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THE ASSOCIATED ULTRASPHERICAL POLYNOMIALS AND THEIR q-ANALOGUES

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1. Introduction. A sequence of polynomials $\{P_n(x)\}$ is orthogonal if $P_n(x)$ is of precise degree n and there is a finite positive measure $d\mu$ such that

$$(1.1) \qquad \int_{-\infty}^{\infty} P_n(x)P_m(x)d\mu(x) = \lambda_n\delta_{mn}, m, n = 0, 1, \ldots .$$

A necessary and sufficient condition for orthogonality [9] is that $\{P_n(x)\}$ satisfies a three term recurrence

$$(1.2) \qquad P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x), n = 0, 1, \ldots ,$$

with

$$(1.3) \qquad P_{-1}(x) = 0, P_0(x) = 1, \text{ and } A_nA_{n+1}C_{n+1} > 0, n = 0, 1, \ldots .$$

Given a sequence of orthogonal polynomials $\{P_n(x)\}$ satisfying (1.2), the associated polynomials $\{P_n^{(\gamma)}(x)\}$, $\gamma > 0$, are defined by

$$(1.4) \qquad P_{n+1}^{(\gamma)}(x) = (A_{n+\gamma}x + B_{n+\gamma})P_n^{(\gamma)}(x) - C_{n+\gamma}P_{n-1}^{(\gamma)}(x), \quad n = 0, 1, \ldots ,$$

with $P_{-1}^{(\gamma)}(x) = 0$, $P_0^{(\gamma)}(x) = 1$, when $A_{n+\gamma}$ and $B_{n+\gamma}$ are well-defined. Another related sequence of polynomials is the sequence $\{P_n^*(x)\}$ satisfying (1.2) with $P_0^*(x) = 0$ and $P_1^*(x) = A_0$. Thus

$$(1.5) \qquad P_n^*(x) = A_0P_{n-1}^{(1)}(x)$$

where $P_n^{(1)}(x)$ is defined by (1.4) with $\gamma = 1$.

The following theorem of Markoff [9] shows how to find the Stieltjes transform of $d\mu(t)$ from the asymptotic behavior of $P_n(x)$ and $P_n^*(x)$.

THEOREM 1.1. *If the support of $d\mu$ is compact then*

$$(1.6) \qquad \lim_{n\to\infty} P_n^*(z)/P_n(z) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{z - t}, z \notin \text{supp } (d\mu).$$

Compactness of supp $(d\mu)$ can be deduced from the coefficients A_n, B_n and C_n in (1.2) as follows ([19]):

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THEOREM 1.2. Supp $(d\mu)$ *is compact if*

$$\operatorname{Sup}_{k\in\mathbf{N}}\left\{\left|\frac{B_n}{A_n}\right|+\frac{C_{n+1}}{A_nA_{n+1}}\right\}<\infty.$$

The measure $d\mu(t)$ can be recovered from (1.6) by applying the Stieltjes inversion theorem ([**23**]):

THEOREM 1.3. *If*

$$F(z)=\int_{-\infty}^{\infty}\frac{d\mu(t)}{z-t}$$

then

$$\mu(t_2)-\mu(t_1)=\lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\int_{t_1}^{t_2}[F(t-i\epsilon)-F(t+i\epsilon)]dt.$$

The following method of Darboux [**20**] can frequently be useful in computing the limit in (1.6).

THEOREM 1.4. *Let* $f(z)=\sum_{n=0}^{\infty}a_nz^n$ *be analytic in* $|z|<r$ *and assume that there is a comparison function* $g(z)=\sum_{n=0}^{\infty}b_nz^n$ *analytic in* $|z|<r$ *such that* $f(z)-g(z)$ *is continuous in* $|z|\leqq r$. *Then* $a_n=b_n+o(r^{-n})$.

Theorems (1.1)–(1.4) provide a technique for determining the measure $d\mu(t)$. Another way of recovering $d\mu(t)/dt$ from the asymptotic behavior of $P_n(x)$ is the following theorem of Nevai [**19**, Corollary 36, p. 141].

THEOREM 1.5. *If*

$$\sum_{n=1}^{\infty}\left\{\left|\frac{B_n}{A_n}\right|+\left|\left(\frac{C_{n+1}}{A_nA_{n+1}}\right)^{1/2}-\frac{1}{2}\right|\right\}<\infty$$

and $\{\tilde{P}_n\}$ *denotes the orthonormal set then*

$$\limsup_{n\to\infty}[\mu'(x)\sqrt{1-x^2}\,\tilde{p}_n^{\,2}(x)]=2/\pi \text{ for almost all } x\in\operatorname{supp}(d\mu). \tag{1.7}$$

Another theorem of Nevai is very useful in determining the measure, [**19**, Theorem 40, p. 143].

THEOREM 1.6. *If*

$$\sum_{n=1}^{\infty}\left\{\left|\frac{B_n}{A_n}\right|+\left|\left(\frac{C_{n+1}}{A_nA_{n+1}}\right)^{1/2}-\frac{1}{2}\right|\right\}<\infty$$

then $d\mu(t)=\mu'(t)dt+d\mu_j(t)$ *where* $\mu'(t)$ *is continuous and positive in* $(-1,1)$, supp $\mu'(t)=[-1,1]$, *and* $\mu_j(t)$ *is a step function constant in* $(-1,1)$.

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The ultraspherical or Gegenbauer polynomials $\{C_n^\nu(x)\}$ satisfy the recursion

$$(1.8)\quad (n+1)C_{n+1}^\nu(x) = 2x(\nu+n)C_n^\nu(x) - (2\nu+n-1)C_{n-1}^\nu(x)$$

while their continuous q-analogues $\{C_n(x;\beta|q)\}$ satisfy, ([**4**, **5**])

$$(1.9)\quad (1-q^{n+1})C_{n+1}(x;\beta|q) = 2x(1-\beta q^n)C_n(x;\beta|q) - (1-\beta^2q^{n-1})C_{n-1}(x;\beta|q).$$

The ultraspherical polynomials are well known and play an important role in harmonic analysis. The continuous q-ultraspherical polynomials were first studied by L. J. Rogers in his well-known memoirs where he proved the Rogers-Ramanujan identities of the theory of partitions, see [**2**]. They also belong to the class of generalized Legendre polynomials, that is polynomials having generating functions

$$(1.10)\quad \sum_{n=0}^{\infty} P_n(\cos\theta)r^n = |F(re^{i\theta})|^2,$$

where $F(z)$ is a real analytic function in a neighborhood of the origin. Fejer and Szego obtained a number of interesting results about these polynomials, ([**26**]). Feldheim [**13**] and Lanzewizky [**18**] proved independently that the only generalized Legendre polynomials that are also orthogonal are the ultraspherical polynomials and their continuous q-analogues. These polynomials have the generating relations

$$(1.11)\quad \sum_{n=0}^{\infty} C_n^\nu(\cos\theta)r^n = |(1-re^{i\theta})^{-\nu}|^2$$

and

$$(1.12)\quad \sum_{n=0}^{\infty} C_n(\cos\theta;\beta|q)r^n = \left|\prod_{n=0}^{\infty}\left\{\frac{1-r\beta e^{i\theta}q^n}{1-re^{i\theta}q^n}\right\}\right|^2.$$

The weight function for the continuous q-ultraspherical polynomials was found very recently by Askey and Ismail [**4**, **5**]. For details and further references on the q-ultraspherical polynomials we refer the interested reader to [**5**]. There is another q-analogue of the ultraspherical polynomials that is orthogonal on $[-1, 1]$ with respect to a purely discrete measure. The continuous q-analogues are orthogonal on $[-1, 1]$ with respect to an absolutely continuous measure.

