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ON FUNCTIONS ATTRACTING POSITIVE ENTROPY

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Abstract

We examine dynamical systems which are ‘nonchaotic’ on a big (in the sense of Lebesgue measure) set in each neighbourhood of a fixed point x_0, that is, the entropy of this system is zero on a set for which x_0 is a density point. Considerations connected with this family of functions are linked with functions attracting positive entropy at x_0, that is, each mapping sufficiently close to the function has positive entropy on each neighbourhood of x_0.

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1. Introduction and preliminaries

The theory of discrete dynamical systems has wide applications in various fields of knowledge (economics, physics, biology, information flow theory etc). Many mathematicians are of the opinion that the basic criterion for determining whether a dynamical system is chaotic is positive entropy of the system (some heuristic justification can be found in [3]). Key considerations on topological entropy for discrete dynamical systems are based on two classical definitions: the first formulated by Adler *et al.* [1] based on the cover theory for compact spaces and the second given, for suitable metric spaces, by Bowen [4] and Dinaburg [10]. Initially, entropy of a function was considered only for continuous functions. This changed in 2004 when entropy was associated with Darboux-like functions [8]. Continuing this line of research, we will associate the concept of entropy with the theory of approximately continuous functions. This is the basis of much contemporary work, mainly due to a group of scientists gathered around Wilczyński (see, for example, [2, 11]).

The analysis of various examples of functions leads us to the interesting observation that entropy of a function may be focused at one point [15, 18, 20]. In a natural manner, these issues can be linked to the approximate continuity of a function at a given point. In this paper we will concentrate on functions 0-approximately continuous at a given point, that is, functions which are approximately continuous at a point and are not chaotic on some set near this point. In September 2016 during the 30th International Summer Conference on Real Functions Theory in Stará Lesná, Wilczyński, in reference to our lecture connected with such functions, formulated the following problem (see also [19]):

> *How big may be the set of approximate continuity points of a function f which are not its 0-approximate continuity points?*

The paper starts with a partial answer to this question. In the second part of the paper we present additional observations which have emerged while working on the question and broadened the scope of our research.

Throughout the paper we will consider only functions from the closed unit interval $\mathbb{I} = [0, 1]$ into $\mathbb{I}$. If $A \subset \mathbb{I}$ and f is a function, then $f \restriction A$ means the restriction of f to A. The cardinality of A will be denoted by $\#(A)$.

For the reasons given above, we will often limit our considerations to Darboux functions, that is, functions having the intermediate value property. The family of all such functions will be denoted by $\mathfrak{D}$. At the same time, for simplicity of proofs, we will use the concept of a left (right)-hand Darboux point and a Darboux point of a function [6, 7, 14]. We rely on the following properties of these concepts.

(D1) A point x_0 is a Daboux point of a function f if and only if it is simultaneously a left- and a right-hand Darboux point of f (obviously if $x_0 = 0$ or $x_0 = 1$, then we consider only one-sided Darboux points).

(D2) If f is right- (left-) hand continuous at a point x_0, then x_0 is a right- (left-) hand Darboux point of f.

(D3) A function f belongs to $\mathfrak{D}$ if and only if each point of $\mathbb{I}$ is a Darboux point of f.

Since our considerations are closely related to entropy of discrete dynamical systems, we recall the basic facts and definitions related to this concept [8, 12]. Let f be a function, $A \subset \mathbb{I}$, $n \in \mathbb{N}$ and $\varepsilon > 0$. A set $M \subset A$ is (f, A, ε, n)-*separated* if, for each $x, y \in M$ with $x \neq y$, there is an i with $0 \leq i < n$ such that $|f^i(x) - f^i(y)| > \varepsilon$ (where $f^n(x) = f(f^{n-1}(x))$ and $f^0(x) = x$ for $x \in \mathbb{I}$). Let $s(f, A, \varepsilon, n)$ denote the cardinality of an (f, A, ε, n)-separated set with the maximum possible number of points. The *topological entropy of a function f on the set A* is the number

$$h(f, A) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(f, A, \varepsilon, n).$$

To examine local aspects of the entropy of a function, it is useful to consider certain families of sets [13]. We will limit our considerations to the family $\mathcal{F}$ of all closed subsets of $\mathbb{I}$ (cf. Lemma 2.1).

Another essential notion is the concept of a density point. Let $\mathcal{L}$ denote the σ-algebra of all Lebesgue measurable sets and λ the Lebesgue measure. For any $x_0 \in \mathbb{I}$ and $A \in \mathcal{L}$, if the limit

$$\lim_{h \to 0^+} \frac{\lambda(A \cap [x_0 - h, x_0 + h])}{2h}$$

exists, we call it the *density of the set A at the point* x_0 and denote it by $d(A, x_0)$. Similarly, we define right-hand and left-hand densities of a set at a point, denoted by $d_+(A, x_0)$ and $d_-(A, x_0)$, respectively. If $x_0 = 0$ or $x_0 = 1$, then we consider a suitable one-sided density. If $d(A, x_0) = 1$, then we say that x_0 is a *density point of a set A*.

