ and $c_k \rightarrow \infty$ as $k \rightarrow \infty$. Let $\{y_k(n), n \geq 0\}$ be a sequence of DBI processes corresponding to the parameters $\{(g_k, h_k)\}$ and assume that $y_k(0) = c_k$. Suppose that, for all $t \geq 0$ and $\lambda \geq 0$, the limits

$$\lim_{k \rightarrow \infty} g_k^{[kt]} (e^{-\lambda/b_k})^{c_k} =: \phi_1(t, \lambda) \quad \text{and} \quad \lim_{k \rightarrow \infty} \prod_{j=0}^{[kt]-1} h_k(g_k^j (e^{-\lambda/b_k})) =: \phi_2(t, \lambda) \tag{1.5}$$

exist and the convergence is locally uniform in $\lambda \geq 0$ for each fixed $t \geq 0$, where g_k^j denotes the j -order composition of g_k and $[kt]$ denotes the integer part of kt .

Theorem 1.1. (Kawazu and Watanabe (1971, Theorem 2.1).) *Suppose that (1.5) holds and that $\phi_1(t, \lambda) < 1$, for some $t > 0$ and $\lambda > 0$. Then $\{y_k([kt])/b_k, t \geq 0\}$ converges in finite-dimensional distributions to a stochastically continuous, conservative CBI process $\{y(t), t \geq 0\}$ with transition semigroup given by (1.3).*

Based on this theorem, Kawazu and Watanabe (1971) showed that, given each stochastically continuous, conservative CBI process $\{y(t), t \geq 0\}$, there is a sequence of positive numbers $\{b_k\}$, where $b_k \rightarrow \infty$, and a sequence of DBI processes $\{y_k(n), n \geq 0\}$ such that $\{y_k([kt])/b_k, t \geq 0\}$ converges in finite-dimensional distributions to $\{y(t), t \geq 0\}$. Their results have become the basis of many studies of branching processes with immigration; see, for example, Pitman and Yor (1982) and Shiga and Watanabe (1973). On the other hand, since (1.5) involves complicated compositions of the probability generating functions, it is sometimes not so easy to verify. In view of the characterizations (1.1), (1.3), and (1.4) of the two classes of process, we naturally expect some simple sufficient conditions for the convergence of the DBI processes to the CBI processes given in terms of the parameters (g, h) and (R, F) . The purpose of this note is to provide a set of conditions of this type. For the convenience of proof, we

shall discuss the convergence of $\{y_k([\gamma_k t])/k, t \geq 0\}$ for some sequence of positive numbers $\{\gamma_k\}$ with $\gamma_k \rightarrow \infty$, which is slightly different from the scaling of Kawazu and Watanabe (1971). Instead of the convergence of finite-dimensional distributions, we shall consider the weak convergence on the space of càdlàg functions $D([0, \infty), \mathbb{R}_+)$.

2. The limit theorem

In this section, we prove a limit theorem for DBI processes on the space $D([0, \infty), \mathbb{R}_+)$. Let F be defined by (1.2). For simplicity we assume that the function R is given by

$$R(\lambda) = \beta\lambda - \alpha\lambda^2 - \int_0^\infty (e^{-\lambda u} - 1 + \lambda u)\mu(du), \quad \lambda \geq 0, \tag{2.1}$$

where $\beta \in \mathbb{R}$ and $\alpha \geq 0$ are constants and $\min\{u, u^2\}\mu(du)$ is a finite measure on $(0, \infty)$. Suppose that $\{y(t), t \geq 0\}$ is a CBI process corresponding to (R, F) . Let $\{y_k(n), n \geq 0\}$ be a sequence of DBI processes corresponding to the parameters $\{(g_k, h_k)\}$ and let $\{\gamma_k\}$ be a sequence of positive numbers. For $0 \leq \lambda \leq k$, set

$$F_k(\lambda) = \gamma_k \left[1 - h_k \left(1 - \frac{\lambda}{k} \right) \right] \tag{2.2}$$

and

$$R_k(\lambda) = k\gamma_k \left[\left(1 - \frac{\lambda}{k} \right) - g_k \left(1 - \frac{\lambda}{k} \right) \right]. \tag{2.3}$$

Let us consider the following set of conditions.

- (A) As $k \rightarrow \infty$, we have $\gamma_k \rightarrow \infty$ and $\gamma_k/k \rightarrow \gamma_0$, for some $\gamma_0 \geq 0$.
- (B) As $k \rightarrow \infty$, the sequence $\{F_k\}$ defined by (2.2) converges to a continuous function.
- (C) The sequence $\{R_k\}$ defined by (2.3) is uniformly Lipschitz on each bounded interval, and converges to a continuous function as $k \rightarrow \infty$.