In the present work we propose to study the associated ultraspherical polynomials and their continuous q-analogues. The associated Legendre polynomials have been studied by Barrucand and Dickinson [**8**]. In Section 2 we study the associated continuous q-ultraspherical polynomials in detail. They are orthogonal on $[-1, 1]$ with respect to an absolutely continuous measure and the measure is found. A generating function and some asymptotic formulas are obtained. This is followed by Section 3 where we study the associated ultraspherical polynomials.

We obtain a generating function, an explicit formula and record the orthogonality relation.

The associated ultraspherical polynomials, which include the associated Legendre polynomials, are special cases of Pollaczek's four parameter orthogonal polynomials [**21**]. Although Barrucand and Dickinson's weight function for the associated Legendre polynomials was contained in Pollaczek's earlier, and more general, work ([**21**]) it is worth mentioning that Barrucand and Dickinson's proof is simpler and more elementary. In Section 4, the associated continuous and discrete q-Hermite and q-Laguerre polynomials are treated. Some results are known about the associated Hermite polynomials, see e.g. [**9**] and [**27**, **28**]. The continuous q-Hermite polynomials appeared in [**5**]. Szego [**25**] evaluated their weight function. The continuous q-Laguerre polynomials were studied in [**7**]. The discrete Hermite polynomials were introduced by Hahn [**17**]. Recently Cigler [**10**] established a Mehler formula for the discrete q-Hermite polynomials.

We prove the positivity of the coefficients in the linearization of some of the orthogonal polynomials treated in the present paper. This is achieved via the following theorem of Askey [**3**].

THEOREM 1.7. *Let $\{\phi_n(x)\}$ be a sequence of monic polynomials, that is $\phi_n(x) = x^n +$ a polynomial of degree $n - 1$ or less, that satisfies*

$$(1.13) \quad \phi_1(x)\phi_n(x) = \phi_{n+1}(x) + a_n\phi_n(x) + b_n\phi_{n-1}(x).$$

If $a_n \geqq 0$, $b_n \geqq 0$ and $a_{n+1} \geqq a_n$, $b_{n+1} \geqq b_n$, then

$$(1.14) \quad \phi_n(x)\phi_m(x) = \sum_{k=|m-n|}^{m+n} \alpha_k\phi_k(x), \quad \text{with } \alpha_k \geqq 0.$$

The present work is a sequel to [**6**] and the basic methods and techniques used in both papers are essentially the same. In fact both papers owe a great deal to Pollaczek's very interesting and surprisingly neglected work [**22**].

Define $(a; q)_0 = 1$, $(a; q)_n = (1 - a)(1 - aq) \dots (1 - aq^{n-1})$, $n > 0$. The basic, or q, hypergeometric function is defined by

$$(1.15) \quad {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(b_1; q)_n \cdots (b_s; q)_n} \frac{z^n}{(q; q)_n}.$$

The ${}_2\phi_1$ basic hypergeometric function occurs so often in this paper that to simplify the notation we will use the notation $\phi(a, b; c; q, z)$ instead of ${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right)$. We will use the standard notation for the hypergeometric series

$${}_2F_1(a, b; c; z) = \sum_{K=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k,$$

with $(a)_0 = 1$, $(a)_k = a(a + 1) \dots (a + k - 1)$, $k > 0$.

2. The associated continuous q-ultraspherical polynomials. The continuous q-ultraspherical polynomials were studied in [**4, 5**]. Denoting these polynomials by $C_n(x; \beta|q)$, they are defined by the recurrence relation

$$(1 - q^{n+1})C_{n+1}(x; \beta|q) = 2x(1 - \beta q^n)C_n(x; \beta|q) - (1 - \beta^2 q^{n-1})C_{n-1}(x; \beta|q)$$

with $C_{-1}(x; \beta|q) = 0$, $C_0(x; \beta|q) = 1$.

The associated continuous q-ultraspherical polynomials then satisfy the recursion

$$(2.1)\quad (1 - q^{n+r+1})C_{n+1}^{(r)}(x; \beta|q) = 2x(1 - \beta q^{n+r})C_n^{(r)}(x; \beta|q) - (1 - \beta^2 q^{n+r-1})C_{n-1}^{(r)}(x; \beta|q),$$

where $C_{-1}^{(r)}(x; \beta|q) = 0$, $C_0^{(r)}(x; \beta|q) = 1$. The parameter r may be allowed any value real or complex for which $q^r \in (-1, 1)$ when $q \in (-1, 1)$. It is convenient to replace q^r by a new parameter $\alpha \in (-1, 1)$. Thus we define the polynomials $C_n^{(\alpha)}(x; \beta|q)$ by

$$(2.2)\quad (1 - \alpha q^{n+1})C_{n+1}^{(\alpha)}(x; \beta|q) = 2x(1 - \alpha\beta q^n)C_n^{(\alpha)}(x; \beta|q) - (1 - \alpha\beta^2 q^{n-1})C_{n-1}^{(\alpha)}(x; \beta|q),$$

with $C_{-1}^{(\alpha)} = 0$, $C_0^{(\alpha)} = 1$. In (2.2) we will take $\alpha \in (-1, 1)$. We will determine values of β for which the polynomials given by (2.2) are orthogonal. Applying the criterion $A_n A_{n+1} C_{n+1} > 0$ from (1.3) to (2.2) we see that the polynomials $C_n^{(\alpha)}(x; \beta|q)$ are orthogonal if and only if

$$(2.3\text{-a})\quad (1 - \alpha\beta q^n)(1 - \alpha\beta q^{n+1})(1 - \alpha\beta^2 q^n) > 0,\ n = 0, 1, 2, \dots.$$

From (2.3-a) we see that orthogonality is obtained in the following cases.

