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UNBOUNDEDNESS OF THE BERGMAN PROJECTIONS ON L^p SPACES WITH EXPONENTIAL WEIGHTS

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Abstract We prove that the Bergman projection on $L^p(w)$ $(p \neq 2)$, where $w(r) = (1-r^2)^A \mathrm{e}^{-B/(1-r^2)^\alpha}$, is not bounded.

Keywords: Bergman kernel; Bergman projection; Laplace method

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1. Introduction and notation

Let Δ denote the unit disc and let w denote a positive continuous function on the interval $[0,1)$. Let $\mathrm{d}A(z)$ denote the Lebesgue measure on Δ and $\mathrm{d}\mu(z)$ the measure on Δ defined by

$$\mathrm{d}\mu(z) = w(|z|)\,\mathrm{d}A(z).$$

Let $L^p(w)$ $(1 \leqslant p < \infty)$ be the space of all measurable functions f on Δ such that

$$\|f\|_p = \left(\int_\Delta |f|^p \,\mathrm{d}\mu\right)^{1/p} < \infty,$$

and let $L^p_{\mathrm{a}}(w)$ be the subspace of $L^p(w)$ consisting of analytic functions. It is known and easy to see that L^p_{a} is a closed subspace of L^p. Let P denote the ortho-projector from $L^2(w)$ onto $L^2_{\mathrm{a}}(w)$. P is called the Bergman projection. Let $\delta_n^2 = 2\pi \int_0^1 r^{2n+1} w(r)\,\mathrm{d}r$. Since the system

$$\left\{\frac{z^n}{\delta_n}\right\}_{n=0}^{\infty}$$

is an orthonormal base of $L^2_{\mathrm{a}}(w)$, the corresponding Bergman kernel (of the projection P) is given by

$$K(z,\xi) = \sum_{n=0}^{\infty} \frac{z^n \bar{\xi}^n}{\delta_n^2}.$$

Therefore,

$$Pf(z) = \int_{\Delta} K(z,\xi) f(\xi)\,\mathrm{d}\mu(\xi), \quad \text{for } f \in L^2(w),$$

and

$$Pf(z) = f(z), \quad \text{for } f \in L^2_{\mathrm{a}}(w).$$

It is of interest to study the boundedness of the projection P on the spaces $L^p(w)$ $(1 < p < \infty)$ because then it is easy to find the dual of $L^p_{\mathrm{a}}(w)$. In [**3**], Lin and Rochberg studied Toeplitz and Hankel operators on $L^2_{\mathrm{a}}(w)$ in the case when $w = \mathrm{e}^{-h}$, where h is a subharmonic function satisfying some additional conditions. As typical weights satisfying these conditions Lin and Rochberg mentioned the functions

$$w_0(r) = (1-r^2)^A \quad (A > 0)$$

and

$$w(r) = (1-r^2)^A \exp\biggl(-\frac{B}{(1-r^2)^\alpha}\biggr) \quad (A \geqslant 0,\ B > 0,\ \alpha > 0).$$

It is known (see, for example, [**4**, pp. 53–55]) that the projection P corresponding to w_0 is bounded on $L^p(w_0)$ for $1 < p < \infty$.

In this paper we shall show that in the case of the weight

$$w(r) = (1-r^2)^A \exp\biggl(-\frac{B}{(1-r^2)^\alpha}\biggr),$$

the corresponding Bergman projection is not bounded on $L^p(w)$, $p \neq 2$.

2. Result

Theorem 2.1. *If*

$$w(r) = (1-r^2)^A \exp\biggl(-\frac{B}{(1-r^2)^\alpha}\biggr), \quad A \geqslant 0,\ B > 0,\ 0 < \alpha \leqslant 1,$$

then the Bergman projection

$$Pf(z) = \int_{\Delta} K(z,\xi) f(\xi)\,\mathrm{d}\mu(\xi) \quad (\mathrm{d}\mu(\xi) = w(|\xi|)\,\mathrm{d}A(\xi))$$

is bounded on $L^p(w)$ only *if $p = 2$.*

For the proof we need the following lemma.