Interval sets at a point play a special role. Let $x_0 \in (0, 1)$ and $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}, \{c_n\}_{n\in\mathbb{N}}, \{d_n\}_{n\in\mathbb{N}} \subset \mathbb{I}$ be sequences such that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = x_0$ and $c_n < d_n < c_{n+1}$ and $b_{n+1} < a_n < b_n$ for any $n \in \mathbb{N}$. An *interval set at a point* x_0 is a set given by the formula

$$\bigcup_{n=1}^{\infty} L_n \cup \{x_0\} \cup \bigcup_{n=1}^{\infty} R_n,$$

where $L_n = [c_n, d_n]$ and $R_n = [a_n, b_n]$ for $n \in \mathbb{N}$. Similarly, we define a right- (left-) hand interval set at a point $x_0 \in \mathbb{I}$. To shorten notation, for $x_0 = 0$ or $x_0 = 1$, an interval set at x_0 means a right-hand or left-hand interval set at this point, respectively.

Finally, we recall the Lusin–Menchoff theorem, which will be useful in the next part of the paper.

THEOREM 1.1 [5]. *Let $E \in \mathcal{L}$ and S be a closed subset of E such that $d(E, x) = 1$ for $x \in S$. There exists a perfect set F such that $S \subset F \subset E$ and $d(F, x) = 1$ for $x \in S$.*

2. 0-approximately continuous functions attracting positive entropy

Approximately continuous functions were introduced in 1915 by Denjoy [9]. The idea is that such a function is continuous on a big (in the sense of measure) set in every neighbourhood of a given point. We will also require that the function not be chaotic on this big set (that is, the entropy of the function on this set is zero) and on the other hand it 'attracts chaos'. As mentioned earlier, the widest class of functions to which the concept of entropy commonly applies is the family $\mathfrak{D}$, so our considerations will often be limited to functions from this class.

Let us start with the definitions. We shall say that f is *approximately continuous* (0-*approximately continuous*) at $x_0 \in \mathbb{I}$ if there exists a set $A \in \mathcal{L}$ such that $d(A, x_0) = 1$ and $\lim_{A\ni x\to x_0} f(x) = f(x_0)$ (and, moreover, $h(f, A) = 0$).

Let $f \in \mathfrak{D}$ and $x_0 \in \mathbb{I}$. We say that f *attracts positive entropy at a point* x_0 if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each function $g \in B_{\rho_u}(f, \delta) \cap \mathfrak{D}$ we have $h(g, B(x_0, \varepsilon)) > 0$. Here $B(x_0, \varepsilon)$ is the open ball with centre x_0 and radius ε with respect to the natural metric and $B_{\rho_u}(f, \delta)$ is the open ball with centre f and radius δ with respect to the metric of uniform convergence.

Obviously, if $f \in \mathfrak{D}$ is such that $h(f, \mathbb{I}) = 0$, then f does not attract positive entropy at any point $x_0 \in \mathbb{I}$. Moreover, even if a function has positive entropy, it may not attract positive entropy at any point. We see at once that there exists a function which is 0-approximately continuous at each point of $\mathbb{I}$. Much more interesting is the problem of the existence of a function approximately continuous at a point $x_0 \in \mathbb{I}$, which is not 0-approximately continuous at this point. Wilczyński's problem, quoted in the introduction, arises in this context. Given the above definitions, this problem should be extended to functions attracting positive entropy at a point.

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Before we state our main results, we formulate some lemmas which will be useful in the next part of the paper. The first two lemmas are easily derived from the considerations in the paper [13] (Lemma 2.1) and the monograph [17] (Lemma 2.2).

Lemma 2.1. *Let $Y \subset \mathbb{I}$ and let $F_0, F_1 \subset Y$ be closed and disjoint sets such that $F_i \xrightarrow[f]{} F_j$ (that is, $F_j \subset f(F_i)$) for $i, j \in \{0, 1\}$. Then $h(f, Y) > 0$.*

Lemma 2.2. *For any set $A \in \mathcal{L}$ with positive Lebesgue measure, there exists a set $B \subset A$ having the cardinality of the continuum and Lebesgue measure zero.*

Lemma 2.3. *Let $[a, b]$ be a nondegenerate closed interval. There exists a function $\Phi_{[a,b]} : [a, b] \to [a, b]$ such that $\Phi_{[a,b]}(a) = a$, $\Phi_{[a,b]}(b) = b$ and $\Phi_{[a,b]}(A) = [a, b]$ for any $A \subset [a, b]$ such that $\lambda(A) > 0$.*

Proof. If $[a, b]$ is a nondegenerate closed interval, then $a < b$. Let $\mathfrak{B}_{(a,b)}$ be the family of all Borel sets contained in (a, b) and having positive Lebesgue measure. Obviously, $\#(\mathfrak{B}_{(a,b)}) = \mathfrak{c}$.

By the well-ordering principle, the family $\mathfrak{B}_{(a,b)}$ can be indexed by the ordinal numbers less than $\omega_{\mathfrak{c}}$, where $\omega_{\mathfrak{c}}$ is the first ordinal having $\mathfrak{c}$ predecessors. That is, $\mathfrak{B}_{(a,b)} = \{B_\alpha\}_{\alpha<\omega_{\mathfrak{c}}}$. We now proceed by transfinite induction and construct the family $\{C_\alpha\}_{\alpha<\omega_{\mathfrak{c}}}$ such that

$$\#(C_\alpha) = \mathfrak{c} \quad \text{and} \quad C_\alpha \subset \Big(B_\alpha \setminus \bigcup_{\beta<\alpha} C_\beta\Big) \quad \text{and} \quad \lambda(C_\alpha) = 0, \tag{2.1}$$

where, in addition, $\bigcup_{\beta<\alpha} C_\beta = \emptyset$ for $\alpha = 0$.