We remark that conditions (B) and (C) parallel the sufficient conditions for the convergence of continuous-time, discrete-state branching processes with immigration; see, for example, Li (1992) for discussion in the setting of measure-valued processes. Based the results of Li (1991), the following lemma can be proved by modifying the arguments of the proofs of Li (1992, Lemmas 3.4 and 4.1).

Lemma 2.1. (i) Under conditions (B) and (C), the limit functions F and R of $\{F_k\}$ and $\{R_k\}$ have representations (1.2) and (2.1), respectively.

(ii) For any (F, R) given by (1.2) and (2.1), there are sequences $\{\gamma_k\}$ and $\{(g_k, h_k)\}$ as above such that conditions (A), (B), and (C) hold with $F_k \rightarrow F$ and $R_k \rightarrow R$.

For $\lambda \geq 0$ we set $S_k(\lambda) = k\gamma_k[(1 - \lambda/k) - g_k(e^{-\lambda/k})]$.

Lemma 2.2. Under conditions (A) and (C), let $R = \lim_{k \rightarrow \infty} R_k$. Then we have

$$\lim_{k \rightarrow \infty} S_k(\lambda) = R(\lambda) - \frac{\gamma_0 \lambda^2}{2} \quad \text{and} \quad \lim_{k \rightarrow \infty} \gamma_k [1 - g_k(e^{-\lambda/k})] = \gamma_0 \lambda \tag{2.4}$$

uniformly on each bounded interval.

Proof. By the mean value theorem, we obtain

$$S_k(\lambda) = R_k(\lambda) - k\gamma_k g'_k(\eta_k) \left(e^{-\lambda/k} - 1 + \frac{\lambda}{k} \right), \tag{2.5}$$

where $1 - \lambda/k < \eta_k < e^{-\lambda/k}$ and g'_k denotes the derivative of g_k . Under condition (C), the sequence $R'_k(\lambda) = \gamma_k [g'_k(1 - \lambda/k) - 1]$ is uniformly bounded on each bounded interval $\lambda \in [0, l]$, $l \geq 0$. Thus, $g'_k(1 - \lambda/k) \rightarrow 1$ uniformly on each bounded interval. In particular, we have $g'_k(\eta_k) \rightarrow 1$, and the first equality in (2.4) follows from condition (A) and (2.5). The second equality in (2.4) follows by a similar argument.

Theorem 2.1. *Suppose that conditions (A), (B), and (C) hold with $F = \lim_{k \rightarrow \infty} F_k$ and $R = \lim_{k \rightarrow \infty} R_k$. If $y_k(0)/k$ converges in distribution to $y(0)$, then $\{y_k([\gamma_k t])/k, t \geq 0\}$ converges in distribution on $D([0, \infty), \mathbb{R}_+)$ to the CBI process $\{y(t), t \geq 0\}$ corresponding to (R, F) with initial value $y(0)$.*

Proof. Let $(P_t)_{t \geq 0}$ denote the transition semigroup of the CBI process corresponding to (R, F) . For $\lambda > 0$ and $x \geq 0$, set $e_\lambda(x) = e^{-\lambda x}$. We denote by D_1 the linear hull of $\{e_\lambda, \lambda > 0\}$. Then D_1 is an algebra which strongly separates the points of \mathbb{R}_+ . Let $C_0(\mathbb{R}_+)$ be the space of continuous functions on \mathbb{R}_+ vanishing at infinity. By the Stone–Weierstrass theorem, D_1 is dense in $C_0(\mathbb{R}_+)$ for the supremum norm; see, for example, Hewitt and Stromberg (1965, pp. 98–99). For $\lambda > 0$ we set

$$Ae_\lambda(x) = -e^{-\lambda x} [xR(\lambda) + F(\lambda)], \quad x \in \mathbb{R}_+, \tag{2.6}$$

and extend the definition of A to D_1 by linearity. Then $A := \{(f, Af), f \in D_1\}$ is a linear subspace of $C_0(\mathbb{R}_+) \times C_0(\mathbb{R}_+)$. Since D_1 is invariant under $(P_t)_{t \geq 0}$, it is a core of A ; see, for example, Ethier and Kurtz (1986, p. 17). In other words, the semigroup $(P_t)_{t \geq 0}$ is generated by the closure of A ; see, for example, Ethier and Kurtz (1986, p. 15 and p. 17). Note that $\{y_k(n)/k, n \geq 0\}$ is a Markov chain with state space $E_k := \{0, 1/k, 2/k, \dots\}$ and one-step transition probability $Q_k(x, dy)$ determined by

$$\int_{E_k} e^{-\lambda y} Q_k(x, dy) = g_k(e^{-\lambda/k})^{kx} h_k(e^{-\lambda/k}).$$

