A NOTE ON INVOLUTIONS WITH A FINITE NUMBER OF FIXED POINTS

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1. Introduction. Let M be a smooth, closed, simply connected manifold of dimension greater than 5. Let T be an involution on M with a positive, finite number of fixed points. Our aim in this paper is to prove the following theorem (which is somewhat like that of Wasserman (**7**)).

THEOREM 1. *There exists an invariant Morse function f on M such that, if x and y are critical points of index r and s, respectively, with $r < s$, then $f(x) < f(y)$. Moreover, f has but one critical point of type zero.*

An interesting consequence of the above theorem is that, in order for M to have such an involution T, the integral homology groups of M must satisfy rather strict conditions; namely, the betti numbers of M bound the number of fixed points in a very specific way (Theorem 2) and if I_m (the integers modulo m) is a factor of $H_q(M)$, it is a factor an even number of times.

2. Notation and definitions. First, let us give some familiar definitions. An n-dimensional euclidean space will be denoted by

$$R^n = \{(x_1, x_2, \ldots, x_n) | x_i \in R, i = 1, 2, \ldots, n\},$$

where R is the set of real numbers. Throughout this paper, I will denote the unit interval. If $x \in R^n$, then $||x|| = (x, x)^{1/2}$, where (,) is the usual euclidean product. Furthermore, $D^n = \{x \in R^n | \, ||x|| \leqq 1\}$ and S^{n-1} is its boundary. If E is a subset of a topological space, then the symbol E° will be used to denote the interior of E. If T is an involution on a manifold M, then S is an invariant set if $S = T(S)$. If f is a smooth, real-valued function on M, then f is called invariant with respect to the involution T or just invariant if $f = fT$; also, it is a Morse function if it does not have degenerate critical points.

3. Consequences of theorem 1. Before proving Theorem 1 we derive some consequences of the theorem. If such a function f exists, we note that it readily follows that every fixed point of T is a critical point of f. Hence, let us call a fixed point of T a fixed point of type r if, as a critical point of f, it is a critical point of type r. We also note that if C is the set of critical points of type r, then T permutes the set C.

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Now let c be a critical value of f. Then for ϵ positive and sufficiently small, $f^{-1}(c - \epsilon)$ and $f^{-1}(c + \epsilon)$ are manifolds which form the boundary of the manifold V_c,

$$V_c = \{x \in M | \, c - \epsilon \leqq f(x) \leqq c + \epsilon\}.$$

Furthermore, there is only one type of spherical modification required to go from $f^{-1}(c - \epsilon)$ to $f^{-1}(c + \epsilon)$ with trace V_c. (For terminology and definitions, see **6**, § 3.) Let F be the family of orthogonal trajectories of f which begin on $f^{-1}(c - \epsilon)$. The union of all curves G of F which end at a critical point ω of f forms a cell with boundary in $f^{-1}(c - \epsilon)$. We shall say that this cell corresponds to the critical point w. Now $T(w)$ is a critical point of f, and it can be seen that, since f is an invariant Morse function, the union of all curves G' of F which end at $T(w)$ is $T(G)$; in fact, the orthogonal trajectory of f with initial condition p is carried into the orthogonal trajectory with initial condition $T(p)$ by T (cf. **7**).

From Wallace's point of view, V_c is the trace of certain spherical modifications performed on directly embedded spheres of $f^{-1}(c - \epsilon)$ (**6**, § 3). The directly embedded spheres in this case are precisely the boundaries of those cells in V_c which correspond to the critical points of f. Thus, in this context, two types of spherical modifications occur. First, a directly embedded sphere S is shrunk together with $T(S)$ if S is the boundary of a cell which corresponds to a critical point w of f which is not a fixed point of T ($T(S)$ corresponds to $T(w)$); and secondly, a directly embedded sphere S is shrunk for which $S = T(S)$ if S is the boundary of a cell which corresponds to a fixed point of T. In addition, if w is a fixed point of type $p + 1$, then we have the following commutative diagram:

(1)
$$\begin{array}{ccc} (D^{p+1}, S^p) & \xrightarrow{\;h\;} & M \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle T} \\ M & \xrightarrow{\;h\;} & M \end{array}$$

where h is an embedding map for D^{p+1} and Δ is the antipodal map.

