ZETA FUNCTIONS ON THE UNITARY SPHERE

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1. Introduction. In an earlier paper [**5**], the author defined a zeta function on the real sphere $x_1^2 + x_2^2 + \ldots + x_{k+1}^2 = 1$, whereas in the present paper it is proposed to define one on the unitary sphere $x_1\bar{x}_1 + x_2\bar{x}_2 + \ldots + x_{k+1}\bar{x}_{k+1} = 1$ where x_i's are complex numbers and $\bar{x}_i$ their complex conjugates. Following E. Cartan, harmonics on the unitary sphere are defined and then a zeta function formed just as in the case of a real sphere. The unitary sphere is seen to behave like an even-dimensional closed manifold, since results similar to the ones proved by the author and A. Pleijel [6] for closed manifolds (of even dimensions) are observed here also.

The zeta function on the real sphere may be viewed as a zeta function associated with the orthogonal group $\mathfrak{O}(n)$ while the one on the unitary sphere as that associated with the unitary group $\mathfrak{U}(n)$. It is clear that one could give a suitable definition of a zeta function on any compact group [**8**]. If the group acts transitively on a closed manifold with a metric, one could use the idea of harmonics on this manifold [**3**] and define a zeta function. One could still define harmonics on groups [**7**] by taking the group itself as the manifold on which the group may act. But in all these cases one should be able to obtain these harmonics as eigenfunctions with associated eigenvalues. That this is possible was shown by Casimir [**4**]. According to him the elements of the unitary representations of a compact group are the eigenfunctions of a second order differential operator, now known as the Casimir operator. If the eigenvalues of the operator are λ_n with the eigenfunctions ϕ_n, then

$$\sum \frac{\phi_n(p)\overline{\phi}_n(q)}{|\lambda_n|^s}$$

will be defined as a zeta function (if the spectrum of the operator is not discrete one will have to use suitable modifications of the definition). Since this series will converge for sufficiently large values of $R(h)$, one may study the analytic continuation of the function so defined.[1] While the discussion of the properties of such functions on abstract groups remains an open question, special cases lend themselves to easy treatment.

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[1]If what Dr. I. Singer has communicated to me is true—that the Casimir operator for a compact Lie group is the Laplace-Beltrami operator—the results that Pleijel and I have proved carry over easily to this case. As pointed out by Hermann Weyl [**7**] the functional equation, if any, will have to be obtained.

2. Harmonics on the unitary sphere. We enumerate a few properties of harmonics on the unitary sphere $x_1\bar{x}_1 + x_2\bar{x}_2 + \ldots + x_{k+1}\bar{x}_{k+1} = 1$. A full account of them is to be found in Cartan's book on projective geometry [**2**, chap. V]. Let $V = V(x_1,x_2, \ldots, x_{k+1},\bar{x}_1, \ldots, \bar{x}_{k+1})$ be an integral polynomial in the $2k + 2$ variables $x_i,\bar{x}_i$, $(i = 1, \ldots, k + 1)$, homogeneous and of degree n in the variables x_i and also homogeneous and of degree n in the variables $\bar{x}_i$. This polynomial is said to be a harmonic of order n, if it satisfies

$$\Delta V \equiv \sum_{i=1}^{k+1} \frac{\partial^2 V}{\partial x_i \partial \bar{x}_i} = 0.$$

There are only a finite number of linearly independent harmonics of order n, their number k_n being given by

$$k_n = k(k + 2n) \left(\frac{(k + 1)(k + 2) \ldots (k + n - 1)}{n!}\right)^2.$$

It is a characteristic property of these polynomials, that any transformation effected on a point of the sphere (leading to another point of the sphere) changes a harmonic of order n to a harmonic of order n which is a linear combination of a basic set of k_n linearly independent harmonics. If we choose the bases to be normal and orthogonal on the sphere and if they are denoted by $U_1^n(M)$, $U_2^n(M)$, $\ldots$, $U_{k_n}^n(M)$, where M is a point on the sphere, then the expression

$$U_1^n(M)\bar{U}_1^n(M') + \ldots + U_{k_n}^n(M)\bar{U}_{k_n}^n(M'),$$

where M' is another point on the sphere, is invariant for the group of transformations on the unitary sphere, viz., the unitary group. Further, the expression is a function of the geodesic distance between the points. In fact, if r is the geodesic distance, it is a polynomial in $\cos 2r - L_n(\cos 2r)$, say, satisfying the differential equation

$$(1 - z^2)L'' - ((k + 1)z + k - 1)L' + n(n + k)L = 0, \qquad z = \cos 2r.$$

We at once identify a polynomial solution of this equation as the Jacobi polynomial $P_n^{k-1,0}(z)$

$$(1 - z)^{k-1}P_n^{k-1,0}(z) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dz^n}\{(1 - z)^{n+k-1}(1 + z)^n\}.$$

More precisely,

$$\sum_{r=1}^{k_n} U_r^n(M)\bar{U}_r^n(M') = \frac{(2k + n)}{kV} \binom{n+k-1}{n} P_n^{k-1,0}(\cos 2r)$$

$$= \frac{(2k + n)}{kV} P_n^{k-1,0}(1)P_n^{k-1,0}(z),$$

where V is the volume of the sphere and $z = \cos 2r$.