$$(2.3\text{-b})\quad \begin{cases} \text{(i) } 0 < \alpha \leqq 1,\ -1 < q < 1,\ -1/\sqrt{\alpha} < \beta < 1/\sqrt{\alpha} \\ \text{(ii) } -1 < \alpha < 0,\ 0 < q < 1,\ \beta > 1/\alpha \\ \text{(iii) } -1 < \alpha < 0,\ -1 < q < 0,\ 1/\alpha < \beta < 1/\alpha q. \end{cases}$$

In the special cases $\beta = 1$, $\beta = q$, the polynomials are particularly simple. When $\beta = q$ then (2.2) reduces to

$$C_{n+1}^{(\alpha)}(x; q|q) = 2xC_n^{(\alpha)}(x; q|q) - C_{n-1}^{(\alpha)}(x; q|q),$$

and hence

$$(2.4)\quad C_n^{(\alpha)}(\cos\theta; q|q) = \frac{\sin (n+1)\theta}{\sin\theta},\quad n = 0, 1, 2, \dots.$$

Similarly when $\beta = 1$ we get

$$(2.5)\quad \frac{(1 - \alpha q^n)}{(1 - \alpha)} C_n^{(\alpha)}(\cos\theta; 1|q) = \frac{\sin (n+1)\theta}{\sin\theta},\quad n = 0, 1, 2, \dots.$$

2.1 *A generating function.* Define $F(x, t)$ by

$$(2.6) \quad F(x, t) = \sum_{n=0}^{\infty} C_n^{(\alpha)}(x; \beta|q)t^n.$$

Multiplying (2.6) by t^{n+1} and summing for $n = 0, 1, 2, \ldots$ we obtain

$$(2.7) \quad F(x, t) = \frac{1 - \alpha}{1 - 2xt + t^2} + \alpha \frac{(1 - 2\beta xt + \beta^2 t^2)}{1 - 2xt + t^2} F(x, qt).$$

Let $H = \{x \in C: |x + \sqrt{x^2 - 1}| \geqq 1\}$ and define $A = A(x)$ and $B = B(x)$ by $AB = 1$, and

$$A = \begin{cases} x + \sqrt{x^2 - 1}, & x \in H \\ x - \sqrt{x^2 - 1}, & x \in H^C. \end{cases}$$

Then $|A| \geqq |B|$ and $1 - 2xt + t^2 = (1 - At)(1 - Bt)$. By iterating (2.7) and noting that $F(x, q^n t) \rightarrow F(x, 0) = 1$ as $n \rightarrow \infty$, we obtain

$$(2.8) \quad \sum_{n=0}^{\infty} C_n^{(\alpha)}(x; \beta|q)t^n = \frac{1 - \alpha}{1 - 2xt + t^2} {}_3\phi_2\left(\begin{matrix}\beta tA, \beta tB, q \\ qtA, qtB\end{matrix}; q, \alpha\right).$$

2.2 *Asymptotic formulas.* We will use Darboux' method, Theorem 1.4. When x lies in the complex plane cut along $[-1, 1]$ the generating function (2.8) is analytic in $|t| < 1/|A|$ with $t = 1/A$ being the only singularity on $|t| = 1/|A|$. Thus

$$\frac{(1 - \alpha)}{(1 - At)(1 - B/A)} \phi\left(\beta, \frac{\beta B}{A}; \frac{qB}{A}; q, \alpha\right)$$

is a comparison function. Hence

$$(2.9) \quad C_n^{(\alpha)}(x; \beta|q) \sim \frac{(1 - \alpha)}{A - B} A^{n+1} \phi\left(\beta, \frac{\beta B}{A}; \frac{qB}{A}; q, \alpha\right),$$

$$x \notin [-1, 1], n \rightarrow \infty.$$

We now determine the asymptotic behavior of $C_n^{(\alpha)}(x; \beta|q)$ for $x \in (-1, 1)$. Write $x = \cos\theta$, and take $A = e^{i\theta}$, $B = e^{-i\theta}$. Observe that the generating function in (2.8) is analytic in $|t| < 1$ and that $e^{i\theta}$ and $e^{-i\theta}$ are the only singularities on $|t| = 1$. Thus we may take

$$(2.10) \quad \frac{(1 - \alpha)\phi(\beta, \beta e^{-2i\theta}; qe^{-2i\theta}; q, \alpha)}{(1 - e^{i\theta}t)(1 - e^{-2i\theta})} + \frac{(1 - \alpha)\phi(\beta, \beta e^{2i\theta}; qe^{2i\theta}; q, \alpha)}{(1 - e^{-i\theta}t)(1 - e^{2i\theta})}$$

as a comparison function. This yields

$$(2.11) \quad C_n^{(\alpha)}(\cos\theta; \beta|q) \sim M(\theta) \frac{\cos[(n + 1)\theta + \psi]}{\sin\theta}, \quad n \rightarrow \infty,$$

where $M(\theta) \geqq 0$, ψ is real, and

$$(2.12) \quad M(\theta)e^{i\psi} = i(\alpha - 1)\phi(\beta, \beta e^{-2i\theta}; qe^{-2i\theta}; q, \alpha).$$

2.3 *The orthogonality relation.* The relationships (1.5), (1.6), and (2.9) imply

$$(2.13) \quad \int_{-\infty}^{\infty} \frac{d\mu(t)}{x-t} = \lim_{n\to\infty} \frac{2(1-\alpha\beta)}{1-\alpha q} \frac{C_{n-1}^{(\alpha q)}(x;\beta|q)}{C_n^{(\alpha)}(x;\beta|q)}, \quad x \notin [-1,1].$$

Applying the results of Section (2.2) we have

$$(2.14) \quad \int_{-\infty}^{\infty} \frac{d\mu(t)}{x-t} = \frac{2(1-\alpha\beta)\phi(\beta, \beta B/A; qB/A; q, \alpha q)}{(1-\alpha q)A\,\phi(\beta, \beta B/A; qB/A; q, \alpha)}, \quad x \notin [-1,1].$$

It is easy to verify that the series in Theorems 1.5 and 1.6 converges in these circumstances and thus by Theorem 1.6 $\mu(t)$ can have no discrete mass points inside $(-1, 1)$. From (2.14) we observe that if the function appearing in the denominator on the right is positive for $x \notin [-1, 1]$ and for given values of α, β, and q, then the function on the left of (2.14) has no poles and hence $d\mu(t)$ can have no discrete mass points outside of $[-1, 1]$. Consequently we will next find conditions under which this function is positive. Write $\omega = \omega(x) = B/A$. Then $0 < \omega < 1$ if $-\infty < x < -1$ or $1 < x < \infty$, and

$$(2.15) \quad \omega(x) = \begin{cases} \dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}, & 1 < x < \infty \\[2ex] \dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}, & -\infty < x < -1. \end{cases}$$

Firstly we observe that each term in $\phi(\beta, \beta\omega; q\omega; q, \alpha)$ is positive if $0 < \omega < 1$, $0 < \beta < 1$, $0 < q < 1$. Thus from (2.14) we conclude that for these values of the parameters $d\mu(t)$ has no discrete mass points outside of $[-1, 1]$. Consider now $\phi(\beta, \beta\omega; q\omega; q, -\alpha)$ for $\alpha > 0$. We will need some preliminary results.