By Lemma 2.2, there exists a set $C_0 \subset B_0$ such that $\#(C_0) = \mathfrak{c}$ and $\lambda(C_0) = 0$. Let $\alpha < \omega_{\mathfrak{c}}$. Suppose that we have already defined C_β for $0 \le \beta < \alpha$ such that

$$\#(C_\beta) = \mathfrak{c} \quad \text{and} \quad C_\beta \subset \Big(B_\beta \setminus \bigcup_{\gamma<\beta} C_\gamma\Big) \quad \text{and} \quad \lambda(C_\beta) = 0.$$

Since $\alpha < \omega_{\mathfrak{c}}$, we have $\lambda(\bigcup_{\beta<\alpha} C_\beta) = 0$, which gives $\lambda(B_\alpha \setminus \bigcup_{\beta<\alpha} C_\beta) = \lambda(B_\alpha) > 0$. By Lemma 2.2, there exists $C_\alpha \subset B_\alpha \setminus \bigcup_{\beta<\alpha} C_\beta$ such that $\#(C_\alpha) = \mathfrak{c}$ and $\lambda(C_\alpha) = 0$, which finishes the proof of (2.1).

Clearly, by the well-ordering principle, each set C_α ($\alpha < \omega_{\mathfrak{c}}$) can be represented as a transfinite sequence $\{x_\gamma^{(\alpha)}\}_{\gamma<\omega_{\mathfrak{c}}}$. Let $A_0 = \{x_0^{(\alpha)} : \alpha < \omega_{\mathfrak{c}}\} \cup ((a, b) \setminus \bigcup_{\alpha<\omega_{\mathfrak{c}}} C_\alpha)$. Moreover, for each $\beta < \omega_{\mathfrak{c}}$ and $\beta > 0$, put $A_\beta = \{x_\beta^{(\alpha)} : \alpha < \omega_{\mathfrak{c}}\}$. Then the family $\mathcal{A} = \{A_\alpha : \alpha < \omega_{\mathfrak{c}}\}$ has the cardinality of the continuum and $A_{\alpha_1} \cap A_{\alpha_2} = \emptyset$ for any $\alpha_1, \alpha_2 < \omega_{\mathfrak{c}}$ and $\alpha_1 \neq \alpha_2$. What is more, $\bigcup_{\alpha<\omega_{\mathfrak{c}}} A_\alpha = (a, b)$.

For each $x \in (a, b)$, let α_x denote the ordinal number such that $x \in A_{\alpha_x}$. Obviously, there is a bijection $\varphi : \mathcal{A} \to [a, b]$. Now we define a function $\Phi_{[a,b]} : [a, b] \to [a, b]$ in the following way: $\Phi_{[a,b]}(a) = a$, $\Phi_{[a,b]}(b) = b$ and $\Phi_{[a,b]}(x) = \varphi(A_{\alpha_x})$ for $x \in (a, b)$. To see that $\Phi_{[a,b]}$ is the required function, it is sufficient to prove that $\Phi_{[a,b]}(A) = [a, b]$ for any $A \subset [a, b]$ such that $\lambda(A) > 0$.

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Let $A \subset [a,b]$ be such that $\lambda(A) > 0$. Since $\lambda(A) = \lambda(A \setminus \{a,b\})$, there is no loss of generality in assuming that $A \subset (a,b)$. Moreover, there is a Borel set $B \subset A$ such that $\lambda(B) > 0$.

Fix $y \in [a,b]$. Clearly, there exists $\alpha_y < \omega_{\mathfrak{c}}$ such that $\varphi(A_{\alpha_y}) = y$. Let $\alpha_B < \omega_{\mathfrak{c}}$ be such that $B = B_{\alpha_B}$. Since $C_{\alpha_B} \subset B_{\alpha_B} = B$ and $C_{\alpha_B} = \{x_\gamma^{(\alpha_B)} : \gamma < \omega_{\mathfrak{c}}\}$, we see that $x_{\alpha_y}^{(\alpha_B)} \in A_{\alpha_y} \cap B \subset A$ and, in consequence, $\Phi_{[a,b]}(x_{\alpha_y}^{(\alpha_B)}) = \varphi(A_{\alpha_y}) = y$. Since y is arbitrary, it follows that $[a,b] \subset \Phi_{[a,b]}(A)$ and, in consequence, $[a,b] = \Phi_{[a,b]}(A)$. □

If $[a,b]$ is a nondegenerate closed interval, then the function $\Phi_{[a,b]} : [a,b] \to [a,b]$ from Lemma 2.3 is a Darboux function which is discontinuous at each point of the interval $[a,b]$.

The next theorem is not only related to Wilczyński's problem (see the note following the proof of the theorem), but it is also related to the problem of finding functions attracting positive entropy at a point.

Theorem 2.4. *There exists a function $f \in \mathfrak{D}$ such that f is continuous at every point of the Cantor set $\mathfrak{C}$, $h(f,A) > 0$ for any set $A \subset \mathbb{I}$ such that $\lambda(A) > 0$, and f attracts positive entropy at every point $x \in \mathbb{I}$.*

Proof. Let $\mathfrak{S}$ be the set of all components of the set $\mathbb{I} \setminus \mathfrak{C}$. Clearly, for any interval $P \in \mathfrak{S}$, there exist a strictly decreasing sequence $\{a_n^P\}_{n \in \mathbb{N}} \subset P$ and a strictly increasing sequence $\{b_n^P\}_{n \in \mathbb{N}} \subset P$ such that $\lim_{n\to\infty} a_n^P = \inf P$, $\lim_{n\to\infty} b_n^P = \sup P$ and $b_1^P - a_1^P > 0$.