This is so since for each fixed point of type $p + 1$ there exists an embedding ϕ of the n-ball $B(2\epsilon)$,

$$B(2\epsilon) = \{x \in R^n | \, ||x|| \leqq 2\epsilon\},$$

into an invariant neighbourhood U of w such that $T\phi(z_1, \ldots, z_n) = \phi(-z_1, -z_2, \ldots, z_n)$ by a well-known theorem of Bochner (see **5**, p. 209). Otherwise stated, the following diagram is commutative:

$$\begin{array}{ccc} B(2\epsilon) & \longrightarrow & U \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle T} \\ B(2\epsilon) & \longrightarrow & U \end{array}$$

where Δ is the antipodal map. We now define h by $h(u, 0) = \phi(\epsilon u, 0)$, where u is in R^{p+1} and 0 is the origin in R^q and $p + q + 1 = n$, $0 \leqq ||u|| \leqq 1$.

We next examine some necessary conditions on M in order that M have such an involution T. We shall show, for example, that every fixed point of type r corresponds to a free generator of $H_r(M)$. But before taking up the study directly, we need some more definitions and observations.

Let $c_0, c_1, \ldots, c_n$ be the critical values of f, where the subscripts correspond to the type number of the critical points, and let $d = \min\frac{1}{2}|c_i - c_j|$ for all $i \neq j$. Define Z_m by $Z_m = \{x \in M | f(x) \leqq c_m + d\}$. We also note that Z_0 is an n-cell. Furthermore, $H_q(Z_m, Z_{m-1}) = 0$ for $q \neq m$ and $H_m(Z_m, Z_{m-1})$ is isomorphic to a free group on k generators, where k is the number of critical points of type m, and the generators of $H_m(Z_m, Z_{m-1})$ are carried by relative cells (cf. **3**, Theorem 3.5).

Now let us examine the following sequence of groups and homomorphisms:

$$(2) \qquad \ldots \to H_m(Z_m, Z_{m-1}) \xrightarrow{\partial} H_{m-1}(Z_{m-1}, Z_{m-2}) \xrightarrow{\partial} H_{m-2}(Z_{m-2}, Z_{m-3}) \to \ldots,$$

where ∂ is the homomorphism obtained in the following way:

$$\begin{array}{ccc} H_m(Z_m, Z_{m-1}) & \xrightarrow{\partial} & H_{m-1}(Z_{m-1}, Z_{m-2}) \\ & \partial' \searrow \quad \nearrow j & \\ & H_m(Z_{m-1}, Z_0) & \end{array}$$

$\partial = j\partial'$, where the maps ∂' and j arise from the homology sequence of the triples (Z_m, Z_{m-1}, Z_0) and (Z_{m-1}, Z_{m-2}, Z_0), respectively. Furthermore, $\partial^2 = 0$. Moreover, it is a trivial calculation to show that the homology groups of (2) are isomorphic to the homology groups of M.

Now, $H_q(Z_q, Z_{q-1}) = F_q \oplus N_q$, where F_q are those elements carried by relative cells which remain invariant under the action of T, and N_q are those elements carried by relative cells which do not. Let $T_\#$ be the chain isomorphism induced by T. Note that $T_\#$ freely permutes the elements of N_q. Furthermore, let us assume that the homomorphisms ∂ have been diagonalized, that is, the matrices associated with the homomorphisms are diagonal matrices. If they are not, we may replace T by an equivariently equivalent involution S and f by a new function g with the same properties as f for which the homomorphisms ∂ are diagonal. To see this last statement let us define

$$A_k = \{x \in M | f(x) \geqq c_k + d\}, \qquad M_k = \{x \in M | f(x) = c_k - d\}$$

and

$$W_k = \{x \in M | c_k - d \leqq f(x) \leqq c_k + d\}.$$

$H_q(W_q, M_q)$ is isomorphic to $H_q(Z_q, Z_{q-1})$ by the excision and homotopy axioms. Now we wish to diagonalize the mapping ∂, where

$$H_{q+1}(Z_{q+} \quad Z_q) \xrightarrow{\partial} H_q(Z_q, Z_{q-1}).$$

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Now, perform a unimodular transformation $A = (a_{ij})$ on the generators of $H_q(Z_q, Z_{q-1})$ so that the homomorphism ∂ is diagonalized. A induces a unimodular transformation on the generators of $H_q(W_q, M_q)$, which, in turn, induces a unimodular transformation on the kernel of the inclusion map of $H_{q-1}(M_q)$ into $H_{q-1}(W_q)$, which is generated by the elements $\alpha_1, \alpha_2, \ldots, \alpha_k$ carried by those spheres which are to be shrunk by the spherical modifications (**6**, Lemma 8.2). Instead of killing the elements α_i by spherical modifications, kill the elements β_i, where $\beta_i = \sum a_{ij}\alpha_j$, $i = 1, \ldots, k$, by spherical modifications. But this is equivalent to saying that there exists a diffeomorphism ϕ of M_q into M_q that can be extended to a diffeomorphism Φ of W_q into W_q such that Φ is the identity on M_{q+1}, M_{q+1} being the "top" boundary component of W_q (cf. **6**, Lemma 7.2). Now, M may be written as $M = Z_{q-1} \cup W_q \cup A_q$, where these sets have been glued together by the appropriate mappings. Define a new smooth function g on M by