LEMMA 2.1. *If* $0 < b < a < 1$ *then*

$$(a;q)_k/(b;q)_k = \int_0^1 t^k d\psi(t)$$

where $\psi(t)$ *is a monotone increasing step function.*

Proof. Set

$$\epsilon_j = \frac{(a;q)_\infty (b/a;q)_j}{(b;q)_\infty (q;q)_j} a^j, \quad j = 0, 1, 2, \ldots.$$

Then $\epsilon_j > 0$ and $\sum_{j=0}^{\infty} \epsilon_j = 1$ by the q-Binomial Theorem, [**18**, p. 92]. Now define $\psi(t)$ as $\psi(t) = 1$ for $t \geqq 1$, $\psi(0) = 0$, and in intervals $q^p \leqq t < q^{p-1}$ set

$$\psi(t) = 1 - \sum_{j=1}^{p} \epsilon_j.$$

$\psi(t)$ is a monotone increasing step function. By definition of $\psi(t)$, and by the q-Binomial Theorem,

$$\int_0^1 t^k d\psi(t) = \frac{(a;q)_\infty}{(b;q)_\infty} \sum_{n=0}^{\infty} \frac{(b/a;q)_n}{(q;q)_n} a^n q^{nk} = \frac{(a;q)_k}{(b;q)_k}$$

which proves the lemma.

LEMMA 2.2. *If* $0 < \beta_1 < \beta_2 < 1$ *and* $\phi(\beta_1\omega, \beta_1; q\omega; q, \alpha) > 0$ *then*

$$\phi(\beta_2\omega, \beta_2; q\omega, q; \alpha) > 0.$$

Proof.

$$\phi(\beta_2\omega, \beta_2; q\omega; q, \alpha) = \sum_{n=0}^{\infty} \frac{(\beta_2\omega;q)_n(\beta_2;q)_n}{(q\omega;q)_n(q;q)_n} \alpha^n$$
$$= \sum_{n=0}^{\infty} \frac{(\beta_1\omega;q)_n(\beta_1;q)_n}{(q\omega;q)_n(q;q)_n} \left[\frac{(\beta_2\omega;q)_n}{(\beta_1\omega;q)_n} \frac{(\beta_2;q)_n}{(\beta_1;q)_n}\right] \alpha^n.$$

By Lemma 2.1 and the convolution theorem for moments we may write the quantity in brackets above as

$$\frac{(\beta_2\omega;q)_n}{(\beta_1\omega;q)_n} \frac{(\beta_2;q)_n}{(\beta_1;q)_n} = \int_0^1 t^n d\psi(t),$$

where $\psi(t)$ is an increasing step function. Hence

$$\phi(\beta_2\omega; \beta_2; q\omega; q, \alpha) = \int_0^1 \phi(\beta_1\omega; \beta_1; q\omega; q, \alpha t) d\psi(t) > 0.$$

This proves the lemma.

THEOREM 2.1. $\phi(\beta\omega, \beta; q\omega; q, -\alpha) > 0$ *for* $0 < \alpha < 1$, $0 < \omega < 1$, $0 < q < 1$, $q^2 \leqq \beta < 1$. *The* q^2 *in* $q^2 \leqq \beta < 1$ *cannot be replaced by any value* $\beta_0 < q^3$.

Proof. To prove the first part of the theorem it suffices due to Lemma 2.2 to prove that

$$\phi(\omega q^2, q^2; q\omega; q, -\alpha) > 0.$$

A simple calculation gives

$$(1-q)(1-\omega q)\phi(\omega q^2, q^2; \omega q; q, -\alpha) = \frac{1}{1+\alpha} - \frac{(1+\omega)q}{1+\alpha q} + \frac{\omega q^2}{1+\alpha q^2}.$$

The right side of this equality is a monotone decreasing function of ω, so it suffices to prove it positive when $\omega = 1$. In this case we have

$$\frac{1}{1+\alpha} - \frac{2q}{1+\alpha q} + \frac{q^2}{1+\alpha q^2} = \frac{(1-q)^2(1-\alpha q)}{(1+\alpha)(1+\alpha q)(1+\alpha q^2)},$$

and the right side of this expression is positive for $0 < \alpha < 1, 0 < q < 1$. This proves the first statement in the theorem.

To prove the second statement it suffices by Lemma 2.2 to prove that $\phi(\omega q^3, q^3; \omega q; q, -\alpha)$ changes sign for certain values of α and q as ω ranges from 0 to 1. When $\omega = 0$ we have after a calculation

$$\phi(0, q^3; 0; q, -\alpha) = \frac{-\alpha(1-q)(1-q^2(1-q^3)}{(1+\alpha)(1+\alpha q)(1+\alpha q^2)(1+\alpha q^3)}.$$

Thus $\phi(0, q^3; 0; q; -\alpha) < 0$ for $0 < \alpha < 1, 0 < q < 1$. Now when $\omega = 1$ we get after a calculation

$$\phi(q^3, q^3; q; q, -\alpha) = \frac{1 - \alpha q - 2\alpha q^2 - \alpha q^3 + \alpha^2 q^4}{(-\alpha; q)_5}.$$

Note that if α and q are close to 0 then the numerator on the right is positive. For such values of α and q then $\phi(\omega q^3, q^3; \omega q; q, -\alpha)$ changes sign as ω ranges from 0 to 1. This proves the theorem.

From the remarks preceding Lemma 2.1 and from Theorem 2.1 we may conclude

THEOREM 2.2. *The measure $d\mu$ given by* (2.14) *has no discrete mass points if* $0 < \beta < 1, 0 < q < 1, 0 < \alpha < 1$ *or if* $q^2 \leqq \beta < 1, 0 < q < 1, -1 < \alpha < 0$.