For any $P \in \mathfrak{S}$, we define the function $f_P : P \to P$ as follows: $f_P(x) = \Phi_Q(x)$ for $x \in Q \in \{[a_1, b_1]\} \cup \{[a_{n+1}^P, a_n^P] : n \in \mathbb{N}\} \cup \{[b_n^P, b_{n+1}^P] : n \in \mathbb{N}\}$, where Φ_Q is the function from Lemma 2.3.

Next put $f(x) = x$ for $x \in \mathfrak{C}$ and $f(x) = f_{P_x}(x)$ for $x \in \mathbb{I} \setminus \mathfrak{C}$, where P_x is the component of the set $\mathbb{I} \setminus \mathfrak{C}$ containing x.

We will show that f is the required function. For that, we will prove first that

$$C(f) = \mathfrak{C}, \tag{2.2}$$

where $C(f)$ denotes the set of all continuity points of f.

Let $x_0 \in \mathfrak{C}$. We will show that f is right-hand continuous at x_0 (for left-hand continuity the proof runs in a similar way). Obviously, in this case we have $x_0 \neq 1$ and $f(x_0) = x_0$. Let $\varepsilon \in (0, \min\{x_0, 1 - x_0\})$.

Assume first that there is $P_0 \in \mathfrak{S}$ such that $x_0 = \inf P_0$. Obviously, one can find $n_1 \in \mathbb{N}$ such that $a_{n_1}^{P_0} < x_0 + \varepsilon$. Put $\delta = a_{n_1}^{P_0} - x_0$, so that $\delta \le \varepsilon$. If $x \in (x_0, x_0 + \delta)$, then $x \in [a_{n+1}^{P_0}, a_n^{P_0}]$ for some $n \ge n_1$. Thus, $f(x) = f_{P_0}(x) \in [a_{n+1}^{P_0}, a_n^{P_0}]$, so $f(x) \in [x_0, x_0 + \varepsilon)$ and, in consequence, $f([x_0, x_0 + \delta)) \subset [x_0, x_0 + \varepsilon)$.

Assume now that $x_0 \neq \inf P$ for every $P \in \mathfrak{S}$ and consider the point $x_0 + \varepsilon \in \mathbb{I}$. If $x_0 + \varepsilon \in \mathfrak{C}$, then put $\delta_1 = \varepsilon$. If $x_0 + \varepsilon \notin \mathfrak{C}$, then put $\delta_1 = \min\{\varepsilon, \inf P_{x_0+\varepsilon} - x_0\}$, where $P_{x_0+\varepsilon} \in \mathfrak{S}$ and $x_0 + \varepsilon \in P_{x_0+\varepsilon}$. Obviously, $\delta_1 \le \varepsilon$. Moreover, if $P \in \mathfrak{S}$ and $P \cap (x_0, x_0 + \delta_1) \neq \emptyset$, then $P \subset (x_0, x_0 + \varepsilon)$. Now it is easy to see that $f([x_0, x_0 + \delta_1)) \subset [x_0, x_0 + \varepsilon)$.

Thus, we have proved that $\mathfrak{C} \subset C(f)$. Taking into account the remark made after the proof of Lemma 2.3, we immediately obtain $\mathfrak{C} = C(f)$.

Now we prove that

$$f \in \mathfrak{D}. \tag{2.3}$$

If $x \in \mathfrak{C}$, then, by (2.2) and condition (D2), x is a Darboux point of f. If $x \notin \mathfrak{C}$, then there is $P \in \mathfrak{S}$ such that $x \in P$. From the construction of f and the remark made after the proof of Lemma 2.3, it follows that also in this case x is a Darboux point of f. Condition (D3) gives (2.3).

Now we show that

$$h(f, A) > 0 \text{ for any set } A \text{ such that } \lambda(A) > 0. \tag{2.4}$$

Let A be as in (2.4) and let $F \subset A$ be a closed set such that $\lambda(F) > 0$. Obviously, there is $P \in \mathfrak{S}$ such that $\lambda(P \cap F) > 0$. Without loss of generality, we can assume that $\lambda([a^P_{n_0+1}, a^P_{n_0}] \cap F) > 0$ for some $n_0 \in \mathbb{N}$. Thus, there are $c, d \in (a^P_{n_0+1}, a^P_{n_0})$ such that $c < d$, $\lambda([a^P_{n_0+1}, c] \cap F) > 0$ and $\lambda([d, a^P_{n_0}] \cap F) > 0$. Now put $F_0 = [a^P_{n_0+1}, c] \cap F$ and $F_1 = [d, a^P_{n_0}] \cap F$. Clearly, F_0, F_1 are closed disjoint sets. Since $f(F_0) = [a^P_{n_0+1}, a^P_{n_0}]$, $f(F_1) = [a^P_{n_0+1}, a^P_{n_0}]$ and $F_0 \cup F_1 \subset [a^P_{n_0+1}, a^P_{n_0}]$, we see that $F_i \xrightarrow[f]{} F_j$ for $i, j \in \{0, 1\}$. Therefore, Lemma 2.1 gives (2.4).