$$g(x) = \begin{cases} f(x) & \text{if } x \notin W_q, \\ f\Phi(x) & \text{if } x \in W_q, \end{cases}$$

and a new involution S by

$$S(x) = \begin{cases} T(x) & \text{if } x \notin W_q, \\ \Phi T \Phi^{-1}(x) & \text{if } x \in W_q. \end{cases}$$

It is clear that $gS = g$.

Remark. Let us recall that the theorems of (**6**, § 7) have a dimensionality condition imposed on them; thus, strictly speaking, we can do the above process for q up to $\frac{1}{2}$ of n. To discuss the connection between the type number of critical points of f and the homology of M larger than $\frac{1}{2}$ of n, replace f by $c_n - f$. For, the critical points of f of type k are critical points of $c_n - f$ of type $n - k$. It is evident that $c_n - f$ is an invariant Morse function.

At this point, the reader may be wondering why we have taken all of this trouble to change involutions and functions. The answer is that the unimodular transformations which diagonalize ∂ may affect the direct sum split $F_q \oplus N_q$ with respect to T; however, it does not affect it with respect to S. Thus, henceforth, we may as well assume that the homomorphisms ∂ have been diagonalized to begin with.

LEMMA 1. *$\partial F_q = 0$ and $\partial(N_{q+1}) \subset N_q$.*

Proof. First of all, let us recall from § 3 that if w is a critical point of type q that is not fixed, then so is $T(w)$. This means that there is an even number of free independent generators of N_q; and if x is the element of N_q which is carried by a relative cell (E, S) which corresponds to w in the fashion of § 3, then $T_\#(x)$ is carried by $T(E, S)$. Thus, x and $T_\#(x)$ are free and independent generators of $H_q(Z_q, Z_{q-1})$.

Secondly, x is a generator of F_q if and only if $T_\#(x) = (-1)^q x$ by (1) and the fact that T freely permutes N_q. Next, since ∂ has been diagonalized, $\partial(x)$ is either in F_{q-1} or N_{q-1}.

Now, let x be in F_q. Then $\partial(x) = dy$, where y is a generator of $H_{q-1}(Z_{q-1}, Z_{q-2})$. Suppose that y is in F_{q-1}. Then, since $\partial T_\# = T_\# \partial$, we have that $(-1)^q \partial(x) = (-1)^{q-1} dy$. Thus $y = -y$, and hence $d = 0$. Now, let y be in N_{q-1}. Then, again by the naturality of $T_\#$, $dy = (-1)^q d T_\#(y)$. But there can be no relation between y and $T_\#(y)$, save the trivial relation, since they are independent generators. Hence, $d = 0$ and the first part of the lemma follows.

Now, let x be a generator of N_{q+1}. Then, $x' = T_\#(x)$ is also an independent generator of N_{q+1}. Since ∂ has been diagonalized, either ∂x and $\partial x'$ are both zero or they go into two different independent generators of $H_q(Z_q, Z_{q-1})$, and thus each cannot lie in F_q, and the lemma follows.

Since $\partial(F_q) = 0$ and $\partial(N_{q+1}) \subset N_q$, this means that each generator of F_q represents a free homology class of $H_q(M)$, and since the generators of F_q are in one-to-one correspondence with the fixed points of type q, the betti number of $H_q(M)$ is an upper bound for the number of such fixed points. Furthermore, it is evident that the number of those homology classes in $H_q(M)$, which do not correspond to fixed points, is even. Combining the above, we have the following theorem and corollaries.