It is clear that the support of $d\mu$ is $[-1, 1]$ because the right side of (2.14) is single valued for $|x| > 1$ and has no poles on the real axis. Furthermore Theorem 1.6 implies the absolute continuity of $d\mu$. At this stage we may compute $\mu'(x)$ either via the Stieltjes inversion theorem, Theorem 1.3, or by applying Theorem 1.5 to (2.11). The orthonormal set is $\{C_n^\infty(x; \beta|q)/\lambda_n^{1/2}\}$. It is well known, see e.g. [**9**], that

(2.16) $\lambda_n = \lambda_0 A_0 C_1 \dots C_n/A_n, \; n > 0,$

hence, since μ is normalized by

$$\int_{-\infty}^{\infty} d\mu(x) = 1,$$

(2.17) $\lambda_n = (1-\alpha\beta)(\alpha\beta^2; q)_n\{(1-\alpha\beta q^n)(\alpha q; q)_n\}^{-1}.$

Therefore

$$\lambda_n \sim (1-\alpha\beta)(\alpha\beta^2; q)_\infty/(\alpha q; q)_\infty \text{ and}$$

$$\frac{\sqrt{1-x^2}M^2(\theta)(\alpha q; q)_\infty}{(1-\alpha\beta)(\alpha\beta^2; q)_\infty \sin^2\theta} \frac{d\mu(x)}{dx} = \frac{2}{\pi},$$

that is

(2.18) $$\frac{d\mu(\cos\theta)}{d\theta} = \frac{2}{\pi} \frac{(1-\alpha\beta)(\alpha\beta^2; q)_\infty}{(\alpha q; q)_\infty M^2(\theta)} \sin^2\theta.$$

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This measure is the same as the measure appearing in (2.14) since the left side is

$$x^{-1}\int_{-\infty}^{\infty} d\mu(t) + O(x^{-2}) \quad \text{as } x \to \infty,$$

but $A \sim 2x$, $B \sim (2x)^{-1}$ so the right side is equal to

$$\frac{2(1-\alpha\beta)}{(1-\alpha)2x}\,{}_1\phi_0(\beta;\,-\,;q,q\alpha)/{}_1\phi_0(\beta;\,-\,;q,\alpha),$$

hence is equal to $x^{-1} + O(x^{-2})$ by the q-binomial theorem. This establishes the orthogonality relation

$$\text{(2.19)} \quad \int_0^\pi C_n^\alpha(x;\beta|q)C_m^\alpha(x;\beta|q)\frac{\sin^2\theta}{M^2(\theta)}d\theta = \frac{\pi}{2}\frac{(\alpha q^{n+1};q)_\infty}{(1-\alpha\beta q^n)(\alpha\beta^2 q^n;q)_\infty}\delta_{m,n}.$$

When $\alpha = 1$ the orthogonality relation (2.19) reduces to the orthogonality relation for the continuous q-ultraspherical polynomials, see (4.3) in [**4**].

2.4 *A Wronskian type formula for some* ${}_2\phi_1$ *functions.* The relationship (2.15) and the Stieltjes inversion formula (1.6) imply

$$\text{(2.20)} \quad \frac{d\mu(x)}{dx} = \frac{2(1-\alpha\beta)}{\pi(1-\alpha)}\operatorname{Im}\left\{\frac{e^{i\theta}\phi(\beta,\beta e^{2i\theta};qe^{2i\theta};q,\alpha q)}{\phi(\beta,\beta e^{2i\theta};qe^{2i\theta};q,\alpha)}\right\}.$$

Comparing (2.18) with (2.20) and using (2.11) we get

$$\text{(2.21)} \quad e^{i\theta}\phi(\beta,\beta e^{2i\theta};qe^{2i\theta};q,\alpha q)\phi(\beta,\beta e^{-2i\theta};qe^{-2i\theta};q,\alpha)$$
$$-e^{-i\theta}\phi(\beta,\beta e^{-2i\theta};qe^{-2i\theta};q,\alpha q)\phi(\beta,\beta e^{2i\theta};qe^{2i\theta};q,\alpha)$$
$$= \frac{-2i(\beta^2\alpha;q)_\infty\sin\theta}{(\alpha;q)_\infty}, \quad -1 < \alpha < 1, 0 \leqq \theta \leqq \pi.$$

Formula (2.21) holds by analytic continuation in any domain not containing the points $\alpha = q^{-n}$, $n = 0, 1, 2, \ldots$. When $\beta = q$, (2.21) reduces to

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta.$$

The case $\alpha = 1$ can be handled by a limiting argument taking an Abel limit as $\alpha \to 1^-$. The result is

$$\text{(2.22)} \quad e^{i\theta}(qe^{2i\theta};q)_\infty(\beta e^{-2i\theta};q)_\infty\phi(\beta,\beta e^{2i\theta};qe^{2i\theta};q,q)$$
$$-e^{-i\theta}(qe^{-2i\theta};q)_\infty(\beta e^{2i\theta};q)_\infty\phi(\beta,\beta e^{-2i\theta};qe^{-2i\theta};q,q)$$
$$= -2i\sin\theta\frac{(\beta^2;q)_\infty}{(\beta;q)_\infty}(qe^{2i\theta};q)_\infty(qe^{-2i\theta};q)_\infty.$$

Formulas (2.21) and (2.22) resemble Wronskian formulas.

2.5 *Linearization of products and a continued fraction.* Write

$$\phi_n(x) = \frac{2^{-n}(\alpha q; q)_n}{(\alpha\beta; q)_n} C_n^{(\alpha)}(x; \beta|q),$$

and rewrite (2.2) in the form

$$\phi_{n+1} = x\phi_n - \frac{(1-\alpha\beta^2 q^{n-1})(1-\alpha q^n)}{4(1-\alpha\beta q^n)(1-\alpha\beta q^{n-1})}\phi_{n-1}.$$

Clearly $\phi_1(x) = x$. The coefficients a_n and b_n of (1.13) are given by $a_n = 0$ and

$$b_n = \frac{(1-\alpha\beta^2 q^{n-1})(1-\alpha q^n)}{4(1-\alpha\beta q^n)(1-\alpha\beta q^{n-1})}.$$

The coefficients b_n remain unchanged if α, β, q are replaced by α^{-1}, β^{-1}, q^{-1} respectively. So we shall assume that $0 < q < 1$, $0 < \beta < 1$, $-1 < \alpha < 1$, and by symmetry this covers the case $q > 1$, $\beta \geqq 1$, $|\alpha| > 1$. Thus $b_n \geqq 0$ and

$$\frac{b_{n+1} - b_n}{b_n} = \frac{(1-\alpha\beta^2 q^n)(1-\alpha q^{n+1})(1-\alpha\beta q^{n-1})}{(1-\alpha\beta^2 q^{n-1})(1-\alpha q^n)(1-\alpha\beta q^{n+1})} - 1.$$

Therefore $b_{n+1} \geqq b_n$ if

$$(1-\alpha\beta^2 q^n)(1-\alpha\beta q^{n+1})(1-\alpha\beta q^{n-1}) \geqq (1-\alpha\beta^2 q^{n-1})(1-\alpha q^n)(1-\alpha\beta q^{n+1}).$$

This last inequality is equivalent to

$$\alpha(1-q)(1-\beta)(q-\beta)(1+\alpha\beta q^n) \geqq 0.$$

Since $0 < q < 1, 0 < \beta < 1, -1 < \alpha < 1$, this reduces to $\alpha(q-\beta) \geqq 0$. We are thus led to

THEOREM 2.3. *The linearization coefficients A_{klm} in the identity*

$$C_k^{(\alpha)}(x; \beta|q) C_l^{(\alpha)}(x; \beta|q) = \sum_{m=|k-l|}^{|k+l|} A_{klm} C_m^{(\alpha)}(x; \beta|q)$$

are positive if $\alpha(q-\beta) \geqq 0$.