3. Zeta function on the unitary sphere. Inasmuch as we can associate the eigenvalue $n(n+k)$ with any harmonic of order n, we define a zeta function as the analytic continuation of the function represented by the Dirichlet's series

$$\sum_{n=1}^{\infty} \frac{1}{n^s(n+k)^s} \sum_{r=1}^{k_n} U_r^n(M) U_r^n(M') \tag{1}$$

$$= \frac{1}{kV} \sum_{n=1}^{\infty} \frac{(2n+k)P_n^{k-1,0}(1)P_n^{k-1,0}(z)}{n^s(n+k)^s}, \qquad s = \sigma + i\tau$$

which has a half-plane of convergence, viz., $R(s) > k$.

We show that *this function is an entire function of s with simple zeros for negative integral values of s provided $M \neq M'$ and a meromorphic function of s with simple poles at $s = 1, 2, \ldots, k$ if $M = M'$.*

The proof proceeds along the same lines as in our earlier paper [**5**] and we briefly indicate it here. We need the following [**1**]

LEMMA.

$$\sum_{n=0}^{\infty} (2n+k)P_n^{k-1,0}(1)P_n^{k-1,0}(\cos 2r)\, t^n$$

$$= \frac{k(1-t)}{(1+t)^{k+1}} F\left(\frac{k+1}{2}; \frac{k+2}{2}; 1; \frac{\cos^2 r}{\rho^2}\right), \qquad \rho = \tfrac{1}{2}(t^{\frac{1}{2}} + t^{-\frac{1}{2}})$$

or

$$\sum_{n=0}^{\infty} (2n+k)P_n^{k-1,0}(1)P_n^{k-1,0}(\cos 2r)e^{-nt} \tag{2}$$

$$= \frac{k}{2^k} e^{\frac{1}{2}kt} \frac{\sinh \frac{1}{2}t}{\cosh \frac{1}{2}t} F\left(\frac{k+1}{2}, \frac{k+2}{2}; 1; \frac{\cos^r r}{\cosh^2 \frac{1}{2}t}\right).$$

From the above lemma we obtain an integral representation of the Dirichlet's series (1) as in [**5**, (15)].

$$\sum_{n=1}^{\infty} \frac{(2n+k)P_n^{k-1,0}(1)P_n^{k-1,0}(\cos 2r)}{n^s(n+k)^s} \tag{3}$$

$$= \left(\frac{4}{k}\right)^{s-\frac{1}{2}} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(2s)} \int_0^{\infty} \left\{\frac{k}{2^k} \frac{\sinh \frac{1}{2}t}{(\cosh \frac{1}{2}t)^{k+1}} F\left(\frac{k+1}{2}, \frac{k+2}{2}; 1; \frac{\cos^2 r}{\cosh^2 \frac{1}{2}t}\right) - e^{-\frac{1}{2}kt}\right\} t^{s-\frac{1}{2}} I_{s-\frac{1}{2}}(\tfrac{1}{2}kt)\,dt.$$

If $R(s) > k$, the series on the left converges absolutely and is represented by the integral on the right which converges absolutely if $R(s) > 0$. So the function represented by the series on the left can be continued up to $R(s) = 0$. We shall show, however, that it can be continued over the whole plane adopting the usual procedure of replacing the integral on the right by a contour integral.

In the complex t plane make a cut along the positive side of the real axis from the origin and take a contour C from ∞ in the upper half of the plane, going round the origin in the anti-clockwise direction and then going back to ∞ in the lower half of the plane (poles of the integrand being in the exterior of C). Now consider the integral along C, viz.,

$$\text{(4)}\quad \begin{gathered}\frac{1}{2\pi i}\int_C \left\{\frac{k}{2^k}\frac{\sinh \frac{1}{2}t}{(\cosh \frac{1}{2}t)^{k+1}} F\left(\frac{k+1}{2}, \frac{k+2}{2}, 1; \frac{\cos^2 r}{\cosh^2 \frac{1}{2}t}\right) - e^{-\frac{1}{2}kt}\right\} \\ (-t)^{s-\frac{1}{2}} I_{s-\frac{1}{2}}(-\tfrac{1}{2}kt)dt = e^{-\pi i(2s-1)}\frac{1}{2\pi i}\int_\infty^0 + \oint + e^{i\pi(2s-1)}\frac{1}{2\pi i}\int_0^\infty .\end{gathered}$$