Then the (discrete) generator A_k of $\{y_k([\gamma_k t])/k, t \geq 0\}$ is given by

$$\begin{aligned} A_k e_\lambda(x) &= \gamma_k [g_k(e^{-\lambda/k})^{kx} h_k(e^{-\lambda/k}) - e^{-\lambda x}] \\ &= \gamma_k [\exp\{xk\alpha_k(\lambda)(g_k(e^{-\lambda/k}) - 1)\} \exp\{\beta_k(\lambda)(h_k(e^{-\lambda/k}) - 1)\} - e^{-\lambda x}], \end{aligned}$$

where

$$\alpha_k(\lambda) = (g_k(e^{-\lambda/k}) - 1)^{-1} \log g_k(e^{-\lambda/k})$$

and $\beta_k(\lambda)$ is defined by the same formula with g_k replaced by h_k . Under conditions (A), (B), and (C), it is easy to show that

$$\lim_{k \rightarrow \infty} (g_k(e^{-\lambda/k}) - 1) = \lim_{k \rightarrow \infty} (h_k(e^{-\lambda/k}) - 1) = 0$$

and

$$\lim_{k \rightarrow \infty} \alpha_k(\lambda) = \lim_{k \rightarrow \infty} \beta_k(\lambda) = 1.$$

Then we have

$$A_k e_\lambda(x) = -e^{-\lambda x} [x\alpha_k(\lambda)S_k(\lambda) + x\gamma_k(\alpha_k(\lambda) - 1)\lambda + H_k(\lambda)] + o(1), \tag{2.7}$$

where

$$H_k(\lambda) = \gamma_k \beta_k(\lambda)(1 - h_k(e^{-\lambda/k})).$$

By elementary calculations we find that

$$\alpha_k(\lambda) = 1 + \frac{1}{2}(1 - g_k(e^{-\lambda/k})) + o(1 - g_k(e^{-\lambda/k})),$$

and so $\lim_{k \rightarrow \infty} \gamma_k(\alpha_k(\lambda) - 1) = \gamma_0 \lambda / 2$ by Lemma 2.2. It follows that

$$\lim_{k \rightarrow \infty} [\alpha_k(\lambda)S_k(\lambda) + \gamma_k(\alpha_k(\lambda) - 1)\lambda] = R(\lambda).$$

By the argument of the proof of Lemma 2.2, we can show that

$$\lim_{k \rightarrow \infty} H_k(\lambda) = \lim_{k \rightarrow \infty} F_k(\lambda) = F(\lambda).$$

In view of (2.6) and (2.7), we obtain

$$\lim_{k \rightarrow \infty} \sup_{x \in E_k} |A_k e_\lambda(x) - A e_\lambda(x)| = 0,$$

for each $\lambda > 0$. This clearly implies that

$$\lim_{k \rightarrow \infty} \sup_{x \in E_k} |A_k f(x) - A f(x)| = 0,$$

for each $f \in D_1$. From Ethier and Kurtz (1986, p. 226 and pp. 233–234) we find that $\{y_k([\gamma_k t])/k, t \geq 0\}$ converges in distribution on $D([0, \infty), \mathbb{R}_+)$ to the CBI process corresponding to (R, F) .

By Lemma 2.1 and Theorem 2.1, for any functions (R, F) given by (1.2) and (2.1), there is a sequence of positive numbers $\{\gamma_k\}$ and a sequence of DBI processes $\{y_k(n), n \geq 0\}$ such that $\{y_k([\gamma_k t])/k, t \geq 0\}$ converges in distribution on $D([0, \infty), \mathbb{R}_+)$ to the CBI process corresponding to (R, F) .

3. Generalized Ray–Knight theorems

As an example of the applications of their limit theorems, Kawazu and Watanabe (1971) re-proved the Ray–Knight theorems of diffusion characterizations of the Brownian local time. In this section, we generalize the results to the case of a Brownian motion with drift. We refer the reader to Le Gall and Le Jan (1998) for another adequate formulation of the Ray–Knight theorems for general Lévy processes.