Theorem 2.3 is far from best possible. It holds for the case $\beta = q^\lambda$, $0 < q \leqq 1, \alpha = 1$. Richard Askey mentioned in a private communication that he suspects Theorem 2.3 to hold for $\alpha > 0$, $0 < \beta < 1$, $0 < q < 1$. That remains an open question.

When a set of polynomials $\{P_n(z)\}$ is orthogonal on a bounded interval the Stieltjes transform of the corresponding measure $\mu(t)$ is related to the continued fraction whose numerators are $P_n^*(z)$ and denominators are

$P_n(z)$. In fact

$$\int_{-\infty}^{\infty} \frac{d(t)}{z-t} = \lim_{n\to\infty} \frac{P_n^*(z)}{P_n(z)} = \frac{A_0|}{|A_0 z + B_0} - \frac{C_1|}{|A_1 z + B_1} - \cdots - \frac{C_n|}{|A_n z + B_n} - \cdots .$$

The continued fraction associated with the associated continuous q-ultraspherical polynomials belongs to the class of continued fractions studied by Frank [**14**, **15**, **16**].

3. The associated ultraspherical polynomials. The associated ultraspherical (or Gegenbauer) polynomials $C_n^{(\gamma)}(x;\beta)$ satisfy the recurrence relation

$$(3.1) \quad (n+\gamma+1)C_{n+1}^{(\gamma)}(x;\beta) = 2x(n+\beta+\gamma)C_n^{(\gamma)}(x;\beta) - (2\beta+n+\gamma-1)\,C_{n-1}^{(\gamma)}(x;\beta),\ n = 1, 2, \ldots$$

with

$$(3.2) \quad C_0^{(\gamma)}(x;\beta) = 1,\ C_{-1}^{(\gamma)}(x;\beta) = 0.$$

The associated Legendre polynomials obtained with $\beta = 1/2$ were studied in [**8**].

The Legendre functions of the first kind $p_\nu^\mu(x)$ and of the second kind $Q_\nu^\mu(x)$ satisfy [**11**, pp. 160–161]

$$(3.3) \quad (2\nu+1)xf_\nu^\mu(x) = (\nu-\mu+1)\,f_{\nu+1}^\mu(x) + (\nu+\mu)\,f_{\nu-1}^\mu(x).$$

We claim that

$$(3.4) \quad C_n^{(\gamma)}(x;\beta) = \frac{i\Gamma(\gamma+1)}{\Gamma(2\beta+\gamma-1)}\,e^{-i\beta\pi} \times \{Q_{\beta+\gamma-3/2}^{\beta-1/2}(x)P_{n+\beta+\gamma-1/2}^{\beta-1/2}(x) - P_{\beta+\gamma-3/2}^{\beta-1/2}(x)Q_{n+\beta+\gamma-1/2}^{\beta-1/2}(x)\}.$$

The expression on the right of (3.4) vanishes when $n = -1$ and satisfies (3.1). In order to prove (3.4) it remains only to show that it holds when $n = 0$, that is

$$(3.5) \quad Q_{\beta+\gamma-3/2}^{\beta-1/2}(x)P_{\beta+\gamma-1/2}^{\beta-1/2}(x) - P_{\beta+\gamma-1/2}^{\beta-1/2}(x)Q_{\beta+\gamma-1/2}^{\beta-1/2}(x) = e^{i(\beta-1/2)\pi}\Gamma(2\beta+\gamma-1)/\Gamma(\gamma+1).$$

Using formula (10), p. 161 of [**11**] we easily obtain

$$(\nu+\mu)\{Q_{\nu-1}^\mu(x)P_\nu^\mu(x) - P_{\nu-1}^\mu(x)Q_\nu^\mu(x)\} = (1-x^2)\left\{P_\nu^\mu(x)\frac{d}{dx}Q_\nu^\mu(x) - Q_\nu^\mu(x)\frac{d}{dx}P_\nu^\mu(x)\right\}$$

which when combined with the Wronskian formula (13), p. 123 of [**13**] and the duplication formula for the gamma function

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2),$$

see e.g. [**11**, p. 5], establishes (3.5). This completes the proof of (3.4).

3.1 *A generating function and connection to the ultraspherical polynomials.* Set

$$(3.6)\qquad G(x,t) = \sum_{n=0}^{\infty} C_n^{(\gamma)}(x;\beta)t^n.$$

Multiplying (3.1) by t^{n+1} and adding the resulting equations for $n = 1, 2, \ldots$, we derive the differential equation

$$t(1-2x+t^2)\frac{\partial G}{\partial t} + [\gamma - 2x(\beta+\gamma)t + (2\beta+\gamma)t^2]G = \gamma,$$

whose solution is

$$(3.7)\qquad G(x,t) = \gamma(1-2xt+t^2)^{-\beta}\int_0^1 u^{\gamma-1}(1-2xtu+u^2t^2)^{\beta-1}du,$$

that is, [**11**, (5), p. 231]

$$(3.8)\qquad G(x,t) = (1-2xr+t^2)^{-\beta}F_1(\gamma;1-\beta,1-\beta;\gamma+1;At,Bt),$$

where F_1 is the Apell function

$$(3.9)\qquad F_1(\alpha;\beta,\beta';\gamma;x,y) = \sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_m}{(\gamma)_{m+n}m!n!}x^m y^n,$$

and

$$(3.10)\quad A = x+\sqrt{x^2-1},\ B = x-\sqrt{x^2-1}.$$

When $x \in [-1, 1]$, (3.8) takes the form

$$(3.11)\quad \sum_{n=0}^{\infty} C_n^{(\gamma)}(\cos\theta;\beta)t^n = (1-2xt+t^2)^{-\beta}F_1(\gamma;1-\beta,1-\beta;\gamma+1;te^{i\theta},te^{-i\theta}).$$

When $\gamma = 0$ the generating function (3.11) reduces to the usual generating function for the ultraspherical polynomials, see (3.12) below. The generating function (3.11) reduces to the generating function for the associated Legendre polynomials when $\beta = 1/2$, see [**8**].