The second integral on the right, taken along a circle round the origin, is zero since the integrand is regular and hence for $R(s) > 0$, we have

$$\frac{1}{2\pi i}\int_C = \frac{\sin (2s-1)\pi}{\pi}\int_0^\infty .$$

Thus

$$\text{(5)}\quad \sum_{n=1}^\infty \frac{(2n+k)P_n^{k-1,0}(1)P_n^{k-1,0}(\cos 2r)}{n^s(n+k)^s} = -\left(\frac{4}{k}\right)^{s-\frac{1}{2}}\frac{\Gamma(s+\frac{1}{2})\Gamma(1-2s)}{2\pi i}\int_C$$

for $R(s)>k$. The integral along C is as in (4). Since the contour integral is finite for all values of s, it represents a regular function of s. Thus the series on the left represents an analytic function which can be continued throughout the plane with the possible exception of the simple poles of $\Gamma(1-2s)\Gamma(s+\frac{1}{2})$, viz., half odd integral and positive integral values of s. But the integral is seen to vanish for these values of s, since the integrand is then a regular function of t. To observe that the negative integral values of s are the "trivial" zeros of the function, we have only to note that the residues of the integral are zero. We split the integral as the difference of two, viz.,

$$\int_C \frac{\sinh \frac{1}{2}t}{(\cosh \frac{1}{2}t)^{k+1}} F\left(\frac{k+1}{2}, \frac{k+2}{2}, 1; \frac{\cos^2 r}{\cosh^2 \frac{1}{2}t}\right)(-t)^{s-\frac{1}{2}} I_{s-\frac{1}{2}}(-\tfrac{1}{2}kt)\,dt$$

and

$$\int_C e^{-\frac{1}{2}kt}(-t)^{s-\frac{1}{2}} I_{s-\frac{1}{2}}(-\tfrac{1}{2}kt)\,dt.$$

That the residues are zero in the first case is proved in view of the fact that the integrand is an even function of t for real t, when s is a negative integer. The residues of the second integral are easily calculated to be zero, when s is a negative integer, using the familiar formulas for $e^{-\frac{1}{2}kt}t^{-n-\frac{1}{2}}I_{-n-\frac{1}{2}}(\frac{1}{2}kt)$.

When the two points M and M' coincide, i.e. when $r = 0$, we obtain

$$\sum_{n=1}^\infty \frac{(2n+k)\{P_n^{k-1,0}(1)\}^2}{n^s(n+k)^s} = \frac{\Gamma(s+\frac{1}{2})\Gamma(1-2s)}{2\pi i}$$

$$\cdot\int_C \left\{\frac{k}{2^k}\frac{\sinh \frac{1}{2}t}{(\cosh \frac{1}{2}t)^{k+1}} F\left(\frac{k+1}{2}, \frac{k+2}{2}, 1; \frac{1}{\cosh^2 \frac{1}{2}t}\right) - e^{-\frac{1}{2}kt}\right\}(-t)^{s-\frac{1}{2}} I_{s-\frac{1}{2}}(-\tfrac{1}{2}kt)dt.$$

As in the previous case we observe that analytic continuation over the whole plane is possible with the possible exception of the simple poles of $\Gamma(s + \frac{1}{2}) \Gamma(1 - 2s)$, viz., half odd integral and positive integral values of s. That part of the integral containing $e^{-\frac{1}{2}kt}$ gives no difficulty, since the integrand is regular for half integral values of s. The first part of the integrand is found to be an even function of t for real t, when s is half an odd integer, taking into account the singularity of the hypergeometric function at $t = 0$. Therefore the residues are zero. Thus half odd integral values do not contribute poles. Further, since we know that the function is regular for $R(s) > k$, we observe that the only poles are $s = 1, 2, \ldots, k$.

References

1. W. N. Bailey, *Generalized hypergeometric series*, Cambridge Tract No. 32 (1935).
2. E. Cartan, *Leçons sur la géométrie projective complexe* (Paris, 1931).
3. ———, *Sur la détermination d'un système orthogonal complet dans un espace de Riemann symétrique clos*, Rend. Circ. Mat. di Palermo, vol. 53 (1929), 217-252.
4. H. B. G. Casimir, *Rotation of a rigid body in quantum mechanics*, Leiden thesis (1931).
5. S. Minakshisundaram, *Zeta function on the sphere*, J. Ind. Math. Soc., vol. 13 (1949), 41-48.
6. S. Minakshisundaram and A. Pleijel, *Some properties of the eigenfunction of the Laplace operator on Riemann manifolds*, Can. J. Math., vol. 1 (1949), 242-286.
7. H. Weyl, *Harmonics on homogeneous manifolds*, Annals of Math., vol. 35 (1934), 485-499.
8. ———, *Ramification, old and new, of the eigenvalue problem*, Bull. Amer. Math. Soc., vol. 56 (1950), 15-139.

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