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AN ANSWER TO A QUESTION OF KEGEL ON SUMS OF RINGS

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ABSTRACT. We construct a ring R which is a sum of two subrings A and B such that the Levitzki radical of R does not contain any of the hyperannihilators of A and B. This answers an open question asked by Kegel in 1964.

Kegel [6] proved that a ring is nilpotent if it is a sum of two nilpotent subrings. Several related results on rings which are sums of their subrings were obtained by a number of authors. We shall mention only a few papers [1], [2], [3], [4], [5], [7], [8], [10], [11], [12]). The aim of this note is to answer another related question which still remains open.

Let R be a ring which is a sum of two subrings R_1 and R_2. In 1964 Kegel asked whether at least one of the hyperannihilators $N(R_1)$ or $N(R_2)$ is contained in the Levitzki radical $L(R)$ ([7, p. 105]). Recall that $L(R)$ is the largest locally nilpotent ideal of R, and the *hyperannihilator* $N(R)$ of R is equal to the union $\bigcup_{\alpha\geq 1} N_\alpha(R)$, where

$$N_1(R) = \{z \in R \mid zR = Rz = 0\};$$
$$N_\lambda(R) = \bigcup_{\alpha<\lambda} N_\alpha(R) \quad \text{for limit ordinals } \lambda;$$
$$N_{\alpha+1}(R)/N_\alpha(R) = N_1\big(R/N_\alpha(R)\big) \quad \text{otherwise.}$$

THEOREM 1. *There exists a ring $R = R_1 + R_2$ such that $L(R)$ does not contain any of the rings $N(R_1)$ and $N(R_2)$.*

PROOF. If F is a semigroup with ideal J, then we write F/J for $(F\cap J)/J$ to simplify the notation. We use a construction similar to the one introduced in [8]. Let $X = \{x_1, x_2, \ldots\}$, $Y = \{y_1, y_2, \ldots\}$, and let F be the free semigroup with the set of free generators $X \cup Y$. For $s \in F$, let $n_x(s)$ ($n_y(s)$) denote the number of letters of s belonging to X (respectively, Y). Let $|s| = n_x(s) + n_y(s)$. Put $h(s) = n_x(s) - n_y(s)$, $G = \{s \in F \mid h(s) > 0\}$, $E = \{s \in F \mid h(s) = 0\}$, $F_1 = G \cup E$, $F_2 = F \setminus G$. Let J be the ideal generated in F by the set

$$Z = \bigcup_{i=1}^{\infty} \{x_i^2 F_1 \cup F_1 x_i^2 \cup y_i^2 F_2 \cup F_2 y_i^2\}.$$

Let $\mathbb{R}$ be the ring of real numbers. Consider the contracted semigroup ring $\mathbb{R}(F/J)$. Then $\mathbb{R}(F/J) = \mathbb{R}(F_1/J) + \mathbb{R}(F_2/J)$.

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It is easily seen that $N_1(R_1)$ is the subring generated by all x_i^2 for $i \geq 1$, because $x_i^2 R_1 = R_1 x_i^2 = 0$ by the definition of Z. Take any $r \in R_1 \setminus N_1(R_1)$. There exists a letter x_j which does not occur in any of the terms of the element r. Then $x_j r \neq 0$. Thus $N_1\big(R_1/N_1(R_1)\big) = 0$. Therefore $N(R_1) = N_1(R_1)$. Similarly, $N(R_2)$ is the subring generated by all y_i^2 for $i \geq 1$.

Consider the ideal K generated in R by $x_1^2 \in N(R_1)$. Put $u = y_1 y_2 y_3 x_1^2 y_4 y_5 y_6$. Then $u \in K$ and $u^k \neq 0$ for any $k > 1$. Hence K is not nil. Therefore $N(R_1) \not\subseteq L(R)$. Similarly, $N(R_2) \not\subseteq L(R)$. ∎

In conclusion we note that an example of a primitive ring which is a sum of two Wedderburn radical subrings was constructed in [9]. This answers negatively all questions asked in [11, Section 2.4] and seriously simplifies the proof of the main theorem of [8] which answered a long standing question considered by several authors.

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