The ultraspherical polynomials have the generating function

$$(3.12)\quad \sum_{n=0}^{\infty} C_n^{\nu}(x)t^n = (1-2xt+t^2)^{-\nu}.$$

Expanding the integrand in (3.7) in powers of t we get, in view of (3.12),

$$\sum_{n=0}^{\infty} C_n^{(\gamma)}(x;\beta)t^n = \gamma\sum_{k=0}^{\infty} C_k^{\beta}(x)t^k\sum_{m=0}^{\infty} C_m^{1-\beta}(x)t^m/(m+\gamma).$$

This leads to the convolution type formula

$$(3.13)\quad C_n^{\gamma}(x;\beta) = \sum_{k=0}^{n} \frac{\gamma}{\gamma+k} C_k^{1-\beta}(x) C_{n-k}^{\beta}(x).$$

3.2. *The measure.* Pollaczek [**21**] has explicitely calculated the measure $\rho(x)dx$ corresponding to a four parameter family of orthogonal polynomials that includes the associated ultraspherical polynomials. The weight function for the associated ultraspherical polynomials is

$$\rho(\cos t) = \frac{(2\sin t)^{2\beta-1}[\Gamma(\lambda+\gamma)]^2}{2\pi\Gamma(2\beta+\gamma)\Gamma(\gamma+1)} |{}_2F_1(1-\beta,\gamma;\gamma+\beta;e^{2it})|^{-2}, \quad 0 \leqq t \leqq \pi.$$

4. The associated basic (or q) Hermite and Laquerre polynomials.

4.1 *The associated continuous q-Hermite polynomials.* The continuous q-Hermite polynomials, [**4**], are defined by

$$(4.1)\quad H_n(x|q) = (q;q)_n C_n(x;0|q),$$

where the $C_n(x;0|q)$ are the continuous q-ultraspherical polynomials defined in Section 2. This suggests defining their associated polynomials by

$$(4.2)\quad H_n^{(\alpha)}(x|q) = (q;q)_n C_n^{(\alpha)}(x;0|q)$$

where the $C_n^{(\alpha)}(x;0|q)$ are defined by (2.2) with $\beta = 0$. Then from the results of Section 2 we find that the associated continuous q-Hermite polynomials have the generating function

$$(4.3)\quad \sum_{n=0}^{\infty} \frac{H_n^{(\alpha)}(x|q)t^n}{(q;q)_n} = \frac{1-\alpha}{1-2xt+t^2}\, {}_3\phi_2\left(\begin{matrix}0, 0, q \\ qtA, qtB\end{matrix}; q, \alpha\right).$$

Their three term recurrence is

$$(4.4)\quad H_{n+1}^{(\alpha)}(x|q) = 2xH_n^{(\alpha)}(x|q) - (1-\alpha q^n)H_{n-1}^{(\alpha)}(x|q).$$

Applying the criterion $A_nA_{n+1}C_n > 0$ from (1.3) we find that $\{H_n^{(\alpha)}(x|q)\}$ is orthogonal if $\alpha < q^{-1}$. Again from Section 2 (see 2.19) we find that these polynomials satisfy the orthogonality relation

$$(4.5)\quad \int_0^{\pi} H_m^{(\alpha)}(\cos\theta|q)H_n^{(\alpha)}(\cos\theta|q)(\sin^2\theta)|\phi(0,0,qe^{-2i\theta};q,\alpha)|^{-2}d\theta = \frac{\pi}{2}\frac{(\alpha q^{n+1};q)_\infty(1-\alpha)^2}{(q;q)_n{}^2}\delta_{mn}.$$

4.2 *The associated continuous q-Laguerre polynomials.* The continuous

q-analogue of the Laguerre polynomials is defined by the recursion

$$(4.6)\quad (1 - q^{n+1})L_{n+1}(x; a, b|q) = [2x - q^n(a + b)]L_n(x; a, b|q) - (1 - abq^{n-1})L_{n-1}(x; a, b|q),\ n = 0, 1, \ldots$$

with $L_{-1}(x; a, b|q) = 0$, $L_0(x; a, b|q) = 1$.

Comparison of (4.6) with the recurrence relation satisfied by the Laguerre polynomials $\{L_n^\alpha(x)\}$ [**26**] yields

$$(-1)^n L_n\left[1 - \frac{x}{2}(1 - q); a, b|q\right] \to L_n^\alpha(x) \quad \text{as } q \to 1^-$$

where $\alpha = a + b - 1$. The polynomials $L_n(x; a, b|q)$ are a special case of the ${}_4\phi_3$ polynomials studied in [**7**]. We define the associated polynomials by

$$(4.7)\quad (1 - \alpha q^{n+1})\, L_{n+1}^{(\alpha)}(x; a, b|q) = [2x - \alpha q^n(a + b)]L_n^{(\alpha)}(x; a, b|q) - (1 - ab\alpha q^{n-1})L_{n-1}^{(\alpha)}(x; a, b|q),\ n = 0, 1, \ldots$$

where $L_{-1}^{(\alpha)} = 0$ and $L_0^{(\alpha)} = 1$.

In the present case the criterion $A_nA_{n+1}C_{n+1} > 0$ given in (1.3) is

$$(1 - ab\alpha q^{n-1})(1 - \alpha q^n) > 0,\ n = 0, 1, \ldots.$$

So the associated continuous q-Laguerre polynomials are orthogonal with respect to a positive measure if and only if the above positivity condition holds. Next we derive a generating function. Let

$$(4.8)\quad F(x, t) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x; a, b|q)t^n.$$

Multiplying (4.7) by t^{n+1} and summing on n yields

$$(4.9)\quad F(x, t) = (1 - 2xt + t^2)^{-1}\{1 - \alpha + \alpha[1 - (a + b)t + abt^2]\, F(x; qt)\}.$$

Iteration of (4.9) yields

$$(4.10)\quad F(x, t) = (1 - \alpha)\sum_{k=0}^{\infty} \frac{(at; q)_k(bt; q)_k}{(tA; q)_{k+1}(tB; q)_{k+1}}\alpha^k, \quad |\alpha| < 1,$$

since $F(q^n x, t) \to F(0, t) = 1$ as $n \to \infty$, and where A and B are as defined in Section 2.1. In the notation of basic hypergeometric functions,

$$(4.11)\quad \sum_{n=0}^{\infty} L_n^{(\alpha)}(x; a, b|q)t^n = (1 - \alpha)(1 - 2xt + t^2)^{-1}\, {}_3\phi_2\left(\begin{matrix} at, bt, q \\ qAt, qBt \end{matrix}; q, \alpha\right).$$

4.3 *Asymptotic formulas and the measure.* The generating function (4.11) is analytic in $|t| < 1/|A|$. For $x \notin [-1, 1]$ Darboux's method

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yields as $n \to \infty$,

$$(4.12) \quad L_{n+1}^{(\alpha)}(x; a, b|q) \sim \frac{1-\alpha}{1 - B/A} A^n \phi(a/A, b/A; qB/A; q, \alpha).$$

For $x \in (-1, 1)$ Darboux's method gives as $n \to \infty$

$$(4.13) \quad L_n^{(\alpha)}(\cos\theta; a, b|q) \sim N(\theta) \frac{\cos[(n+1)\theta + \psi]}{\sin\theta},$$

where $0 < \theta < \pi$, $N(\theta) \geqq 0$, and where

$$(4.14) \quad N(\theta)e^{i\psi} = i(\alpha - 1)\phi(ae^{-i\theta}, be^{-i\theta}; qe^{-2i\theta}; q, \alpha).$$