In order to complete the proof, we will show that

$$f \text{ attracts positive entropy at every point } x \in \mathbb{I}. \tag{2.5}$$

Let $x_0 \in \mathbb{I}$ and $\varepsilon > 0$. Assume first that $x_0 \in \mathfrak{C}$. Let $P \in \mathfrak{S}$ be such that $P \subset B(x_0, \varepsilon)$. Put $\delta = \frac{1}{3}\lambda([a^P_1, b^P_1])$ and consider $g \in B(f, \delta) \cap \mathfrak{D}$, $c_1 = a^P_1 + \delta$ and $c_2 = b^P_1 - \delta$. Let F_0, F_1 be closed disjoint sets with positive Lebesgue measure such that $F_0 \cup F_1 \subset [c_1, c_2]$. Naturally, $f(F_i) = [a^P_1, b^P_1]$ for $i = 0, 1$. Thus, one can find $x^i_1 \in F_i$ and $x^i_2 \in F_i$ such that $f(x^i_1) = a^P_1$ and $f(x^i_2) = b^P_1$ for $i = 0, 1$. Hence, $g(x^i_1) < f(x^i_1) + \delta = a^P_1 + \delta = c_1$ and $g(x^i_2) > f(x^i_2) - \delta = b^P_1 - \delta = c_2$ for $i = 0, 1$. Since $g \in \mathfrak{D}$, we have $[c_1, c_2] \subset g([x^i_1, x^i_2])$ if $x^i_1 < x^i_2$ or $[c_1, c_2] \subset g([x^i_2, x^i_1])$ if $x^i_2 < x^i_1$. Finally, $F_i \subset [c_1, c_2] \subset g(F_j)$ for $i, j = 0, 1$, which means that $F_i \xrightarrow[g]{} F_j$ for $i, j = 0, 1$. By Lemma 2.1, we conclude that $h(g, B(x_0, \varepsilon)) > 0$, which proves (2.5) in this case.

Now assume that $x_0 \notin \mathfrak{C}$. Let $P \in \mathfrak{S}$ be such that $x_0 \in P$. There is no loss of generality in assuming that there is $n_0 \in \mathbb{N}$ such that $x_0 \in [a^P_{n_0+1}, a^P_{n_0}]$. Put $\delta = \frac{1}{3}(a^P_{n_0} - a^P_{n_0+1})$. Let $g \in B(f, \delta) \cap \mathfrak{D}$ and $d_1, d_2 \in [a^P_{n_0+1} + \delta, a^P_{n_0} - \delta] \cap B(x_0, \varepsilon)$ be such that $d_1 < d_2$. There are closed and disjoint sets F_0, F_1 having a positive Lebesgue measure and such that $F_0 \cup F_1 \subset [d_1, d_2]$. We see at once that $f(F_i) = [a^P_{n_0+1}, a^P_{n_0}]$ for $i = 0, 1$. An argument similar to that in the proof of the first case establishes (2.5). □

Referring to Wilczyński's problem mentioned at the beginning of the paper, the classical Cantor set is an uncountable set consisting of continuity (so also approximate continuity) points of the function f from the above theorem. On the other hand, none of these points is a point of 0-approximate continuity of f. (In fact, the function f is not 0-approximately continuous at any point.) The problem of the existence of a set A of positive Lebesgue measure and a function g such that every point of A is a point of approximate continuity of g and is not a point of 0-approximate continuity of g is still open.

3. Attracting positive entropy in some equivalence classes

Many papers connected with the entropy of a function are related to functional structures (for example, semi-groups, envelopes etc). We will relate our considerations to the equivalence classes with respect to relations based on density points. We first define these equivalence relations.

We shall say that two functions f and g are *approximately equal at a point* $x_0 \in \mathbb{I}$ if there is a set $M \in \mathcal{L}$ such that $x_0 \in M$, $d(M, x_0) = 1$ and $M \subset \{x \in \mathbb{I} : f(x) = g(x)\}$. Equivalently: f and g are approximately equal at a point x_0 if there is $A \in \mathcal{L}$ such that $x_0 \notin A$, $d(A, x_0) = 0$ and $\mathrm{diff}(f, g) = \{x \in \mathbb{I} : f(x) \neq g(x)\} \subset A$.

Let $\varepsilon > 0$. We shall say that two functions f and g are *approximately ε-close at a point* $x_0 \in \mathbb{I}$ if there is a set $M \in \mathcal{L}$ such that $d(M, x_0) = 1$ and $\rho_u(f \restriction M, g \restriction M) < \varepsilon$ (where ρ_u is the metric of uniform convergence).

Let us now introduce two equivalence relations related to the above definitions. Let f, g be functions.

(1) $f \underset{x_0}{\simeq} g$ if and only if f and g are approximately equal at a point x_0.

(2) $f \underset{x_0}{\asymp} g$ if and only if for any $\varepsilon > 0$ the functions f and g are approximately ε-close at a point x_0.

From now on the symbol $[f]_{x_0}$ will stand for the equivalence class of f with respect to the relation $\underset{x_0}{\simeq}$ and the symbol $\langle f \rangle_{x_0}$ for the equivalence class of f with respect to the relation $\underset{x_0}{\asymp}$. We see at once that $[f]_{x_0} \subset \langle f \rangle_{x_0}$ for any f. We note two basic properties.

Property 3.1.

(a) *A function f is approximately continuous (0-approximately continuous) at a point $x_0 \in \mathbb{I}$ if and only if each function $g \in [f]_{x_0}$ is approximately continuous (0-approximately continuous) at x_0.*

(b) *For any function f and any $x_0 \in \mathbb{I}$, there exists $g \in \langle f \rangle_{x_0}$ such that g is not approximately continuous (so also is not 0-approximately continuous) at the point x_0. Moreover, if $f \in \mathfrak{D}$, then $g \in \mathfrak{D}$.*

It is easy to see that we cannot replace approximate continuity with continuity in (a).