Lemma 2.2. *If*

$$w(r) = (1-r^2)^A \exp\biggl(-\frac{B}{(1-r^2)^\alpha}\biggr), \quad A \geqslant 0,\ B > 0,\ 0 < \alpha \leqslant 1,$$

and

$$\Phi(\lambda) = \int_0^1 r^\lambda w(r)\,\mathrm{d}r,$$

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then the following asymptotic formula holds:

$$\Phi(\lambda) \sim C\lambda^D \mathrm{e}^{-E\lambda^{\alpha/(\alpha+1)}}, \quad \lambda \to \infty,$$

where C, D, E *are constants depending only on* A, B, α *and* $E > 0$. (*We write* $f(\lambda) \sim g(\lambda)$, $\lambda \to \infty$, *to denote that* $\lim_{\lambda\to\infty}(f(\lambda)/g(\lambda)) = 1$.)

Proof. Consider the case $0 < \alpha < 1$. (The case $\alpha = 1$ is similar.)
Let

$$S(t) = -(\alpha B)^{1/(\alpha+1)}t - B(\alpha B)^{-\alpha/(\alpha+1)}t^{-\alpha}$$

and

$$H(\mu) = \int_0^\infty t^A \mathrm{e}^{\mu S(t)}\,\mathrm{d}t.$$

Since S attains its maximum for $t = 1$, an application of the Laplace method gives (see [**1**, pp. 66, 67])

$$H(\mu) \sim \mathrm{e}^{\mu S(1)}\sqrt{\frac{2\pi}{-\mu S''(1)}}, \quad \mu \to \infty.$$

Having in mind that

$$S(1) = -(\alpha B)^{1/(\alpha+1)} - B(\alpha B)^{-\alpha/(\alpha+1)} < 0 \quad \text{and} \quad S''(1) < 0$$

and taking

$$E = -S(1), \qquad F = \sqrt{-\frac{2\pi}{S''(1)}},$$

we obtain

$$H(\mu) \sim \frac{F}{\sqrt{\mu}}\mathrm{e}^{-E\mu}, \quad \mu \to \infty, \quad E, F > 0. \tag{2.1}$$

Consider now the asymptotic behaviour of the function

$$G_0(\lambda) = \int_0^\infty x^A \mathrm{e}^{-(\lambda+1)x - Bx^{-\alpha}}\,\mathrm{d}x, \quad \lambda \to \infty.$$

Introducing the substitution

$$x = t\left(\frac{\alpha B}{\alpha+1}\right)^{1/(\alpha+1)}$$

in the previous integral, we get

$$G_0(\lambda) = (\alpha B)^{(A+1)/(\alpha+1)}(\lambda+1)^{-(A+1)/(\alpha+1)}H((\lambda+1)^{\alpha/(\alpha+1)}),$$

and therefore, from (2.1), there follows the asymptotic formula

$$G_0(\lambda) \sim F_1\lambda^D \mathrm{e}^{-E\lambda^{\alpha/(\alpha+1)}}, \quad \lambda \to \infty,$$

where the constants F_1, D can easily be determined but their exact values are not of importance here.

Let

$$\Phi_0(\lambda) = \int_0^1 t^\lambda (1-t)^A \exp\biggl(\frac{-B}{(1-t)^\alpha}\biggr)\,dt.$$

Let us show that

$$\lim_{\lambda\to\infty} \frac{\Phi_0(\lambda)}{G_0(\lambda)} = 1. \tag{2.2}$$

Since

$$\Phi_0(\lambda) - G_0(\lambda) = \int_0^\infty e^{-(\lambda+1)x - B(1-e^{-x})^{-\alpha}} x^A \biggl(\biggl(\frac{1-e^{-x}}{x}\biggr)^A - 1\biggr)\,dx + \int_0^\infty e^{-(\lambda+1)x} x^A (e^{-B(1-e^{-x})^{-\alpha}} - e^{-Bx^{-\alpha}})\,dx$$

and $1 - e^{-x} \leqslant x$, we have

$$|\Phi_0(\lambda) - G_0(\lambda)| \leqslant \int_0^\infty e^{-(\lambda+1)x - Bx^{-\alpha}} x^A \biggl|\biggl(\frac{1-e^{-x}}{x}\biggr)^A - 1\biggr|\,dx + \int_0^\infty e^{-(\lambda+1)x - Bx^{-\alpha}} x^A |e^{B(x^{-\alpha} - (1-e^{-x})^{-\alpha})} - 1|\,dx. \tag{2.3}$$

