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CARTAN-WHITEHEAD DECOMPOSITION AS ADAMS COCOMPLETION

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Abstract

Deleanu, Frei and Hilton have developed the notion of generalized Adams completion in a categorical context; they have also suggested the dual notion, namely, the Adams cocompletion of an object in a category. In this paper the different stages of the Cartan-Whitehead decomposition of a 0-connected space are shown to be the cocompletions of the space with respect to suitable sets of morphisms.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 18 A 40, 55 P 60, 55 S 45.

1. Adams cocompletion

Let $\mathscr{C}$ be an arbitrary category and S a set of morphisms of $\mathscr{C}$. Let $\mathscr{C}[S^{-1}]$ denote the category of fractions of $\mathscr{C}$ with respect to S and

$$F\colon \mathscr{C} \to \mathscr{C}[S^{-1}]$$

the canonical functor. Let $\mathscr{S}$ denote the category of sets and functions. Then for a given object Y of $\mathscr{C}$,

$$\mathscr{C}[S^{-1}](Y,-)\colon \mathscr{C} \to \mathscr{S}$$

defines a covariant functor. If this functor is representable by an object Y_S of $\mathscr{C}$, that is, if

$$\mathscr{C}[S^{-1}](Y,-) \approx \mathscr{C}(Y_S,-),$$

then Y_S is called the (generalized) Adams cocompletion of Y with respect to the set of morphisms S or simply the S-cocompletion of Y. We shall often refer to Y_S simply as the cocompletion of Y.

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Given a set S of morphisms of $\mathscr{C}$, we define $\bar{S}$, the saturation of S, as the set of all morphisms u in $\mathscr{C}$ such that $F(u)$ is an isomorphism in $\mathscr{C}[S^{-1}]$. Further, S is said to be saturated if $S = \bar{S}$.

Deleanu, Frei and Hilton ((1974), dual of Theorem 1.2) have shown that if the set of morphisms S is saturated then the Adams cocompletion of a space is characterized by a certain couniversal property. In most applications, however, the set of morphisms S is not saturated. We therefore present a stronger version of Deleanu, Frei and Hilton's characterization of Adams cocompletion in terms of a couniversal property.

Proposition 1.1. *Let S be a set of morphisms of $\mathscr{C}$ admitting a calculus of right fractions. Then the object Y_S is the cocompletion of the object Y with respect to S if and only if there exists a morphism e: $Y_S \to Y$ in $\bar{S}$ which is couniversal with respect to morphisms in S: given s: $Z \to Y$ in S, there exists a unique morphism t: $Y_S \to Z$ in $\bar{S}$ such that $st = e$.*

The above proposition turns out to be essentially the dual of Theorem 1.2 (Deleanu, Frei and Hilton (1974)) if we assume S to be saturated; hence the Proposition can be proved by recasting the dual of the proof of Theorem 1.2 (Deleanu, Frei and Hilton (1974)) with minor changes. The details are omitted.

2. Description of the category $\tilde{\mathscr{C}}$

Let $\tilde{\mathscr{C}}$ denote the category of 0-connected based spaces and homotopy classes of based maps. We assume that the category $\tilde{\mathscr{C}}$ is a small $\mathscr{U}$-category. Let S_n denote the set of all maps α in $\tilde{\mathscr{C}}$ which have the following property that α: $A \to B$ is in S_n if and only if α_*: $\pi_k(A) \to \pi_k(B)$ is an isomorphism for $k > n$ and a monomorphism for $k = n$.

Proposition 2.1. *S_n admits a calculus of right fractions.*

Proof. This follows from Theorem 1.3* (Deleanu, Frei and Hilton (1974)).

In fact, the set S_n admits a strong calculus of right fractions.

A set S of morphisms of a small $\mathscr{V}$-category $\mathscr{C}$ admits a strong calculus of right fractions if (i) S admits a calculus of right fractions and (ii) for any set $\{s_i$: $B_i \to A, i \in I, I$ is a $\mathscr{V}$-set$\}$, there exists a commutative completion $\{f_i$: $C \to B_i$, $i \in I\}$ such that $s_i f_i \in S$ for every $i \in I$.

Proposition 2.2. *S_n admits a strong calculus of right fractions.*

PROOF. Let $\{s_i\colon B_i \to A, i \in I\}$ be a given set of morphisms and $I \in \mathscr{U}$. We have a map $A \to P^nA$, where P^nA is the nth Postnikov section of A. Convert this into a fibration; let A_n be its fibre $A_n \xrightarrow{e_n} A \to P^nA$. Considering the exact homotopy sequence of this fibration, we conclude that $\pi_k(A_n) = 0$ for $k \leqslant n$, $\pi_k(A_n) \simeq \pi_k(A)$ for $k > n$. Thus $e_n \in S_n$. Moreover, since $\pi_1(A_n) = 0$, e_n has a lifting f_i

$$\begin{array}{ccc} & & B_i \\ & f_i \nearrow & \\ & & \downarrow s_i \\ A_n & \underset{e_n}{\to} & A \end{array}$$

as shown by the dotted arrow and the proposition is proved.

REMARK 2.3. Note that the morphism $e_n\colon A_n \to A$ is independent of the index i.

3. Cartan-Whitehead decomposition as Adams cocompletion

Now for a given object X in $\tilde{\mathscr{C}}$, let S_X denote the set of morphisms $S_X = \{s\colon Y \to X\colon s \in S_n, Y \text{ is an object of } \tilde{\mathscr{C}}\}$. It has been proved in (Nanda (1980)) that S_X is an element of $\mathscr{U}$. Thus, in view of Proposition 2.2 and Remark 2.3, we have a commutative diagram

$$\begin{array}{ccc} & & Y \\ & f_s \nearrow & \downarrow s \\ X_n & \underset{e_n}{\to} & X \end{array}$$

where $s \in S_X$ is arbitrary, e_n is the map as constructed in Proposition 2.2 and f_s is the lifting of e_n corresponding to s. Observe that (i) $e_n \in S_n$ and (ii) with respect to any $s \in S_X$, e_n has couniversal property. Thus by Proposition 1.1, we obtain the following

THEOREM 3.1 *X_n is the S_n-cocompletion of X. Moreover, $e_n\colon X_n \to X$ is in S_n and X_n is n-connected.*

Since $e_n \in S_n \subset S_{n+1}$, it follows from the couniversal property of e_{n+1} that there exists a unique morphism $\theta_{n+1}\colon X_{n+1} \to X_n$ such that $e_{n+1} = e_n \circ \theta_{n+1}$. The maps $\{\theta_n\}$ can of course be replaced by fibrations in the usual manner. Therefore

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we have a tower of spaces

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
 & & X_{n+1} & & \\
 & & \downarrow \theta_{n+1} & & \\
 & & X_n & \searrow e_{n+1} & \\
 & & \downarrow \theta_n & \searrow e_n & \\
 & & \vdots & & \\
 & & \downarrow & & \\
* & = & X_0 & \xrightarrow[e_0]{} & X
\end{array}
$$

Thus we get the Cartan-Whitehead decomposition of a 0-connected space in $\tilde{\mathscr{C}}$.

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