Proof. In both cases it suffices to prove necessity.

(a) Assume that f is approximately continuous (0-approximately continuous) at a point x_0. Then there exists a set $B \in \mathcal{L}$ such that $d(B, x_0) = 1$ and $\lim_{B \ni x \to x_0} f(x) = f(x_0)$ (and, moreover, $h(f, B) = 0$). Let $g \in [f]_{x_0}$. Thus, $f \underset{x_0}{\simeq} g$, so one can find a set $A \in \mathcal{L}$ such that $x_0 \notin A$, $d(A, x_0) = 0$ and $\mathrm{diff}(f, g) \subset A$. We see at once that $g(x_0) = f(x_0)$. With $C = B \setminus A$, we see that $d(C, x_0) = 1$ and $\lim_{C \ni x \to x_0} f(x) = f(x_0)$ (and, moreover, $h(g, C) = 0$). Thus, g is approximately continuous (0-approximately continuous) at x_0.

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(b) Let $x_0 \in \mathbb{I}$. We need only consider the case where f is approximately continuous at x_0. Let $P = \bigcup_{n=1}^{\infty} L_n \cup \{x_0\} \cup \bigcup_{n=1}^{\infty} R_n$, where $L_n = [c_n, d_n]$ and $R_n = [a_n, b_n]$ for $n \in \mathbb{N}$, be an interval set at x_0 such that $d(P, x_0) = 1$ (if $x_0 = 0$ or $x_0 = 1$, then we consider a one-sided interval set).

Fix $\beta \in \mathbb{I}\backslash\{f(x_0)\}$. Define a function g by $g(x_0) = \beta$, $g(x) = f(x)$ for $x \in P\backslash\{x_0\}$, $g(x) = 0$ for $x \in \{\frac{1}{3}(2b_{n+1} + a_n) : n \in \mathbb{N}\} \cup \{\frac{1}{3}(2c_{n+1} + d_n) : n \in \mathbb{N}\}$ and $g(x) = 1$ for $x \in \{\frac{1}{3}(b_{n+1} + 2a_n) : n \in \mathbb{N}\} \cup \{\frac{1}{3}(c_{n+1} + 2d_n) : n \in \mathbb{N}\}$, and let g be linear on each interval $[b_{n+1}, \frac{1}{3}(2b_{n+1} + a_n)]$, $[\frac{1}{3}(2b_{n+1} + a_n), \frac{1}{3}(b_{n+1} + 2a_n)]$, $[\frac{1}{3}(b_{n+1} + 2a_n), a_n]$, $[d_n, \frac{1}{3}(c_{n+1} + 2d_n)]$, $[\frac{1}{3}(c_{n+1} + 2d_n), \frac{1}{3}(2c_{n+1} + d_n)]$ and $[\frac{1}{3}(2c_{n+1} + d_n), c_{n+1}]$. If $x_0 \neq 0$ ($x_0 \neq 1$), then, in addition, put $g(x) = f(x)$ for $x \in [0, c_1)$ ($g(x) = f(x)$ for $x \in (b_1, 1]$).

It is easy to see that $g \in \langle f\rangle_{x_0}$. Moreover, if we assume in addition that $f \in \mathfrak{D}$, then, by (D1)–(D3), we obtain immediately that $g \in \mathfrak{D}$. □

The next theorem establishes the interesting fact that each equivalence class $[f]_{x_0}$ for Darboux functions contains a function that attracts positive entropy at x_0.

Theorem 3.2. *If $f \in \mathfrak{D}$ and $x_0 \in \mathbb{I}$, then there exists $g \in [f]_{x_0}$ which attracts positive entropy at the point x_0.*

Proof. Let $x_0 \in \mathbb{I}$ and let $P = \bigcup_{n=1}^{\infty} L_n \cup \{x_0\} \cup \bigcup_{n=1}^{\infty} R_n \subset \mathbb{I}$ be an interval set at x_0 such that $d(P, x_0) = 0$. If $x_0 \in \{0, 1\}$, then as usual we consider a one-sided interval set. Without restriction of generality, we can assume that for any $n \in \mathbb{N}$ there is a closed interval $H_n = [a_n^*, b_n^*] \subset (\inf R_n, \sup R_n)$. For each $n \in \mathbb{N}$, fix points $c_n^*, d_n^*, s_n^*, t_n^* \in H_n$ such that $a_n^* < c_n^* < d_n^* < s_n^* < t_n^* < b_n^*$. We define the function $g : \mathbb{I} \to \mathbb{I}$ in the following way: $g(x) = f(x)$ for $x \in \mathbb{I}\backslash \operatorname{int}(\bigcup_{n=1}^{\infty} R_n)$, $g(x) = 0$ for $x \in \{a_n^*, d_n^*, t_n^*\}, n \in \mathbb{N}$, $g(x) = 1$ for $x \in \{b_n^*, c_n^*, s_n^*\}, n \in \mathbb{N}$ and g is linear otherwise. It is easy to see that $f \underset{x_0}{\simeq} g$. Moreover, by (D1)–(D3), we see at once that $g \in \mathfrak{D}$. Now we will show that

$$g \text{ attracts positive entropy at } x_0.$$

Let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $R_{n_0} \subset B(x_0, \varepsilon)$. Put $\delta = \min\{1 - t_{n_0}^*, c_{n_0}^*\}$ and consider $g_* \in B(g, \delta) \cap \mathfrak{D}$. If $F_0 = [c_{n_0}^*, d_{n_0}^*]$ and $F_1 = [e_{n_0}^*, t_{n_0}^*]$, then, since $g_* \in \mathfrak{D}$, it is easy to see that $F_i \underset{g_*}{\to} F_j$ for $i, j = 0, 1$. From that and Lemma 2.1, we conclude that $h(g_*, B(x_0, \varepsilon)) > 0$. Since $\varepsilon > 0$ is arbitrary, it follows that g attracts positive entropy at x_0. □