Since

$$\lim_{x\to0^+} \biggl(\biggl(\frac{1-e^{-x}}{x}\biggr)^A - 1\biggr) = 0 \quad\text{and}\quad \lim_{x\to0^+} \biggl(\frac{1}{x^\alpha} - \frac{1}{(1-e^{-x})^\alpha}\biggr) = 0 \quad (\text{for } 0<\alpha<1),$$

then for given ε there is $\delta > 0$ such that

$$\biggl|\biggl(\frac{1-e^{-x}}{x}\biggr)^A - 1\biggr| < \tfrac{1}{3}\varepsilon \quad\text{and}\quad |-1 + e^{B(x^{-\alpha} - (1-e^{-x})^{-\alpha})}| < \tfrac{1}{3}\varepsilon \quad (0<x<\delta),$$

and from (2.3) it follows that

$$\begin{aligned} &|\Phi_0(\lambda) - G_0(\lambda)| \\ &\quad \leqslant \tfrac{2}{3}\varepsilon \int_0^\delta x^A e^{-(\lambda+1)x - Bx^{-\alpha}}\,dx + \int_\delta^\infty e^{-(\lambda+1)x - Bx^{-\alpha}} x^A \biggl|\biggl(\frac{1-e^{-x}}{x}\biggr)^A - 1\biggr|\,dx \\ &\qquad + \int_\delta^\infty e^{-(\lambda+1)x - Bx^{-\alpha}} x^A |-1 + e^{B(x^{-\alpha} - (1-e^{-x})^{-\alpha})}|\,dx. \end{aligned} \tag{2.4}$$

Since

$$\int_\delta^\infty e^{-(\lambda+1)x - Bx^{-\alpha}} x^A \biggl|\biggl(\frac{1-e^{-x}}{x}\biggr)^A - 1\biggr|\,dx = O(e^{-\lambda\delta})$$

and

$$\int_\delta^\infty e^{-(\lambda+1)x - Bx^{-\alpha}} x^A |-1 + e^{B(x^{-\alpha} - (1-e^{-x})^{-\alpha})}|\,dx = O(e^{-\lambda\delta}),$$

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we get from (2.4) that

$$|\Phi_0(\lambda) - G_0(\lambda)| \leqslant \tfrac{2}{3}\varepsilon \int_0^\infty x^A \mathrm{e}^{-(\lambda+1)x - Bx^{-\alpha}}\,\mathrm{d}x + O(\mathrm{e}^{-\lambda\delta}),$$

i.e.

$$\left|\frac{\Phi_0(\lambda)}{G_0(\lambda)} - 1\right| \leqslant \tfrac{2}{3}\varepsilon + O\biggl(\frac{\mathrm{e}^{-\lambda\delta}}{G_0(\lambda)}\biggr) < \varepsilon \quad \text{for } \lambda \geqslant \lambda_0,$$

which proves (2.2).

Since $\Phi(\lambda) = \frac{1}{2}\Phi_0(\frac{1}{2}(\lambda - 1))$, the assertion of the lemma follows from the asymptotics of the function Φ_0 (i.e. of G_0). □

Remark 2.3. The proof of the previous lemma could be derived from Theorem 3 in [**2**], but the function

$$w(r) = (1 - r^2)^A \exp\biggl(-\frac{B}{(1 - r^2)^\alpha}\biggr)$$

should first be replaced by

$$g(r) = \biggl(\ln\frac{1}{r^2}\biggr)^A \exp\biggl(\frac{-B}{(\log(1/r^2))^\alpha}\biggr).$$

Then the function $\Phi(\lambda)$ and $m(\lambda)$ ($m(\lambda) = \int_0^1 r^\lambda g(r)\,\mathrm{d}r$) have the same asymptotics when $\lambda \to \infty$. Furthermore, it is necessary to check whether the function $v(t) = B_0 t^{-\alpha} - A_0 t$ ($A_0, B_0 > 0$) satisfies all conditions of Theorem 3 in [**2**].