THEOREM 2. *The number of fixed points f_q of type q is bounded by the betti number b_q of $H_q(M)$. Moreover, $b_q = f_q$ (Mod 2).*

COROLLARY 1. $x(M) = \sum_{r=0}^{n} f_r$ (Mod 2), *where $x(M)$ is the Euler characteristic of M.*

Proof. Corollary 1 in our setting is the same as (**2**, Corollary 4.3). The proof follows by observing that

$$\sum(-1)^r b_r = \sum(-1)^r f_r \text{ (Mod 2)} \quad \text{and} \quad \sum(-1)^r f_r = \sum f_r \text{ (Mod 2)}.$$

It is known that an even-dimensional manifold M with $x(M)$ odd cannot have an involution with a finite number of fixed points (**1**, p. 71). Hence, in order for M to have an involution T with a finite number of fixed points, $x(M)$ must be even. Therefore, we have the following corollaries.

COROLLARY 2. *The number of fixed points of T is even.*

COROLLARY 3. *A necessary condition that the fixed points of T coincide with the critical points of an invariant Morse function is that the homology groups (integral) of M be torsion free.*

Since T freely permutes N_q for all q, it is evident that if I_m (the integers modulo m) is a factor of $H_q(M)$, then it is a factor an even number of times. The above leads to the following proposition.

PROPOSITION. *Let T_q be the qth torsion group of $H_q(M)$. If I_m is a factor of*

T_q an odd number of times for any q, then the fixed-point set of any involution on M is either empty or infinite.

4. Proof of Theorem 1. The proof of Theorem 1 will be based upon an equivariant version of arguments in (**3**, § 6; **4**, Theorem 4.1). It will be broken down into several lemmas, the first of which is the following.

LEMMA 2. *There exists an invariant function $f\colon M \to R$ such that each fixed point of T is a non-degenerate critical point of f. Furthermore, f has but one critical point of type zero.*

Proof. First of all, there is an embedding ϕ of M in R^{2n} (for sufficiently large n) so that the following diagram commutes.

$$\begin{array}{ccc} M & \xrightarrow{\ \phi\ } & R^{2n} \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \delta} \\ M & \xrightarrow{\ \phi\ } & R^{2n} \end{array}$$

where δ is defined by $\delta(p, q) = \delta(q, p)$, p and q are n-tuples. If D is the subset of R^{2n} defined by $D = \{(p, p) | p$ is an n-tuple$\}$, then the fixed points of T lie in D. For p in D (p now a $2n$-tuple), define $L_p\colon M \to R$ by $L_p(q) = ||p - q||^2$. Note that L_p is an invariant mapping on M, and according to Milnor (**3**, Lemma 6.5), q in M is a degenerate critical point of L_p if and only if p is a focal point of (M, q). Since T has only a finite number of fixed points, only a finite number of points of D can be a focal point for any fixed point of T. Hence, a p in D can be chosen so that no fixed point of T is a degenerate critical point of L_p. But it is clear that the fixed points are critical points. Hence, by choosing a p in D sufficiently close to a fixed point of T, but not a focal point of any of them, such an L_p satisfies the condition of the lemma.

Although the fixed points of T are non-degenerate critical points of L_p, it is still possible that L_p may have degenerate critical points. Our next step is to construct a function which closely approximates L_p and satisfies the first part of Theorem 1. But before doing this we need the following lemma.

LEMMA 3. *Let M be a compact, smooth, simply connected manifold of dimension greater than 4. Let T be an involution on $M \times I$ without fixed points and assume that $M \times (0)$ is carried into $M \times (0)$ by T. Then, if T' is the involution induced on M by T, T is equivariently equivalent to the involution $S = T' \times i$, where i is the identity map of I into itself.*

Proof. The orbit spaces $M \times I/T$ and $M \times I/S$ are h-cobordisms for the manifold M/T', whose fundamental group is Z_2. Hence, by the s-cobordism theorem, $M \times I/T$ and $M \times I/S$ are diffeomorphic, say by a mapping h.

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h may be lifted to a diffeomorphism H between $M \times I$ and itself so that the following diagram commutes:

$$\begin{array}{ccc} M \times I & \xrightarrow{\eta} & M \times I/T \\ H \uparrow & & \uparrow h \\ M \times I & \xrightarrow{\eta^1} & M \times I/S, \end{array}$$

where η and η^1 are the orbit maps. It is a trivial matter to show that $T = HSH^{-1}$.