Since $[f]_{x_0} \subset \langle f\rangle_{x_0}$, we can replace the equivalence class $[f]_{x_0}$ by $\langle f\rangle_{x_0}$ in the above theorem. It is interesting to ask about the existence of a function f such that the equivalence class $\langle f\rangle_{x_0}$ consists only of functions attracting positive entropy at x_0. Note that such a function f cannot be approximatively continuous at x_0 (it is sufficient to consider the constant function assuming the value $f(x_0)$). Taking into account this observation and the fact that each approximately continuous function $f: \mathbb{I} \to \mathbb{I}$ is a derivative function (that is, a derivative of some function), the following theorem seems to be particularly interesting.

Theorem 3.3. *There exist a derivative function f and $x_0 \in \mathbb{I}$ such that $f(x_0) = x_0$ and, if $g \in \langle f\rangle_{x_0}$, then g attracts positive entropy at x_0.*

PROOF. Fix $x_0 = \frac{1}{2}$. Let A, B be interval sets at x_0 such that $d(A, x_0) = \frac{1}{2} = d(B, x_0)$, $d_+(A, x_0) = \frac{1}{2} = d_+(B, x_0)$ and $A \cap B = x_0$. (Such sets are constructed, for example, in the proof of Theorem 5.5(d) in [5, Ch. 2].) Let $f : \mathbb{I} \to \mathbb{I}$ be a function satisfying the following conditions: $f(x_0) = x_0$, $f(x) = 1$ for $x \in A \backslash \{x_0\}$, $f(x) = 0$ for $x \in B \backslash \{x_0\}$ and $C(f) = \mathbb{I} \backslash \{x_0\}$ (where $C(f)$ denotes the set of all continuity points of f). Such a function is a derivative function that is not approximately continuous at x_0 (see [5, page 35]). Note that if $\sigma > 0$ and $D \in \mathcal{L}$ is such that $d(D, x_0) = 1$, then

$$\text{there exist } x_A^{D,\sigma} \in A \cap D \cap (x_0, x_0 + \sigma) \text{ and } x_B^{D,\sigma} \in B \cap D \cap (x_0, x_0 + \sigma). \tag{3.1}$$

Let $g \in \langle f \rangle_{x_0}$. Choose $\varepsilon \in (0, \frac{1}{4})$ and $\delta = \frac{1}{8}$. There exists $C \in \mathcal{L}$ such that $d(C, x_0) = 1$ and $\rho_u(f \restriction C, g \restriction C) < \delta$. Then (3.1) implies that there are $x_1, x_2, x_3, x_4 \in (\frac{1}{2}, \frac{1}{2} + \varepsilon)$ such that $x_4 < x_3 < x_2 < x_1$ and $x_1, x_3 \in A \cap C$ and $x_2, x_4 \in B \cap C$.

Consider $g_* \in B_{\rho_u}(g, \delta) \cap \mathfrak{D}$. Putting $F_0 = [x_2, x_1]$ and $F_1 = [x_4, x_3]$, we obtain immediately that $F_i \underset{g_*}{\to} F_j$ for $i, j = 0, 1$. Therefore, by Lemma 2.1, we conclude that $h(g_*, B(x_0, \varepsilon)) > 0$, which implies that g attracts positive entropy at x_0. □

4. 0-approximate stable points of a function

The concept of a stable point of a function or dynamical system is considered in [15, 16]. Obviously if x_0 is a stable point of f then x_0 is a fixed point of f (the set of all fixed points of f will be denoted by $\mathrm{Fix}(f)$). We introduce an analogous concept.

We say that $x_0 \in \mathbb{I}$ is a 0-*approximate stable point* of f if $x_0 \in \mathrm{Fix}(f)$ and there exists a set $A \in \mathcal{L}$ such that $d(A, x_0) = 1$, $h(f, A) = 0$ and, for any $\varepsilon > 0$, there is $\delta > 0$ such that for each $n \in \mathbb{N}$ and $x \in A$, if $|x - x_0| < \delta$, then $|f^n(x) - x_0| < \varepsilon$.

Obviously, if $x_0 \in \mathbb{I}$ is a 0-approximate stable point of f, then f is approximately (0-approximately) continuous at x_0. On the other hand, it is easy to find a continuous function such that $x_0 \in \mathrm{Fix}(f)$ and x_0 is not a 0-approximate stable point of f. The following theorem is of interest in connection with stability and for the observation that a function for which x_0 is a 0-approximate stable point may attract positive entropy at this point.

THEOREM 4.1. *Let $f \in \mathfrak{D}$ be a function approximately continuous at $x_0 \in \mathrm{Fix}(f)$. There exists $g \in \langle f \rangle_{x_0}$ such that g attracts positive entropy at x_0 and x_0 is a 0-approximate stable point of g.*

PROOF. Since f is approximately continuous at $x_0 \in \mathrm{Fix}(f)$, one can find a set $B \in \mathcal{L}$ such that $d(B, x_0) = 1$ and $\lim_{B \ni x \to x_0} f(x) = x_0$. There is no loss of generality in assuming that $x_0 \in B$. By Theorem 1.1, there is a perfect set A such that $\{x_0\} \subset A \subset B$ and $d(A, x_0) = 1$.

