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SUB-DIRECT PRODUCT CLOSED FITTING CLASSES

R.A. BRYCE

It is shown that in the Fitting class of all finite p-by-q groups, where p and q are different primes, there is among the sub-direct product closed sub-Fitting classes a unique maximal one : it consists of the groups whose minimal normal subgroups are central.

Several investigations, for example [1,2,3,5] consider the problem of classifying metanilpotent Fitting classes closed under an additional closure operation. To date the least satisfactory results concern sub-direct product closure. In contrast to subgroup closure, quotient group closure and saturation there are sub-direct product closed metanilpotent Fitting classes which are not primitive saturated formations: C, the class of finite soluble groups whose minimal normal subgroups are all central, is a sub-direct product closed Fitting class, so $S_pS_q \cap C$ is a non-quotient group closed, and therefore non-primitive, Fitting class when p,q are different primes. The purpose of this note is to prove the following result.

THEOREM. *If p,q are different primes then $S_pS_q \cap C$ is the unique maximal sub-direct product closed Fitting class contained in S_pS_q.*

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A proof of this result in the case $q | p-1$ is given in [3] and the present proof closely follows that given there. To prove the theorem it suffices to show that if $\mathfrak{F}$ is a sub-direct product closed Fitting class contained in $S_p S_q$, and if $\mathfrak{F} \not\subseteq \mathfrak{C}$, then $\mathfrak{F} = S_p S_q$.

The following lemma will be used several times in the proof.

(1) *Suppose that* A_1, A_2 *are operator groups for a nilpotent group* B, *and suppose that* $\mathfrak{X}$ *is a sub-direct product closed Fitting class. If* $(A_1 \times A_2)B \in \mathfrak{X}$ *then* $A_1 B \in \mathfrak{X}$.

For, $H = (A_1 \times A_1 \times A_2)B \in R_0\{(A_1 \times A_2)B\} \subseteq \mathfrak{X}$. Since $H/(A_1 \times 1 \times 1) \in \mathfrak{X}$ it follows from Lemma 1.1 of [2] that $H/(1 \times A_1 \times A_2) \cong A_1 B \in \mathfrak{X}$.

With $\mathfrak{F}$ as above we show that

(2) $$D_p^q \in \mathfrak{F}$$

For, since $\mathfrak{F} \not\subseteq \mathfrak{C}$ there is a group $G \in \mathfrak{F}$ with a non-central minimal normal subgroup M. M is a p-group and $G/\mathfrak{C}_G(M)$ is a q-group. Consider the natural splitting extension $R = MG$. There are two natural homomorphisms of this group onto G, one defined by $(g,m) \to g$, the other defined by $(g,m) \to gm$. Since their kernels intersect trivially it follows that $R \in \mathfrak{F}$. However, if

$$C = \{(g,1) : g \in \mathfrak{C}_G(M)\}$$

then $C \trianglelefteq R$, $C \cap M = 1$ and R/MC is nilpotent, so by Lemma 1.1 of [2], $R/C \in \mathfrak{F}$. It follows that $\mathfrak{F}$ contains a group S with the following structure : $S = MD$ where M is an elementary abelian p-subgroup, the unique minimal normal subgroup of S, and D is a non-trivial q-group. Consequently if Q is a subgroup of order q contained in $\zeta_1(D)$ then $MQ \in S_n \mathfrak{F} = \mathfrak{F}$; and if M_1 is an irreducible component of M_Q then $M = M_1 \oplus M_2$ where M_2 admits Q. Hence by (1) $D_p^q \cong M_1 Q \in \mathfrak{F}$.

Let $\mathfrak{U}(p,q)$ denote the class of all groups with a normal p-subgroup complemented by a subgroup of order q. The next result is an immediate corollary of (2).

(3) *If* $H \in \mathfrak{U}(p,q)$ *and* $O_{\bar{p}}(H)$ *is elementary abelian then* $H \in \mathfrak{F}$.

(4) *If* $\mathfrak{U}(p,q) \subseteq \mathfrak{F}$ *then* $\mathfrak{F} = S_p S_q$.

Let $H = AB$, $A \trianglelefteq H$, $A \cap B = 1$, $A \in S_p$, $B \in S_q$. We show by induction on $|B|$ that $H \in \mathfrak{F}$. If B is not cyclic then B is a product of normal subgroups B_1, B_2 of smaller order than $|B|$ so by induction

$$AB \in N_0\{AB_1, AB_2\}$$

$$\subseteq \mathfrak{F} .$$

If B is cyclic, say of order q^{α} then the case $\alpha = 1$ is covered by the hypothesis of (4); so suppose $\alpha > 1$. In this case we may regard B as a subgroup of the group $W = C_q \text{ wr } C_{q^{\alpha-1}}$. Form the twisted wreath product (see pp.227-8 of [4])

$$T = A \text{ twr}_B W .$$

Then since W is a product of subnormal subgroups of order less then $|B|$, $T \in \mathfrak{F}$. But then

$$(A|^W)B \in S_n\mathfrak{F} = \mathfrak{F} .$$

Since by Lemma 3.3 of [4]

$$A|^W|B \cong A \times K$$

where K admits B we conclude from (1) that $AB \in \mathfrak{F}$ to complete the induction.