LEMMA 4. *Let w in M be a fixed point of T and suppose that E_1 and E_2 ($E_1 \not\subseteq E_2$) are two invariant disc neighbourhoods of w, each subsets of a coordinate neighbourhood U with coordinate function ϕ. Let f be an invariant function on M such that*

$$\sum_{i=1}^{n} \left| \frac{\partial f \phi^{-1}}{\partial x^i} \right| \geqq \delta$$

for all points in $\phi(E_2 - E_1{}^\circ)$, and let g be an invariant function on $M - E_2{}^\circ$, such that

$$\left| \frac{\partial f \phi^{-1}}{\partial x^i} - \frac{\partial g \phi^{-1}}{\partial x^i} \right| < \frac{\delta}{2n} \quad (\text{for } i = 1, 2, \ldots, n)$$

for all points in $\phi(E_2 - E_1{}^\circ)$. Then there exists an invariant function f' on M such that $f' = g$ on $M - E_2{}^\circ$ and f on E_1, and does not have any critical points on $E_2 - E_1{}^\circ$.

Proof. $E_2 - E_1{}^\circ$ is diffeomorphic to $S^{n-1} \times I$ and Lemma 3 implies that the following diagram commutes:

$$\begin{array}{ccc} E_2 - E_1{}^\circ & \xrightarrow{\phi} & S^{n-1} \times I \\ T \downarrow & & \downarrow \tau \\ E_2 - E_1{}^\circ & \xrightarrow{\phi} & S^{n-1} \times I \end{array}$$

where $\tau(x, t) = (-x, t)$ and ϕ is a diffeomorphism. Define f' by the rule as follows:

$$f'(p) = \begin{cases} g(p) & \text{if } p \in M - E_2{}^\circ, \\ h\,\phi(p) & \text{if } p \in E_2 - E_1{}^\circ, \\ f(p) & \text{if } p \in E_1, \end{cases}$$

where $h\phi(p) = h(x, t) = tg\phi^{-1}(x, t) + (1 - t)f\phi^{-1}(x, t)$. An easy calculation shows that f' satisfies the conclusion of the lemma.

Let us now consider the function $f = L_p$ of Lemma 2. Since each fixed point of T is a non-degenerate critical point of f, there exists pairs of invariant disc neighbourhoods about each fixed point of T in the manner of Lemma 4; that is, each pair of disc neighbourhoods is contained in some coordinate neighbourhood of the fixed point in question. Let there be k fixed points and let $\phi_1, \ldots, \phi_k$ be the coordinate functions for the coordinate neighbourhoods containing the pairs of invariant cells. It is evident that there exists a real number $\delta > 0$ such that, for $j = 1, \ldots, k$,

$$\sum_{i=1}^{n} \left| \frac{\partial f \phi_j^{-1}}{\partial x^i} \right| \geqq \delta \quad \text{for every point in } \phi_j(E_2 - E_1^\circ).$$

Consider the manifold W with boundary that is obtained from M by removing the interiors of each of the "inner" invariant cells about each fixed point. T restricted to W is an involution and it carries each boundary component onto itself. Let W/T be the orbit space with orbit map η. It is evident that W/T is a manifold with boundary. A smooth mapping f': $W/T \to R$ can be defined by setting $f'(p') = f(p)$, where $p \in W$ and $p' = \eta(p)$. f' is well-defined and smooth since f is an invariant function of W. Now, f' can be approximated by a smooth function g' without degenerate critical points as close as we like (**3**, Corollary 6.8), so close that the function $g = g'\eta$ is in a $\delta/2n$ neighbourhood of f in the C^2 topology on the set of smooth real-valued functions on W. We should observe that g is an invariant function of W. Hence, from the above and Lemma 4, it follows that there exists an invariant Morse function on M such that it has but one critical point of type zero.

Theorem 1 follows from the observation that the orthogonal trajectories induced by an invariant Morse function are carried into each other by T; thus, the proof of (**4**, Theorem 4.1) may be imitated.

Remark. If M is a compact, connected smooth manifold of dimension greater than 5, whose fundamental group $\pi_1 M$ has finite order $O(\pi_1 M)$, then we may study involutions of the type that we have been considering by examining the universal covering space M' of M. For, if T is an involution on M with at least one fixed point p, then T may be lifted to an involution T' on M' in the following manner: Let p' be a point over p and let T' be that unique map over T for which $T'(p') = p'$. If W is the set of points q for which $T'^2(q) = q$, it may easily be shown that W is both open and closed in M, and hence is all of M. Evidently, if k is the number of fixed points of T, the number of fixed points of T' must be $kO(\pi_1 M)$. In particular, this implies that the fixed point set of any involution on a smooth manifold for which the sphere is a non-trivial covering space must either be empty or infinite if $O(\pi_1 M) > 2$. If $O(\pi_1 M) = 2$, then the fixed point set is either empty, infinite, or consists of exactly one point.

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