Let $A = \alpha d^2/dx^2 + \beta d/dx$ for given constants $\alpha > 0$ and $\beta \in \mathbb{R}$. Then A generates a one-dimensional Brownian motion with drift $(X_t, \mathcal{F}_t, P_x)$. The local time of $\{X_t, t \geq 0\}$ is a continuous two-parameter process $\{l(t, x), t \geq 0, x \in \mathbb{R}\}$ such that the following property holds almost surely:

$$2 \int_B l(s, x) dx = \int_0^t \mathbf{1}_B(X_s) ds, \quad B \in \mathcal{B}(\mathbb{R}).$$

Here $\mathcal{B}(\mathbb{R})$ denotes the Borel σ -algebra of \mathbb{R} and $\mathbf{1}_B$ is the indicator function of B . For a fixed $a \geq 0$, let

$$l^{-1}(u, a) = \inf\{t \geq 0: l(t, a) = u\}.$$

Theorem 3.1. *The process*

$$\xi_u(t) = l(l^{-1}(u, a), a + t), \quad t \geq 0,$$

is a diffusion generated by

$$x \frac{d^2}{dx^2} + \frac{\beta}{\alpha} x \frac{d}{dx}.$$

Proof. We follow the ideas of Kawazu and Watanabe (1971, Example 2.2). For $c \in \mathbb{R}$ let $\sigma_c = \inf\{t \geq 0: X_t = c\}$. Let $\delta > 0$ and let $u_\delta(x) = P_x\{\sigma_\delta < \sigma_{-\delta}\} = 1 - P_x\{\sigma_\delta > \sigma_{-\delta}\}$. Then $u_\delta(\cdot)$ satisfies

$$\alpha \frac{d^2}{dx^2} u_\delta(x) + \beta \frac{d}{dx} u_\delta(x) = 0, \quad |x| \leq \delta,$$

with $u_\delta(\delta) = 1$ and $u_\delta(-\delta) = 0$. Solving this boundary value problem, we find that

$$u_\delta(x) = \frac{\exp\{\beta\delta/\alpha\} - \exp\{-\beta x/\alpha\}}{\exp\{\beta\delta/\alpha\} - \exp\{-\beta\delta/\alpha\}}.$$

By a δ -downcrossing at $x \in \mathbb{R}$ before time $T > 0$ we mean an interval $[u, v] \subset [0, T]$ such that $X_u = x + \delta$, $X_v = x$, and $x < X_t < x + \delta$ for all $u < t < v$. Let η_δ denote the number of δ -downcrossings at 0 before time $\sigma_{-\delta}$. By the property of independent increments of the Brownian motion with drift, we have

$$E_0[z^{\eta_\delta}] = \sum_{i=0}^{\infty} (1 - p)(pz)^i = \frac{q}{1 - pz},$$

where $p = u_\delta(0)$, $q = 1 - p$, and E_0 denotes the expectation under P_0 . Let $x_i = a + i/k$ for $i \geq 0$ and $k \geq 1$ and let $Z_k(i)$ denote the number of $(1/k)$ -downcrossings at x_i before time $l^{-1}(u, a)$. It is easy to see that $Z_k(i + 1)$ is the sum of $Z_k(i)$ independent copies of $\eta_{1/k}$. Thus, $\{Z_k(i), i = 0, 1, \dots\}$ is a DB process corresponding to the generating function

$$g_k(z) = \frac{q_k}{1 - p_k z},$$

where $p_k = u_{1/k}(0)$ and $q_k = 1 - p_k$. By a standard result for local times of diffusion processes, we obtain

$$\lim_{k \rightarrow 0} \frac{Z_{1/k}([kt])}{k} = l(l^{-1}(u, a), a + t) = \xi_u(t);$$

see Itô and McKean (1965, p. 48 and p. 222). Then Theorem 2.1 implies that the limit $\{\xi_u(t), t \geq 0\}$ is a CB process corresponding to

$$R(\lambda) = \lim_{k \rightarrow \infty} k^2 \left[\left(1 - \frac{\lambda}{k}\right) - g_k\left(1 - \frac{\lambda}{k}\right) \right] = \frac{\beta}{\alpha} \lambda - \lambda^2.$$

This proves the desired result.

Kawazu and Watanabe (1971, Theorem 2.3 and Example 2.2) proved the result of Theorem 3.1 in the special case $\beta = 0$. In that case the generating function g_k is actually independent of $k \geq 1$. In the general case, it seems difficult to check condition (1.5) for the sequence $\{g_k\}$. By arguments similar to those used above, we obtain the following theorem.

Theorem 3.2. *The process*

$$\eta_u(t) = l(l^{-1}(u, a), a - t), \quad 0 \leq t \leq a,$$

is a diffusion generated by

$$x \frac{d^2}{dx^2} + \frac{\beta}{\alpha} x \frac{d}{dx} + \frac{d}{dx}.$$

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