Let $P = \bigcup_{n=1}^{\infty} L_n \cup \{x_0\} \cup \bigcup_{n=1}^{\infty} R_n \subset \mathbb{I}$ be an interval set at x_0 such that $d(P, x_0) = 1$. Without loss of generality, we can assume that $\bigcup_{n=1}^{\infty} L_n \neq \emptyset$, $\bigcup_{n=1}^{\infty} R_n \neq \emptyset$, $\sup R_1 < 1$ and $\inf L_1 > 0$. Obviously, there exists $\delta > 0$ such that $P \subset (x_0 - \delta, x_0 + \delta)$. Moreover,

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we see at once that $d(A \cap P, x_0) = 1$, $A \cap P$ is a closed set and $x_0 \in A \cap P$. Put $C = A \cap P$.

Let $A_n = R_n \cap A$ for $n \in \mathbb{N}$. There is no loss of generality in assuming that $A_n \neq \emptyset$ for $n \in \mathbb{N}$. Put $g_n(x) = x$ for $x \in A_n$ and $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, the set A_n is closed, so there exists a continuous function $g_n^* : R_n \to R_n$ such that $g_n^* \restriction A_n = g_n$. Similarly, for each $n \in \mathbb{N}$, there is a continuous function $\beta_n^* : L_n \to L_n$ such that $\beta_n^*(x) = x$ for $x \in L_n \cap A$.

For $n \in \mathbb{N}$, fix a nondegenerate interval $H_n = [a_n^*, b_n^*] \subset (\sup R_{n+1}, \inf R_n)$ and $c_n^*, d_n^*, s_n^*, t_n^* \in H_n$ such that $a_n^* < c_n^* < d_n^* < s_n^* < t_n^* < b_n^*$. Define the function g as follows: $g(x) = g_n^*(x)$ for $x \in R_n, n \in \mathbb{N}$; $g(x) = \beta_n^*(x)$ for $x \in L_n, n \in \mathbb{N}$; $g(x) = f(x)$ for $x \in \{x_0\} \cup \mathbb{I} \setminus (x_0 - \delta, x_0 + \delta)$; $g(x) = a_n^*$ for $x \in \{a_n^*, d_n^*, t_n^*\}, n \in \mathbb{N}$; $g(x) = b_n^*$ for $x \in \{b_n^*, c_n^*, s_n^*\}, n \in \mathbb{N}$ and g is linear otherwise.

First we will show that

$$g \in \langle f \rangle_{x_0}.$$

Let $\varepsilon > 0$. There is $\delta \in (0, \frac{1}{3}\varepsilon)$ such that $f(B \cap [x_0 - \delta, x_0 + \delta]) \subset (x_0 - \frac{1}{3}\varepsilon, x_0 + \frac{1}{3}\varepsilon)$, so $f(C \cap [x_0 - \delta, x_0 + \delta]) \subset (x_0 - \frac{1}{3}\varepsilon, x_0 + \frac{1}{3}\varepsilon)$. Moreover, $d(C \cap [x_0 - \delta, x_0 + \delta], x_0) = 1$ and $g(C \cap [x_0 - \delta, x_0 + \delta]) \subset (x_0 - \frac{1}{3}\varepsilon, x_0 + \frac{1}{3}\varepsilon)$. From this,

$$\rho_u(f \restriction (C \cap [x_0 - \delta, x_0 + \delta]), g \restriction (C \cap [x_0 - \delta, x_0 + \delta])) < \varepsilon,$$

which shows that f and g are approximately ε-close at x_0 and, in consequence, that $g \underset{x_0}{\asymp} f$. Since $\varepsilon > 0$ is arbitrary, it follows that $g \in \langle f \rangle_{x_0}$.

Now we will prove that x_0 is a 0-approximate stable point of g. We see at once that $x_0 \in \mathrm{Fix}(g)$. Fix $\varepsilon_* > 0$ and put $\delta_* = \varepsilon_*$. Let $n \in \mathbb{N}$, $x \in C$ and $|x_0 - x| < \delta_*$. Thus, $g^n(x) = x \in (x_0 - \varepsilon_*, x_0 + \varepsilon_*)$. Obviously, since $g \restriction C$ is an identity function, we have $h(g, C) = 0$.

To complete the proof, it suffices to show that g attracts positive entropy at x_0. By conditions (D1)–(D3), we deduce that $g \in \mathfrak{D}$.

Let $\varepsilon_1 > 0$. Obviously, there is $n_0 \in \mathbb{N}$ such that $[\sup R_{n_0+1}, \inf R_{n_0}] \subset B(x_0, \varepsilon_1)$. Put $\delta = \min\{\frac{1}{2}(c_{n_0}^* - a_{n_0}^*), \frac{1}{2}(b_{n_0}^* - t_{n_0}^*)\}$ and consider $g_* \in B(g, \delta) \cap \mathfrak{D}$. Since $g_* \in \mathfrak{D}$, it follows that for $F_0 = [c_{n_0}^*, d_{n_0}^*]$ and $F_1 = [e_{n_0}^*, t_{n_0}^*]$, we have $F_i \underset{g_*}{\to} F_j$ for $i, j = 0, 1$. From Lemma 2.1, $h(g_*, B(x_0, \varepsilon)) > 0$ and so g attracts positive entropy at x_0. □

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