Since

$$L_n^{(\alpha)*}(x; q, b|q) = 2(1 - \alpha q)^{-1} L_{n-1}^{(\alpha q)}(x; a, b|q)$$

Markoff's theorem gives for $x \in [-1, 1]$

$$(4.15) \quad \int_{-\infty}^{\infty} \frac{d\psi(t)}{x - t} = \frac{2B}{1-\alpha} \frac{\phi(aB, bB; qB^2; q, \alpha q)}{\phi(aB, bB; qB^2; q, \alpha)}.$$

An application of Nevai's theorem, Theorem 1.6, shows that $\psi(t)$ has no discrete mass points in $(-1, 1)$. Since the function in the denominator of (4.15) is positive if $x \notin [-1, 1]$, and $|a| < 1$, $|b| < 1$, $0 < q < 1$, $0 < \alpha < 1$ we conclude that for these ranges of a, b, q, α; $s\psi(t)$ is absolutely continuous with support $[-1, 1]$.

To calculate the measure we use Theorem 1.5. Let

$$\lambda_n = (1 - \alpha q^{n+1})(ab\alpha q; q)_n (1 - \alpha q)^{-1}.$$

Then the polynomials $\lambda_n^{-1/2} L_n^{(\alpha)}(\cos\theta; a, b|q)$ are orthonormal. Nevai's Theorem then yields

$$(4.16) \quad \psi(t) = \frac{2}{\pi} (ab\alpha q; q)_\infty (1 - \alpha q)^{-1} N^{-2}(\theta) \sin\theta,$$

where $N(\theta)$ is defined by (4.14).

4.4 *The associated discrete q-Hermite polynomials.* The discrete q-Hermite polynomials [**17**] are generated by

$$H_{n+1}(x: q) = xH_n(x: q) - q^{n-1}(1 - q^n) H_{n-1}(x: q)$$

and

$$H_0(x:q) = 1, H_1(x:q) = x.$$

The associated polynomials $H_n(x:q)$ are defined by

$$H_0^\alpha(x:q) = 1, H_1(x:q) = x,$$
$$H_{n+1}^\alpha(x:q) = x H_n^\alpha(x:q) - q^{n-1}(1 - q^n\alpha) H_{n-1}^\alpha(x:q),$$
$$0 < q < 1, \alpha q < 1.$$

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It is easy to derive

$$\sum_0^\infty \frac{H_n^\alpha(x;q)}{(\alpha q;q)_n} t^n = (1-\alpha) \sum_0^\infty \frac{\alpha^n (t^2, q^2)_n}{(xt;q)_{n+1}},$$

or equivalently

$$\sum_0^\infty \frac{H_n^\alpha(x:q)}{(\alpha q;q)_n} t^n = \frac{(1-\alpha)}{(1-xt)}\, {}_3\phi_2\left(\begin{matrix} t, -t, q; q, \alpha \\ qxt, 0 \end{matrix}\right).$$

Darboux' method yields

$$\begin{aligned} H_n^\alpha(x:q) &\sim (\alpha;q)_\infty x^n \sum_0^\infty \frac{\alpha^j (x^{-2};q^2)}{(q;q)} \\ &= (\alpha;q)_\infty x^n \phi(x^{-1}, x^{-1}; 0; q, \alpha). \end{aligned}$$

Krein's theorem, [**9**, p. 142] implies that the H_n^α's are orthogonal on a bounded interval with respect to a purely discrete measure, say σ, and the only limit point of the support of σ is the origin. We now come to the Stieltjes transform of the measure. Clearly $(H_n^\alpha(x:q))^*$ is $H_{n-1}^{\alpha q}(x:q)$. Thus

$$\int_{-\infty}^\infty \frac{d\sigma(t)}{x-t} = x^{-1}(1-\alpha)^{-1} \frac{\phi(x^{-1}, x^{-1}; 0; q, q\alpha)}{\phi(x^{-1}, x^{-1}; 0; q, \alpha)}, \quad \alpha q < 1.$$

As functions of α, $\phi(x^{-1}, x^{-1}; 0; q, q\alpha)$ and $\phi(x^{-1}, x^{-1}; 0, q, \alpha)$ satisfy a second order q-difference equation, hence have no common zeros. The point masses of σ are located at the poles of the Stieltjes transform of σ, hence as a function of x, $\phi(x, -x; 0; q, \alpha)$ has only real and simple zeros, as a function of x. Furthermore the only limit points of these zeros are $\pm\infty$. One can also show that when $\alpha q < 1$ then every two consecutive zeros (x's) of $\phi(x, -x; 0; q, \alpha)$ enclose an odd number of zeros of $\phi(x, -x; 0; q, q\alpha)$. For details of this type of argument see [**1**].

Note added in proof. We show that $x = \pm 1$ are not mass points for the associated q-ultraspherical polynomials as follows. From Darboux's method and the generating function we find that

$$C_n^{(\alpha)}(1, \beta|q) \sim (1-\alpha)(n+1)\,{}_2\phi_1\left(\begin{matrix}\beta, \beta \\ q\end{matrix}; q, \alpha\right),$$

$$C_n^{(\alpha)}(-1, \beta|q) \sim (1-\alpha)(n+1)(-1)^{n+1}\,{}_2\phi_1\left(\begin{matrix}\beta, \beta \\ q\end{matrix}; q, \alpha\right).$$

Now if $\{P_n(x)\}$ is a set of orthonormal polynomials then x_0 is a mass point for $\{P_n(x)\}$ if and only if

$$\sum_{n=0}^\infty P_n^2(x_0) < \infty$$

([**23**], pp. 45–46). For the associated q-ultraspherical polynomials we have

$$P_n^2(\pm 1) \sim \frac{[C_n^{(\alpha)}(\pm 1)]^2(\alpha q; q)_\infty}{(1-\alpha\beta)(\alpha\beta^2; q)_\infty} \sim K(n+1)^2$$

where K is constant. Hence ± 1 are not mass points.

The situation is similar for the associated q-Laguerre polynomials. In this case

$$L_n^\alpha(1, a, b|q) \sim (1-\alpha)(n+1)_2\phi_1\left(\begin{matrix} a, b \\ q \end{matrix}; q, \alpha\right),$$

$$L_n^\alpha(-1, a, b|q) \sim (-1)^n(1-\alpha)(n+1)_2\phi_1\left(\begin{matrix} -a, -b \\ q \end{matrix}; q, \alpha\right).$$

Again we find that the orthonormal set satisfies $P_n^2(\pm 1) \sim K(n+1)^2$ where K is constant and hence ± 1 are not mass points for the q-Laguerre associated polynomials.

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