(5) *Let* $H \in \mathfrak{U}(p,q)$ *and let* V *be a* Z_pH*-module having a submodule* U *with* V/U *being completely reducible. If* p *has order* d *modulo* q *then* $(dV)H \in \mathfrak{F}$ *if* $UH \in \mathfrak{F}$ *and* $(dU)H \in \mathfrak{F}$ *if* $VH \in \mathfrak{F}$.

Write Γ for a cyclic group of order q and let N be a faithful irreducible $Z_p\Gamma$-module. Since V/U is completely reducible for H we may write

$$V\#N/U\#N \simeq \bigoplus_{i=1}^{r} X_i/U\#N$$

where each $X_i/U\#N$ is irreducible for $H \times \Gamma$. Then, $O_p(H)$ being in the kernel of $X_i/U\#N$, and faithfully and irreducible represented abelian

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groups being cyclic, K_i, the kernel of $X_i/U\#N$ in $H \times \Gamma$, satisfies $O_p(H) \leq K_i$ and $H\Gamma = K_i\Gamma$. It follows that if we write $Y_0 = U\#N$ and

$$Y_i = \sum_{j=1}^{i} X_j , \quad 1 \leq i \leq r ,$$

then $$Y_iK_{i+1} \trianglelefteq Y_{i+1}K_{i+1} \in N_0\{Y_iK_{i+1}, Y_{i+1}\} , \quad 0 \leq i \leq r-1 .$$

Consequently

(6) $Y_iK_{i+1} \in F$ if and only if $Y_{i+1}K_{i+1} \in F$, $0 \leq i \leq r-1$.

Also

(7) $$Y_iK_{i+1} \trianglelefteq Y_i(H\times\Gamma) \in N_0\{Y_iK_{i+1}, Y_i\Gamma\}$$

and

(8) $$Y_{i+1}K_{i+1} \trianglelefteq Y_{i+1}(H\times\Gamma) \in N_0\{Y_{i+1}K_{i+1}, Y_{k+1}\Gamma\} .$$

Then, since $Y_j\Gamma \in F$ for all j by (3), we have that if $Y_iH \in F$ then $Y_iK_{i+1} \in F$ by (7) whence $Y_{i+1}K_{i+1} \in F$ by (6) and then that $Y_{i+1}H \in F$ by (8). A similar argument shows the converse and hence $Y_iH \in F$ if and only if $Y_{i+1}H \in F$ $(0 \leq i \leq r-1)$. By induction therefore $(U\#N)H \in F$ if and only if $(V\#N)H \in F$. However $(U\#N)_H$ is isomorphic to a direct sum of d copies of U, $(V\#N)_H$ is isomorphic to a direct sum of d copies of V and therefore since F is R_0-closed (5) is proved.

(9) *Let* $H \in U(p,q)$ *and let* V *be a* Z_pH*-module. Then* $H \in F$ *if and only if* $VH \in F$.

For, there is a chain of submodules

$$V = V_0 > V_1 > V_2 > \ldots > V_s = 0$$

in which the factors V_{i-1}/V_i $(1 \leq i \leq s)$ are completely reducible. It follows from (5) by induction on i that $(d^iV)H \in F$ if $V_iH \in F$ and that $(d^iV_i)H \in F$ if $VH \in F$. In particular when $i = s$ we have that

(10) $$H \in F \quad \textit{if} \quad VH \in F ,$$

which is one half of what we want, and also that $(d^sV)H \in F$ if $H \in F$.

Now

$$d^s V = (d^s - 1)V \oplus V$$

and so it follows from (10) that $(d^s V)H \in \mathfrak{F}$ implies $VH \in \mathfrak{F}$ (replacing H by VH and V by $(d^s - 1)V$). This completes the proof of (9).

Proof of Theorem. We use induction on the order of groups in $U(p,q)$. Let $G \in U(p,q)$ have a minimal normal subgroup M. We show that $G \in \mathfrak{F}$ if $H = G/M \in \mathfrak{F}$. The hypothesis $H \in \mathfrak{F}$ means that

$$W = M \text{ wr } H \in \mathfrak{F} :$$

this because of (9). Let A be the base group of W. By the Krasner-Kaloujnine embedding G may be regarded as a subgroup of W supplementing $A : W = AG$. There is a natural semi-direct product $X = AG$ from which there are two natural homomorphisms onto W :

$$(g,x) \mapsto (gM)x \quad \text{and} \quad (g,x) \mapsto gx , \quad g \in G , \; x \in A ,$$

whose kernels intersect trivially. Consequently $X \in \mathfrak{F}$, if $H \in \mathfrak{F}$. A final application of (9) yields that $G \in \mathfrak{F}$ if $X \in \mathfrak{F}$. This completes the induction and the proof of the theorem.

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Department of Pure Mathematics,
Australian National University,
G.P.O. Box 4,
CANBERRA,
ACT 2601.
Australia.