The proof of Theorem 3 in [**2**] is based on detailed analysis of the Legendre–Fenchel transform of the convenient class of functions (one such example is $v(t) = B_0 t^{-\alpha} - A_0 t$).

Our proof is different and it is based on Laplace's method. Since the weight is the particular function $w(r)$, our proof gives the conclusion more directly.

Proof of Theorem 2.1. It suffices to prove the unboundedness of P on $L^p(w)$ for $1 < p < 2$. Then, unboundedness on $L^p(w)$ for $p > 2$ follows by duality. Let $1 < p < 2$. Consider the system of functions

$$f_n(z) = z^{\omega n}\bar{z}^n,$$

where ω is a fixed positive integer.

By direct computation we get from the definition of P

$$Pf_n(z) = z^{\omega n - n} a_n,$$

where

$$a_n = \frac{\int_0^1 w(r) r^{1+2\omega n}\,\mathrm{d}r}{\int_0^1 w(r) r^{1+2(\omega n - n)}\,\mathrm{d}r}.$$

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Hence, by the lemma, we have

$$\frac{\|Pf_n\|_p}{\|f_n\|_p} \sim \text{const.} \times \exp\biggl(En^{\alpha/(\alpha+1)}\biggl((2(\omega-1))^{\alpha/(\alpha+1)} - (2\omega)^{\alpha/(\alpha+1)} + \frac{1}{p}(p(\omega+1))^{\alpha/(\alpha+1)} - \frac{1}{p}(p(\omega-1))^{\alpha/(\alpha+1)}\biggr)\biggr),\quad n \to \infty. \tag{2.5}$$

Let us show that for large ω, the inequality

$$\biggl((2(\omega-1))^{\alpha/(\alpha+1)} - (2\omega)^{\alpha/(\alpha+1)} + \frac{1}{p}(p(\omega+1))^{\alpha/(\alpha+1)} - \frac{1}{p}(p(\omega-1))^{\alpha/(\alpha+1)}\biggr) > 0 \tag{2.6}$$

holds, i.e.

$$L(\omega) = \biggl(1 - \frac{1}{\omega}\biggr)^{\alpha/(\alpha+1)} - 1 + \frac{1}{p}(\tfrac{1}{2}p)^{\alpha/(\alpha+1)}\biggl(\biggl(1 + \frac{1}{\omega}\biggr)^{\alpha/(\alpha+1)} - \biggl(1 - \frac{1}{\omega}\biggr)^{\alpha/(\alpha+1)}\biggr) > 0.$$

By the binomial formula we have

$$L(\omega) = \frac{\alpha}{\alpha+1}\biggl(\frac{2}{p}(\tfrac{1}{2}p)^{\alpha/(\alpha+1)} - 1\biggr) + O\biggl(\frac{1}{\omega}\biggr).$$

Since $1 < p < 2$, we have $\frac{1}{2}p < (\frac{1}{2}p)^{\alpha/(\alpha+1)}$ and hence $L(\omega) > 0$ if ω is a sufficiently large integer. From (2.5) and (2.6) it follows that the quotient $\|Pf_n\|_p/\|f_n\|_p$ is not bounded, i.e. that the operator P is not bounded. $\square$

Remark 2.4. In the case $p = 2$ instead of (2.6) (which holds for ω large enough), for every $\omega \geqslant 1$ the reverse inequality

$$\tfrac{1}{2}(2(\omega-1))^{\alpha/(\alpha+1)} - (2\omega)^{\alpha/(\alpha+1)} + \tfrac{1}{2}(2(\omega+1))^{\alpha/(\alpha+1)} \leqslant 0$$

holds, which is a consequence of the concavity of the function $x \mapsto x^{\alpha/(\alpha+1)}$. This, of course, agrees with the boundedness of the operator P on $L^2(w)$.

Problems 2.5.

(a) Describe all weights w for which the corresponding Bergman projection is bounded on $L^p(w)$ for every $p \in (1, \infty)$.

(b) What is the dual of $L^p_{\mathrm{a}}(w)$ in the case when the Bergman projection is not bounded on $L^p(w)$?

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